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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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SPECTRAL ANALYSIS IN VIBRATION MECHANICS
(Part II)

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Part II : Vibrations in linear mechanics

A : the abstract problem.

II-1- The general equations in linear oscillations

1- Hypothesis

(i) • Consider two real separable Hilbert spaces V and H such that : $V \subset_{\circ} H$ (with density)
The mapping from V into H is moreover supposed compact

(ii) • A denotes an operator on V with the following properties :
A is linear continuous ($A \in \mathcal{L}(V; V')$), V elliptic (α denotes the coercivity constant) and self adjoint

• We associate to A, the bilinear symmetric continuous and V. elliptic form:

(iii) $$\langle Au, v\rangle = a(u, v) \qquad \forall u, v \in V$$

2- The abstract problem.

$(\cdot,\cdot)$ denotes an inner product over the space H
$((\cdot,\cdot))$ " " over the space V
$\langle\cdot,\cdot\rangle$ is the duality product between V and its dual space V'.
The [illegible] of our chapter is to solve the following

problem :

Find u solution of the equation :

$$(1)\begin{cases} \dfrac{d^2u}{dt^2} + Au = 0 \\ u(0) = u_0 & \text{(initial conditions)} \\ u'(0) = u_1 \end{cases}$$

u_0, u_1 are given functions and T is a known arbitrary positive number.

The variational formulation is

⊛ See note p 41

$$(2)\begin{cases} \left\langle \dfrac{\partial^2 u}{\partial t^2}(t), v \right\rangle + a(u(t), v) = 0 & \forall v \in V;\ \text{a.e } t \in]0,T[\\ u(0) = u_0 \\ u'(0) = u_1 \end{cases}$$

Examples will be described in part B, in order to apply the abstract method for solving practical problems in vibration mechanics.

3. Preliminary lemmas.

lemma 2.1. The form $(u,v) \to a(u,v)$ is an inner product over the space V and the associated norm is equivalent to the given norm $\|\cdot\|$

The proof is trivial. We have

$$\underbrace{\alpha\|u\|^2 \leq a(u,u)}_{\text{v ellipticity}} \underbrace{\leq M\|u\|^2}_{\text{continuity}}$$

Lemma 2.2: The operator A has an infinite sequence $\{\lambda_i\}$ of positive eigenvalues increasing to infinity. The corresponding eigenfunctions form a complete orthogonal sequence in V. That means:

$$a(w_i, \varphi) = \lambda_i (w_i, \varphi) \qquad \forall \varphi \in V \quad \text{for each } i$$

(3) $$a(w_i, w_j) = \lambda_i \delta_{ij}$$

This is exactly theorem 1.5.

Lemma 2.3 By lemma 2.2 and Bessel inequality we get the inequality

(4) $$a(\varphi, \varphi) \geq \sum_{j=1}^{\infty} \left| a\left(\varphi, \frac{w_j}{\sqrt{\lambda_j}}\right)\right|^2 = \sum_{j=1}^{\infty} \lambda_j |(\varphi, w_j)|^2$$

II-2- Existence and uniqueness theorem

1- Theorem 2.1-

Let $u_0 \in V$ and $u_1 \in H$. Then there exist a unique function u, having the following properties.

(5) $u \in \mathcal{C}^0([0\,T]; V)$ T is given arbitrary

(6) $u' = \frac{du}{dt} \in \mathcal{C}^0([0\,T]; H)$

(1) u satisfies (1) $\frac{d^2u}{dt^2} + Au = 0$ with the initial condition

u is an infinite sum of elementary solutions that is.

$$u(x,t) = \sum^{\infty} \left[(u_0, w_i)\cos\sqrt{\lambda_i}\, t + \frac{1}{\sqrt{\lambda_i}} (u_1, w_i) \sin\sqrt{\lambda_i}\, t\right] w_i(x)$$

2- Proof of theorem 2.1

We begin by the usual Faedo Galerkin's procedure so we have 3 parts in the proof

* Finite dimensional solution

For each m let V_m be the space spanned by the eigenfunctions $w_1, w_2, \dots w_m$. We define an approximate solution as follows

(8) $$u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j$$

(9) $$(u''_m(t), w_i) + \langle A u_m(t), w_i \rangle = 0 \qquad i = 1, 2 \dots m.$$

(10) $$u_m(0) = u_{0m} \in V_m \; ; \; u'_m(0) = u_{1m} \in V_m.$$

with $u_{0m} \to u_0$ in V as $m \to \infty$

and $u_{1m} \to u_1$ strongly in H as $m \to \infty$

(This is possible as V_m is dense in V and in H)

The equations (9) (10) form a linear differential system for the functions $g_{1m}, g_{2m}, \dots g_{mm}$:

(11) $$\begin{cases} g''_i(t) + \lambda_i g_i(t) = 0 & i = 1, \dots m. \\ g_i(0) = (u_0, w_i) \\ g'_i(0) = (u_1, w_i) \end{cases}$$

λ_1 is different from zero because A is V-elliptic. The linear differential system (11) has only one solution equal to.

$$(12) \qquad u_m(t) = \sum_{i=1}^{m} \left[(u_0, w_i) \cos\sqrt{\lambda_i}\, t + \frac{1}{\sqrt{\lambda_i}} (u_1, w_i) \sin\sqrt{\lambda_i}\, t \right] w_i(x)$$

A calculation gives:

$$(13) \qquad u_m'(t) = \sum_{1}^{m} \left[\sqrt{\lambda_i}\, (u_0, w_i) \sin\sqrt{\lambda_i}\, t + (u_1, w_i) \cos\sqrt{\lambda_i}\, t \right] w_i(x)$$

<u>lemma 2-4.</u> : For each m

There exists one and only one solution u_m of (9) (10). Moreover $u_m \in \mathcal{C}^0([0\,T]; V)$, $u_m' \in \mathcal{C}^0([0\,T]; H)$.

The last property is a easy consequence of formulas (12) & (13). The lemma 2-4 can be extended to the case $t \in [0, \infty[$

* <u>A priori estimates</u>

From (12) we obtain $\forall t \in [0, T]$:

$$\|u_m(t)\|_V \leq \sum_{1}^{m} \left| (u_0, w_i) \cos\sqrt{\lambda_i}\, t + \frac{1}{\sqrt{\lambda_i}} (u_1, w_i) \sin\sqrt{\lambda_i}\, t \right| \|w_i\|,$$

As $\|\cdot\|$ is the norm on V associated to the inner product $a(u, v)$, we have $\|w_i\|_V = \sqrt{\lambda_i}$. So

$$\|u_m(t)\|_V \leq \sum_{1}^{m} \sqrt{\lambda_i}\, |(u_0, w_i)| + \sum_{1}^{m} |(u_1, w_i)|$$

$$\|u_m(t)\|_V \leq \sum_{1}^{\infty} \sqrt{\lambda_i}\, |(u_0, w_i)| + \sqrt{2}\, |u_1|_H$$

We obtain by (4)

$$(14) \qquad \|u_m(t)\|_V \leq \sqrt{2} \left(|u_0|_V + |u_1|_H \right)$$

In the same way we get by $|w_i|_H = 1$

$$|u'_m(t)|_H \leq \sum_1^m \sqrt{\lambda_i}\,|(u_0, w_i)| + \sum_1^\infty |(u_1, w_i)|$$

(15) $|u'_m(t)|_H \leq \sqrt{2}\,\left(\|u_0\|_V + |u_1|_H\right)$

These estimates (14) (15) enable us to say

<u>lemma 2.5</u> :

| The sequence u_m remains in a bounded set of $\mathcal{C}^0([0\,T]; V)$ and u'_m remains in a bounded set of $\mathcal{C}^0([0\,T]; H)$, independantly of m

* <u>Passage to the limit</u>

- The sequences $\{u_m\}$ and $\{u'_m\}$ are cauchy sequences of bounded and continuous functions.

u_m denotes the m th partial sum of (7)

so $u_m(t)$ converges in V uniformly with respect to t, as we have,

$$\forall t, \quad \|u(t) - u_m(t)\|_V \leq \sum_{m+1}^\infty \sqrt{\lambda_i}\,|(u_0, w_i)| + \sum_{m+1}^\infty |(u_1, w_i)|$$

(these terms converge to 0 for $m \to \infty$ because u_{0m} converges strongly to u_0 in V and u_{1m} converges strongly to u_1 in H.)

It follows that <u>$u \in \mathcal{C}^0([0\,T], V)$</u> (5)

- a similar calculation implies that u'_m converges to u in H uniformly with respect moreover $Au_m(t)$ converges to $Au(t)$ in V' uniformly with respect to t.

- It remains to prove that u defined by (7) satisfies the ... in (2). We can pass to the limit in (5) by standard procedure.

Another way consists to prove by direct calculation that (7) implies:

$$(16) \quad \langle \int_0^T [u'(t)\,\varphi'(t) + Au(t)\,\varphi(t)]\,dt,\ w_i \rangle = 0 \qquad \forall \varphi \in \mathcal{D}(]0,T[)$$

for each i [illegible], μ is an arbitrary constant. $\mathcal{D}(]0,T[)$ is the set of functions indefinitely derivable with compact support in $]0,T[$.

The equality (16) which holds for each i holds, by a density argument, for each v which is a finite linear combination of w_i. Since each term in (16) depends linearly and continuously on v (for the norm of V), the equality is still valid by continuity for each v in V. And this implies:

$$(17) \quad \frac{d^2u}{dt^2} + Au = 0 \qquad \text{in } \mathcal{D}'(]0,T[;V')$$

$\mathcal{D}'(]0,T[;V')$ is the space of distributions

We remark that $Au \in \mathcal{C}^0(0T; V')$.

(17) implies

$$\left\langle \frac{\partial^2 u(t)}{\partial t^2} + Au(t), v \right\rangle = 0 \qquad \forall v \in V \text{ and a.e. } t \in [0T]$$

This proves (1); moreover by (7) we have, of course $u(0) = u_0$ and $u'(0) = u_1$ -

* To prove theorem 2.2 it remains now to study the uniqueness problem, that is the

<u>lemma 2.6</u> : Let w be a solution of (2) with initial conditions $w(0) = w'(0) = 0$. Then $w = 0$.

Two proofs will be given according to the assumptions of regularity.

<u>Proof 1</u> : Suppose that w' belongs to $L^2(0T, V)$ and that w'' belongs to $L^2(0T; H)$

$L^2(0T, H)$ denotes the space of L^2 integrable functions from $[0T]$ into H.

As w belongs to $L^2(0T; V)$ we have :

$$(w'', w') - \langle Aw, w' \rangle = 0$$

So $\quad \frac{1}{2} \frac{d}{dt} |w'|_H^2 + \frac{d}{dt} a(w, w) = 0$

By integrating between 0 and s, for arbitrary $s \geq 0$ we obtain

$$|w'(s)|^2 + 2a(w(s), w(s)) = 0$$

Proof 2 without regularity assumptions.

we define a function $\varphi(t)$ by the following conditions

$$\varphi(t) = \begin{cases} -\int_t^s w(\tau)\, d\tau & t \leq s \\ 0 & \text{as } t > s \quad s > 0 \text{ fixed} \end{cases}$$

we have therefore

$$\int_0^s \langle w'', \varphi(t)\rangle dt + \int_0^s \langle A w(t), \varphi(t)\rangle dt = 0$$

$$\int_0^s \langle A w(t), \varphi(t)\rangle dt = \int_0^s a(\varphi'(t), \varphi(t))\, dt = \frac{1}{2}\left[a(\varphi(t), \varphi(t))\right]_0^s$$

$$= -\frac{1}{2}\, a(\varphi(0), \varphi(0))$$

$$\int_0^s \langle w''(t), \varphi(t)\rangle dt = \underbrace{\langle w'(t), \varphi(t)\rangle\Big|_0^s}_{=0} - \int_0^s \langle w'(t), w(t)\rangle dt$$

$$= -\frac{1}{2}\, |w(s)|_H^2$$

Then: $\frac{1}{2}\left(|w(s)|_H^2 + a(\varphi(0), \varphi(0))\right) = 0 \qquad \forall s > 0$

and $|w(s)|^2 \leq 0$. we deduce that $w(s) = 0$.

Theorem 2.1 holds.

3- Remarks

Remarks 2.1. Similar results hold when T becomes infinity

* if $u_0 \in D(A)$ and $u_1 \in V$, then $\frac{d}{dt}(u'_m, v)$ converges to $\frac{d}{dt}(u', v)$ uniformly with respect to t. Then the equation (2) is verified for all $t \in [0, T]$

Remark 2.2 : we prove the additional property :

Property 2.1. The solution u of theorem 2.1 is almost periodic in t.

Proof :

- From H. Bohr definition a function F is called almost periodic if [10] :

For each $\varepsilon > 0$ there exists $l > 0$ such that every interval of l length contains at least a value T such that : $|F(t+T) - F(t)| < \varepsilon$, $0 < t < \infty$

- By (7) we have :

$$\| u(t+T) - u(t) \| \leq \sqrt{2} \, |\sin \tfrac{T}{2}| \, \{ \|u_0\|_V + |u_1|_H \}$$

Thanks to the periodicity of function sin we can define l associated to each $\varepsilon > 0$ such that $|\sin \frac{T}{2}| \leq \dfrac{\varepsilon}{\sqrt{2}\,[\|u_0\|_V + |u_1|_H]}$

So u is almost periodic.

II - 3. mechanical significance

The last properties have some mechanical significance

1- definition 2.1.

a solution $u(x,t)$ is called a stationnary solution if it is equal to the product of a function with x as only parameter by a function with t as only parameter.

Then theorem 2.1 can be translated in a short form :

2- Theorem 2.2-

Problem (2) has one and only one solution. The solution is equal to the infinite sum of the stationnary solutions.

3- mechanical properties

(2) is the abstract general problem in linear vibrations. The stationnary solutions are defined if the eigenvalues λ_i and the associated eigenfunctions w_i are known.

- definition 2.2-

The set $\{\sqrt{\lambda_i}, w_i\}$ which caracterizes a stationnary solution is called a "normal mode".

- Practically we only calculate the stationnary solutions with the formula $u(x,t) = e^{j\omega t} u(x)$

The equality (7) is called the "modal schema"

- The last assumptions imply that, under the hypothesis of II.1 the modal schema is convergent.

- In conclusion.

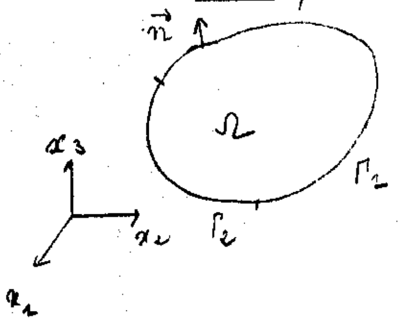
a free oscillation problem in linear vibration mechanics is a special problem for self adjoint positive compact operators.

Vibrations in linear mechanics

B - Some classical applications.

II - h - Example 2.1: Free vibrations in linear elasticity

1 - the problem.



Notations:

For $n = 3$, the elastic body is represented by a bounded open set Ω with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 = \Gamma$. $\vec{n}(n_i)$ represents the outer normal to $\partial\Omega$.

ρ is the density. As the system is incompressible, ρ can be supposed equal to 1.

$u(x,t) = (u_1, u_2, u_3)$ is the displacement field vector referred to the equilibrium configuration.

σ is the tensor whose components $\sigma_{ij}(u)$ represent the stress variations between the actual and the initial state.

We recall that the constitutive equations for linear elasticity are of the form:

$$\sigma_{ij}(u) = \sum_{h,k} a_{ijhk}\, \varepsilon_{hk}(u) \tag{1.8}$$

a_{ijhk} are the elasticity coefficients and

$$\varepsilon_{hk}(u) = \frac{1}{2}\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right) \qquad \text{(strain tensor)}$$

In the example 2.1:
the coefficients a_{ijhk} will be assumed to satisfy the following conditions [4]:

$a_{ijhk} \in L^{\infty}(\Omega)$ ($L^{\infty}(\Omega)$ is the space of essentially bounded functions in Ω)

$a_{ijhk} = a_{ijkh} = a_{khij}$ (symmetric property)

There exists $\alpha > 0$ such that:

$$\sum_{ijhk} a_{ijhk}\, \varepsilon_{ij}\, \varepsilon_{hk} \geq \alpha \sum_{ij} \varepsilon_{ij}\, \varepsilon_{ij}$$ (V-ellipticity)

The study of small vibrations is defined by the following dynamic equations and the problem is:

Find $u(x,t)$ in $\Omega \times]0,T[$ solution of

(19) $$\frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(u)}{\partial x_j}$$ in $\Omega \times]0,T[$, $i = 1, 2, 3$

(20) $$u_i = 0$$ on $\Gamma_1 \times]0,T[$

(21) $$\sum_{j=1}^{3} \sigma_{ij} n_j = 0$$ on $\Gamma_2 \times]0,T[$

(mixed boundary conditions)

with the initial conditions

(22) $$u(x,0) = u_0(x)$$

(23) $$u'(x,0) = u_1(x)$$

T, u_0, u_1 are given

2- Functional properties:

We begin to recall some important results, ... referred to [4].

* Let u and v be two functions such that $v \in (H^1(\Omega))^3$ and $u \in (H^1(\Omega))^3$ with $\frac{\partial}{\partial x_j}\sigma_{ij}(u) \in L^2(\Omega)$ for $i=1,2,3$. Then $\sum_{j=1}^{3} \sigma_{ij}(u) n_j$ belongs to the space $H^{-1/2}(\Gamma)$ and we have the

<u>lemma 2.7</u> (under the last assumptions)

The following Green's formula holds

$$(24)\quad \boxed{(Au, v)} = a(u,v) - \sum_{i=1}^{3} \left\langle \sum_{j=1}^{3} \sigma_{ij}(u) n_j , \gamma_0 v_i \right\rangle_{H^{-1/2}, H^{1/2}}$$

with $\quad (Au)_i = - \dfrac{\partial \sigma_{ij}}{\partial x_j}$

$$a(u,v) = \sum_{i,j,h,k} \int_\Omega a_{ijhk}\, \varepsilon_{ij}(v)\, \varepsilon_{hk}(u)\, dx$$

$$a(u,v) = \sum_{i,j} \int_\Omega \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, dx = a(v,u)$$

($dx = dx_1\, dx_2\, dx_3$ is the Lebesgue's measure in Ω)

γ_0 is the trace operator ($\gamma_0 v_i = v_i|_\Gamma$ and $\gamma_0 v_i \in H^{1/2}(\Gamma)$ if $v_i \in H^1(\Omega)$)

$(\cdot,\cdot)$ is the scalar product in $(L^2(\Omega))^3$. Recall that the associated norm is $|u| = \left(\sum_{i=1}^{3} |u_i|^2_{L^2(\Omega)} \right)^{1/2}$

$((\cdot,\cdot))$ is the scalar product in $(H^1(\Omega))^3$

$$\|u\| = \left(|u|^2 + \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right|^2 \right)^{1/2}$$

$\langle \cdot,\cdot \rangle$ is the duality product between $H^{1/2}(\Gamma)$ and its dual space $H^{-1/2}(\Gamma)$

The proof of formula (24) is not trivial (see [3])

Exercise: By using symmetric properties of stress and strain tensors (24) can be easily verified at least formally

* The following theorem is the most important for the functional analysis of elasticity problems.

Theorem 2.3.

For all $v = (v_1, v_2, v_3) \in (H^1(\Omega))^3$, there exist a constant $c(\Omega)$ such that.

(25) $$\sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(v)\, \varepsilon_{ij}(v)\, dx + \sum_{i=1}^{3} \int_\Omega v_i\, v_i\, dx \geqslant c(\Omega) \|v\|^2$$

(25) is called Korn's inequality. For the proof, the reader can be referred to Friedrichs [5] and Duvaut. Lions [4].

(26) * let $V = \{ v \, ; \, v \in (H^1(\Omega))^3 , \, v_i = 0 \text{ on } \Gamma_1 \}$

V is equipped with the norm $\|\cdot\|$ and is a closed space in $(H^1(\Omega))^3$. By (26) and (25) we obtain

Theorem 2.4.

There exists a positive constant α_0 such that

$a(v, v) \geqslant \alpha_0 \|v\|^2, \quad \forall v \in V$

Proof: see [4] p. 22.

3 - Variational formulation of problem (15)-(20)

Using (26) we obtain that:

(15)-(25) is equivalent to the following problem

Find $u(t) \in V$ solution of

(27) $$\frac{d}{dt}(u'(t), v) + a(u(t), v) = 0 \quad \forall v \in V, \ (\text{equality in } \mathcal{D}'(]0,T[))$$

$u(0) = u_0$

Proof of theorem 2.4.

By Korn's inequality and V-ellipticity (18) we verify that to prove theorem 2.4, we must prove the existence of a constant K such that

(i) $$\sum_{ij} \int_\Omega \varepsilon_{ij}(v)\, \varepsilon_{ij}(v)\, dx \geq K |v|^2 \quad ; \quad \forall v \in V; \ |v| = 1$$

(the problem is homogeneous with respect to v, so we can choose $|v| = 1$)

Suppose the inverse: there exists a set of constants c_n, $c_n \xrightarrow[n\to\infty]{} 0$ and a set $\{v_n\}$, $v_n \in V$, $|v_n| = 1$ such that:

(ii) $$\sum_{ij} \int_\Omega \varepsilon_{ij}(v_n)\, \varepsilon_{ij}(v_n)\, dx = c_n$$

Then the sequence $\{v_n\}$ is bounded in V (thanks to Korn's inequality) and we can substract a subsequence (still noted v_n) such that

(iii) $$v_n \rightarrow v \quad \text{weakly in } V$$

V is a closed convex space in $(H^1(\Omega))^3$ and the form: $u \to \sum_{ij} \int_\Omega \varepsilon_{ij}(u)\, \varepsilon_{ij}(u)\, dx$ is weakly inf semi continuous (by convexity and continuity [illegible]) so:

$$\lim_{n\to\infty} c_n \geq \liminf_{n\to\infty} \sum_{ij} \int_\Omega \varepsilon_{ij}(v_n)\, \varepsilon_{ij}(v_n)\, dx \geq \sum_{ij} \int_\Omega \varepsilon_{ij}(v)\, \varepsilon_{ij}(v)\, dx$$

(iv) $$0 \geq \sum_{ij} \int_\Omega \varepsilon_{ij}(v)\, \varepsilon_{ij}(v)\, dx$$

Then $\varepsilon_{ij}(v) = 0$ a.e., $\forall i, j$. This property means that v is a rigid displacement. From $v_i = 0$ on Γ, it follows that $v = 0$. ∎

(v) $$v_n \rightarrow 0 \quad \text{weakly in } V$$

Moreover the mapping from $H^1(\Omega)$ into $L^2(\Omega)$ is compact (Theorem 1.2). By (Eu) we get:

(vi) $$v_n \rightarrow v \quad \text{strongly in } (L^2(\Omega))^3$$

That means $|v_n| \rightarrow |v| = 0$; this property contradicts the condition $|v_n| = 1$

The theorem is proved

<u>Remark</u>. $\sqrt{a(v,v)}$ defines on V a norm equivalent to the usual norm $\|\cdot\|$

In this setting, the variational equation (27) represent the principle of virtual work valid for all kinematically admissible displacement v (i.e which satisfies the boundary condition $v=0$ on Γ_1)

h- Resolution

- We notice that hypothesis II-1 are satisfied

$H = (L^2(\Omega))^3$

V is given by (26) and the mapping from V into H is compact (theorem 1.1)

A is the given operator of elasticity

$a(u, v)$ is bilinear, symmetric, continuous (verify it, as exercise), and by theorem 2-4, it is V-elliptic on V.

- So, theorem 2.2 is valid here, we get the well known result

Theorem 2.5. In linear elasticity, elastic bodies have an infinity of "normal modes" which give periodic oscillations with respect to t.

Remark 2.2

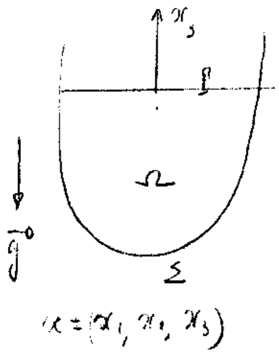
By introducing viscoelasticity, we obtain a non linear problem. This study is not our aim but it can be referred to [12] for example.

II.5. Example 2.2. Linear vibrations on hypersurface

we study the vibrations of a fluid confined in a rigid shell. This example seems important in a mechanical point of view because it leads to computation of elastic tanks partially filled with liquids [1] and even to computation of launch vehicles.

1. The problem.

Notations:



The liquid occupies the initial volume $\Omega \subset \mathbb{R}^3$ with the boundary (supposed smooth) $\partial\Omega = \Sigma \cup \Gamma$; Γ is the free surface at rest and Σ the wetted surface of the tank (see schema)

The liquid is supposed to be inviscid, incompressible irrotational.

$\Phi(x,t)$ is unknown and represents the velocity potential of the liquid.

equations. In the linear approximation corresponding to the sudy of small vibrations the problem is:

Find a function $\Phi(x,t)$ solution of

$$\begin{array}{lll}
(28) & \Delta \Phi = 0 & \text{in } \Omega \times]0,T[\\
(29) & \dfrac{\partial \Phi}{\partial n} = 0 & \text{on } \Sigma \times]0,T[\\
(30) & \dfrac{\partial \Phi}{\partial n} + \dfrac{1}{g}\dfrac{\partial^2 \Phi}{\partial t^2} = 0 & \text{on } \Gamma \times]0,T[
\end{array}$$

with the initial conditions

$$\begin{array}{lll}
(31) & \Phi(x',0) = \Phi_0(x') & x' = (x_1,x_2) \in \Gamma \\
(32) & \Phi'(x',0) = \Phi_1(x') & x' \in \Gamma
\end{array}$$

Φ_0 and Φ_1 are given on Γ, $\vec{g}$ is the gravity vector (30) represents the Bernoulli equation on the free surface, (in linearized theory)

2- Variational formulation:

we recall two lemmas whose proof is due to Friedman [6]:

Lemma 2.8:

For a so given domain Ω (the boundary $\partial\Omega$ has a part Γ horizontal), the restriction to Γ of every function in $H^1(\Omega)$ is in $H^{1/2}(\Gamma)$; moreover the restriction mapping of $H^1(\Omega)$ into $L^2(\Gamma)$ is compact

lemma 2.9

There exists a constant c, such that, for any $\varphi \in H^1(\Omega)$

$$\int_\Omega |\varphi|^2 dx \leq c\left(\int_\Gamma |\varphi|^2 dx' + \int_\Omega |\nabla\varphi|^2 dx\right)$$

By the lemmas and by the Green's formula we have now.

lemma 2.10

Problem (28) – (32) is equivalent to:

Find $\Phi(t) \in H^1(\Omega)$ solution of:

$$(33)\quad \begin{cases} a(\Phi(t), \Psi) + \frac{1}{g} \left\langle \frac{\partial^2 \Phi_{|\Gamma}}{\partial t^2}, \Psi_{|\Gamma} \right\rangle = 0 & \forall \Psi \in H^1(\Omega) \\ & \text{a.e. } t \in]0,T[\\ \Phi(x',0) = \Phi_0(x') & \\ \Phi'(x',0) = \Phi_1(x') & \end{cases}$$

with $a(\Phi, \Psi) = \sum_{i=1}^{3} ((D_i\Phi, D_i\Psi)) = \sum_{i=1}^{3} \left(\left(\frac{\partial \Phi}{\partial x_i}, \frac{\partial \Psi}{\partial x_i}\right)\right)$

$((\cdot,\cdot))$ denotes the scalar product in $L^2(\Omega)$

$(\cdot,\cdot)$ " " $L^2(\Gamma)$

$\langle \cdot,\cdot \rangle$ the duality product between $\mathcal{D}(\Gamma)$ and $\mathcal{D}'(\Gamma)$

3. Resolution on the manifold Γ

There are 3 parts in the proof. [2]

part 1: we begin to solve the following stationnary problem:

For f in $H^{1/2}(\Gamma)$, find $\Phi \in H^1(\Omega)$ solution of

$$(34)\quad \begin{cases} \Delta \Phi = 0 & \text{in } \Omega \\ \frac{\partial \Phi}{\partial n} = 0 & \text{on } \Sigma \\ \Phi = f & \text{on } \Gamma \end{cases}$$

lemma 2.12 Problem (34) admits a unique solution $\Phi \in H^1(\Omega)$.

For proof we remark that (34) is a standard elliptic problem with mixed boundary conditions

From $f \in H^{1/2}(\Gamma)$, we obtain $\frac{\partial \Phi}{\partial n}\Big|_\Gamma \in H^{-1/2}(\Gamma)$
and we can define the operator $A, [\Phi]$:

(35) $A \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ such that $Af = \frac{\partial \Phi}{\partial n}\Big|_\Gamma$

A is a pseudo-differential operator on Γ, and
we have

$$d(\Phi, \Psi) - \langle Af, \Psi|_\Gamma \rangle = 0 \qquad \forall \Psi \in H^1(\Omega)$$

(compare A with the classical Neumann's operator
[9; p 237])

part 2

Let λ be an arbitrary positive value and
$B = A + \lambda I$ (I is the identity)

The preceding lemma implies that for every φ (resp ψ) belonging to $H^{1/2}(\Gamma)$, there exists one and only one Φ in $H^1(\Omega)$ (resp Ψ in $H^1(\Omega)$)
solution of:

(36)
$$\left|\begin{array}{ll} \Delta \Phi = 0 & \text{in } \Omega \\ \frac{\partial \Phi}{\partial n} = 0 & \text{on } \Sigma \\ \Phi = \varphi & \text{on } \Gamma \end{array}\right. \qquad \text{resp} \quad \begin{array}{l} \Delta \Psi = 0 \\ \frac{\partial \Psi}{\partial n} = 0 \\ \Psi = \psi \end{array}$$

Then

(37) $$\langle B\varphi, \psi \rangle = d(\Phi, \Psi) + \lambda(\varphi, \psi) = \langle A\varphi, \psi \rangle + \lambda(\varphi, \psi)$$

Lemma 2.12.

The form $(\varphi, \psi) \to b(\varphi, \psi) = \langle B\varphi, \psi \rangle$ is bilinear , ... on $H^{1/2}(\Gamma)$

Proof; We apply (35) and the continuity of the mapping from $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$. The continuity of the form follows.
Lemma 2.9 implies that there exists a constant $\alpha > 0$ such that $b(\varphi, \varphi) \geq \alpha \|\Phi\|^2_{H^1(\Omega)}$: As the restriction mapping from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$ is continuous, the coercivity is obvious.
Moreover the mapping from $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$ is compact (By $b(\varphi, \varphi) \geq \alpha \|\Phi\|^2_{H^1(\Omega)}$ and lemma 2.8 we verify that B^{-1} is compact on $L^2(\Gamma)$).
So by theorem 2.5 we have:

Lemma 2.1.

The operator B has an infinite sequence of increasing eigenvalues $\mu_i \in [\lambda, +\infty[$. The corresponding eigenfunctions form a complete, orthonormal sequence in $L^2(\Gamma)$, a complete orthogonal sequence in $H^{1/2}(\Gamma)$:

$$\langle B\varphi_i, \varphi_j \rangle = \mu_i (\varphi_i, \varphi_j) = \mu_i \delta_{ij} \qquad \forall i, j$$

Remark 2.4. The smallest eigenvalue is $\mu_1 = \lambda$.
The corresponding eigenfunction satisfies:

$$d(\nabla \phi_1, \nabla \psi) = 0 \qquad \forall \psi \in H^1(\Omega)$$

so:

$$\begin{cases} \Delta \phi_1 = 0 & \text{in } \Omega \\ \dfrac{\partial \Phi_1}{\partial n} = 0 & \text{in } \partial\Omega \end{cases}$$

The solution is $\Phi_1 =$ constant (to avoid this case we could introduce in the variational formulation the subspace $\{ w \in H^1(\Omega) ; \int_\Omega w \, dx = 0 \}$)

Part 3

Thanks to the extension (36), we easily verify that the first problem (33) is equivalent to the following one, on the manifold Γ.

Find $\varphi(t) \in H^{1/2}(\Gamma)$ solution of:

$$
\begin{array}{ll}
(38) & \langle A\,\varphi(t), \psi\rangle + \frac{1}{g}\left\langle \frac{\partial^2\varphi(t)}{\partial t^2}, \psi\right\rangle = 0 \quad \forall \psi \in H^{1/2}(\Gamma), \text{ ae } t \in]0,T[\\
(39) & \varphi(0) = \Phi_0 \\
(40) & \varphi'(0) = \Phi_1
\end{array}
$$

And we are exactly in agreement with the proof of Theorem 2.1, then:

Theorem 2.6

Let $\Phi_0 \in H^{1/2}(\Gamma)$ and $\Phi_1 \in L^2(\Gamma)$. There exists a unique function $\varphi(x';t)$ such that:

$\varphi \in \mathcal{C}^0([0\,T]; H^{1/2}(\Gamma))$; $\frac{d\varphi}{dt} \in \mathcal{C}^0([0\,T], L^2(\Gamma))$

φ is solution of (38) (39) (40)

φ is an infinite sum of stationnary solutions

$$(41) \quad \varphi(x';t) = \sum_{i=1}^{\infty} \left[(\Phi_0, \varphi_i) \cos \omega_i t + \frac{1}{\omega_i} (\Phi_1, \varphi_i) \sin \omega_i t \right] \varphi_i(x')$$

with $\omega_i^2 = g\,(\mu_i - \lambda)$

$\frac{\sin \omega_1 t}{\omega_1} = t$ (because $\omega_1 = 0$)

Moreover the function $\varphi(x';t) - t(\Phi_1, \varphi_1)\,\varphi_1(x')$ is almost

4- Solution of problem (33)

By the extension (36) we return now to the first problem (28)-(32), or the equivalent one (33). The theorem, for solving (33) is finally (refer also to [6])

Theorem 2.7

Let $\Phi_0 \in H^{1/2}(\Gamma)$ and $\Phi_1 \in L^2(\Gamma)$. There exists a unique function $\Phi(x,t)$ having the following properties:

$$\Phi \in \mathcal{C}^0([0T]; H^1(\Omega)) \quad ; \quad \frac{d\Phi}{dt} \in \mathcal{C}^0([0T]; L^2(\Omega))$$

Φ is solution of (33)

$$\Phi(x,t) = \sum_{i=1}^{\infty} \left[(\Phi_0, \varphi_i) \cos \omega_i t + \frac{1}{\omega_i} (\Phi_1, \varphi_i) \sin \omega_i t \right] \Phi_i(x) \tag{42}$$

$\forall i$, $\Phi_{i/\Gamma} = \varphi_i$ and $\Phi_{/\Gamma} = \varphi$ (42)

Remark 2.5.

* There exists, of course additional properties for the regularity of the solution.
* It is realistic to try to compute the normal modes for a given physical system, it is then necessary to get a finite dimensional problem. So the numerical study of the

problem will need :

- to approximate the functional spaces by finite dimensional spaces
- then, to compute the eigenvalues and eigenfunctions for the discrete finite dimensional problem.
- at last, to study (if possible) the convergence of approximate solutions to the true solution.

As examples, some numerical results from [2] are shown in fig. 1 and fig 2.
The equations have been discretized by the finite element method.

* A more general case is the hydro-elastic problem for vibrations of fluid confined in a deformable shell [1]. Then the equations to solve, are coupled equations of elasticity (example 2.1) and fluid dynamics with a free surface (example 2.2).

B. We recall that : if $x, t \to f(x,t)$ denotes a mapping defined on $Q = \Omega \times]0,T[$ we put
$f(t)$: "$x \to f(x,t)$" is a function of the variable t, with values
Then $f : t \to f(t)$ [illegible]

Spherical rigid tank filled with liquid

normal modes

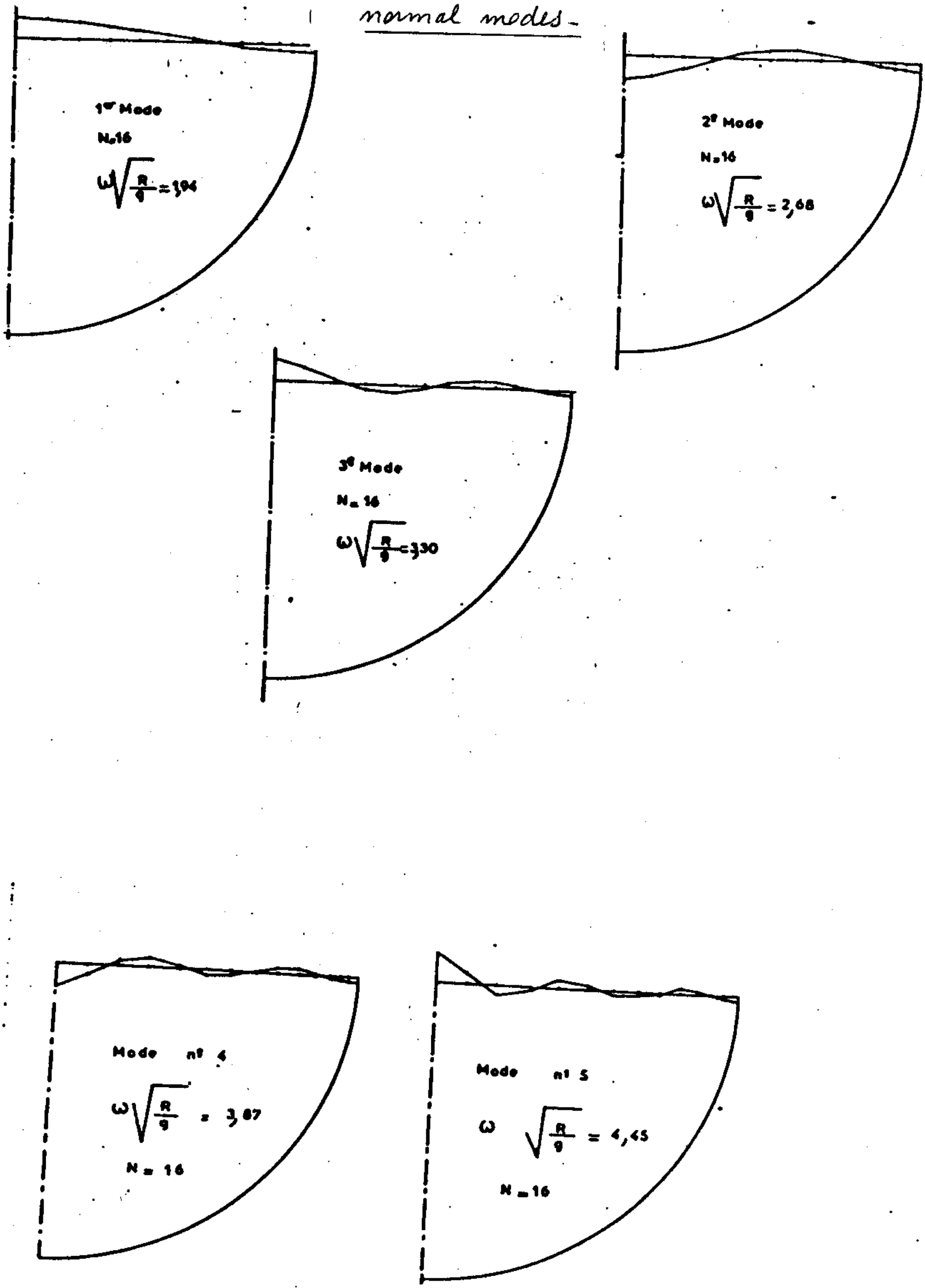
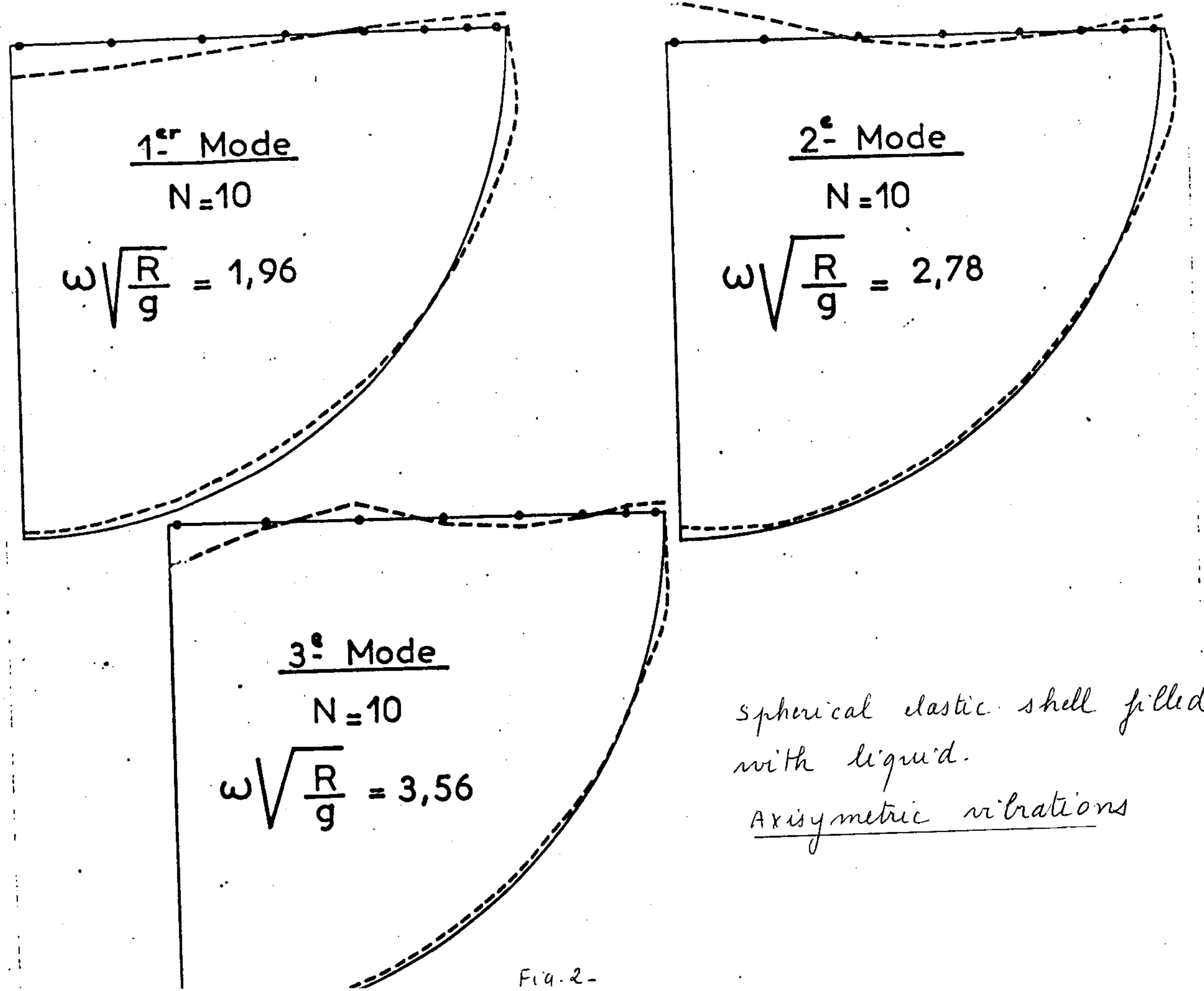


Fig 1



spherical elastic shell filled
with liquid.
Axisymetric vibrations

Fig. 2-

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Vibrations in linear mechanics

C – Other applications –

II.7 – Quadratic eigenvalue problems

Quadratic eigenvalue problems arise in mechanics an physics particularly to study hydrodynamic stability and vibrations with damping.

As example, we only consider the following case [14] :

1 – problem.

Let V be a separable Hilbert space with inner product $(\cdot,\cdot)$

A and B two positive definite operators, symmetric and compact from V into V.

We consider the eigenvalue problem : find λ and $u \in V$ solution of :

$$u - \lambda A u - \lambda^2 B u = 0 \qquad (43)$$

In fact (43) appears as a non linear problem.

2 – Resolution

By introducing (λ is different from zero in (43))

$\mu = \lambda^{-1}$

$v = \lambda B^{1/2} u$

... the equivalent form

(44) $$K\,W = \mu\,W$$

where $w = \begin{pmatrix} u \\ v \end{pmatrix}$ is in $V \times V$,

and $K = \begin{pmatrix} A & B^{1/2} \\ B^{1/2} & 0 \end{pmatrix}$ is compact symmetric and

one to one on $V \times V \to V \times V$.

It follows from theorem 13 that:

<u>lemma 2.14</u>: Problem (44) has at most a denumerable number of solutions

$$K w_i = \mu_i w_i$$

These eigenvectors may be selected so that

$$(((w_i, w_j))) = ((u_i, u_j)) + ((v_i, v_j)) = \delta_{ij}$$

V being separable they form a basis for $V \times V$.

<u>lemma 2.15</u> Problem (43) has a denumerable number of solutions:

(45) $$u_i - \lambda_i A u_i - \lambda_i^2 B u_i = 0 \qquad i = 1, 2 \ldots$$

which may be selected so that

$$((u_i, u_j)) + \lambda_i \lambda_j ((B u_i, u_j)) = \delta_{ij}$$

<u>Remark 2.6</u>

A special study [47] is necessary to obtain the distribution of eigenvalues on the real axis. We recall that: The eigenvalues are both positive or negative and have precisely two ... 0 and infinity.

zero if order of A > order of B

± infinity if order of B > order of A.

3 – Remarks about other quadratic eigenvalue problems

* (43) implies a more general equation in hydrodynamics:

(46) $$Au - \lambda^2 Bu - \lambda Cu = 0$$

(C is a symmetric operator on V)

* In vibration mechanics the general case of vibrations with damping leads to the equation:

$$\frac{d^2u}{dt^2} = Au - s(x)\frac{du}{dt}$$

and to the eigenvalue problem:

(47) $$Au + \lambda Bu + \lambda^2 u = 0$$

$$\left(Bu = \int_\Omega s(x)\, u(x)\, dx \right)$$

We only want to mention that we cannot apply the last method because it leads to a non symmetric operator:

$$\begin{pmatrix} 0 & A^{-1/2} \\ -A^{-1/2} & -A^{-1/2} B A^{-1/2} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \mu \begin{pmatrix} v \\ w \end{pmatrix}$$

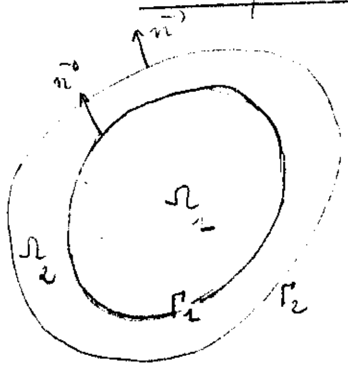
with $\mu = \lambda^{-1}$, $v = A^{1/2} u$, $w = \frac{A^{-1/2} v}{\mu} = \frac{u}{\mu}$

However Galerkin's method may be applied

II.8. Coupled problems

Example 2.3 Spectral [illegible] problem in elasto acoustics

1- the problem.



the aim of this paper is to study the vibrations of a compressible fluid confined in a deformable shell

The fluid occupies the initial volume Ω_1 with the boundary (supposed smooth) $\partial\Omega_1 = \Gamma_1$

The shell supposed thick is represented by a domain Ω_2 with the boundary $\partial\Omega_2 = \Gamma_1 \cup \Gamma_2$

$\vec{n}^0$ is the outward normal.

Exercise it is easy to prove that the equations give a quadratic eigenvalue problem which can be solved by the method of II-7.

Then, the numerical approximation is an eigenvalue problem with a non symmetric matrix. There is some difficulty from a numerical point of view to use standard codes (generally suitable for symmetric schemes)

Another way to solve the problem is to introduce a mixed variational formulation [12]. Then the problem remains symmetric.

2 - The equations of the coupled problem

- notations :

The unknown functions are the following
$\tilde{\Phi}(x,t)$ is the displacement potential of the fluid
$\Phi(x,t) = \frac{\partial}{\partial t}\tilde{\Phi}(x,t)$ is the velocity potential of the fluid

$p(x,t)$ represents the pressure inside the fluid (remark $p(x,t)$ and $\Phi(x,t)$ are two dual functions)

$u(x,t) = (u_1, u_2, u_3)$ is the displacement field vector referred to the equilibrium static configuration of the shell.

σ_{ij} and ε_{ij} are the same as in II-4-

- In the linear approximation corresponding to the study of the small vibrations, the dynamic equations are the following

For the fluid

(48) $$\rho_0 c^2 \Delta \Phi + \frac{\partial p}{\partial t} = 0$$ in $\Omega_1 \times]0,T[$

c is the sound velocity
ρ_0 is the density of fluid at rest

(49) $$p + \rho_0 \frac{\partial \Phi}{\partial t} = 0$$ in $\Omega_1 \times]0,T[$

(50) $$\frac{\partial \Phi}{\partial n} = \frac{\partial u}{\partial t}\cdot n$$ on $\Gamma_1 \times]0,T[$

non penetration condition

For the shell

(51) $$\rho_c \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(u)}{\partial x_j} \quad \text{in } \Omega_2 \times]0,T[$$

ρ_c is the density of the shell

(52) $$\sum_{j=1}^{3} \sigma_{ij}\, n_j = \rho_0 \frac{\partial \Phi}{\partial t} n_i \quad \text{on } \Gamma_1 \times]0,T[$$

(53) $$\sum_{j=1}^{3} \sigma_{ij}\, n_j = 0 \quad \text{on } \Gamma_2 \times]0,T[$$

with initial conditions.

Then we get a problem with coupling (50) (52)
This problem is elastic in Ω_2 and acoustic in Ω_1 (eliminate P between (48) and (49) we get the fundamental equation of acoustics)

3 - Variational formulation

- We have already seen the interest of stationary solutions, which are related to the notion of spectral problem. We therefore search the stationary solutions of the problem under the form:

(54) $$\begin{cases} \Phi(x,t) = i\omega e^{i\omega t} \varphi(x) \\ U(x,t) = u(x)\, e^{i\omega t} \\ P(x,t) = p(x)\, e^{i\omega t} \end{cases}$$

Then the problem (45) - (53) leads to the variational form (ω is supposed $\neq 0$):

[Notations: $\int_\Omega \nabla\varphi \cdot \nabla\psi \, d\Omega = \sum_{i=1}^{3} \left(\int_\Omega \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_i} \, d\Omega \right)$
$u \cdot v = \sum_{i=1}^{3} u_i v_i$]

(155) $\displaystyle\int_{\Omega_2} \rho_0 \nabla\varphi \cdot \nabla\psi \, d\Omega - \frac{1}{c^2}\int_{\Omega_2} p\,\psi \, d\Omega - \int_{\Gamma_2} \rho_0 (u.n)\,\psi \, d\Gamma = 0$

(156) $\displaystyle\int_{\Omega_2} p\, q \, d\Omega - \omega^2 \int_{\Omega_2} \rho_0 \varphi\, q \, d\Omega = 0$

(157) $\displaystyle\sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, d\Omega - \omega^2 \int_{\Omega_2} \rho_c (u.v)\, d\Omega - \omega^2 \int_{\Gamma_2} \rho_0 \varphi (v.n)\, d\Gamma = 0$

for each smooth "test function" (ψ, q, v)

- Define the following functional spaces:

$\dot{L}^2(\Omega_2)$ = the subspace of $L^2(\Omega_2)$ orthogonal to constant functions
(resp $\dot{H}^1(\Omega_2)$) - - - $(H^1(\Omega_2))$ - -

$\tilde{V} = \{ u \,/\, u \in (H^1(\Omega_2))^3/R \}$ where R is the set of rigid displacements (refer to [13])

$\dot{H}^1(\Omega_2)$ is equipped with the norm $\|\varphi\| = \left(\int_{\Omega_2} |\nabla\varphi|^2 d\Omega\right)^{1/2}$ which is equivalent to the usual norm [16]

$\tilde{V}$ is equipped with the inner product:
$((u,v)) = \sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, d\Omega$. The corresponding norm is equivalent to the usual norm on $(H^1(\Omega_2))^3/R$

Let $\mathcal{H}$ be the cartesian product of the three sets:

(158) $\mathcal{H} = \dot{H}^1(\Omega_2) \times \dot{L}^2(\Omega_2) \times \tilde{V}$

$X = (\varphi, p, u)$ and $Y = (\psi, q, v)$ denote functions of $\mathcal{H}$ -

<u>lemma 2-16</u>

Problem (48)-(53) is equivalent to find ω and $X \in \mathcal{H}$ solution of equations (55), (56), (57) for each Y belonging to $\mathcal{H}$

<u>Remark 2-7</u>

- the choice of spaces (55) eliminate the solution $\omega = 0$
- (55), (56), (57) is a mixed variational formulation

4- <u>Spectral problem associated with the initial problem</u>

let $\mathcal{K}$ be the subspace of $\mathcal{H}$ such that:

(59) $$\mathcal{K} = \left\{ Y \in \mathcal{H} \; ; \; \rho_0 c^2 \Delta \psi + q = 0 \text{ in } \Omega_1 \, , \; \frac{\partial \psi}{\partial n} = (v.n) \text{ on } \Gamma_1 \right\}$$

$\mathcal{K}$ is a closed subspace of $\mathcal{H}$.

We remark than on $\mathcal{K}$ we have:

$$\int_{\Omega_1} \rho_0 \nabla \psi . \nabla \varphi \, d\Omega - \frac{1}{c^2} \int_{\Omega_1} q \varphi \, d\Omega - \int_{\Gamma_1} \rho_0 (v.n) \varphi \, d\Gamma = 0 \qquad \forall (\psi, q, v) \in \mathcal{K}$$

By (56) (57) we obtain after reduction:

<u>theorem 2.8</u> (55) (56) (57) is equivalent to the following spectral problem:

Find $\nu \; (= \omega^2)$ and $X \in \mathcal{K}$, solution of:

(60) $$\mathcal{A}(X, Y) = \nu \, \mathcal{B}(X, Y) \qquad \forall Y \in \mathcal{K}$$

with:

(61) $$\mathcal{A}(X, Y) = \sum_{i,j} \int_{\Omega_2} \sigma_{ij}(u) \, \varepsilon_{ij}(v) \, d\Omega + \frac{1}{\rho_0 c^2} \int_{\Omega_1} p \, q \, d\Omega$$

(62) $$\mathcal{B}(X, Y) = \int \rho_c \, u.v \, d\Omega + \int \rho_0 \nabla \varphi . \nabla \psi \, d\Omega$$

We recognize the standard formulation introduced in chapter I -

5- Solution of problem (60)

Define the linear continuous operators A and B on $\mathcal{H}$ by:

$$A, B \in \mathcal{L}(\mathcal{H}; \mathcal{H}')$$

$$(63) \begin{cases} \langle AX, Y \rangle = \mathcal{A}(X, Y) \\ \langle BX, Y \rangle = \mathcal{B}(X, Y) \end{cases}$$

By (63) the operators A and B are moreover positive symmetric and invertible - In order to apply theorem 1.5 we must prove compactness.

Theorem 2.9.

| A^{-1} is B compact on $\mathcal{H}$.

By definition 1-1, theorem 2.9 means that: from every sequence $\{Y_n\} \in \mathcal{H}$ such that $\langle AY_n, Y_n \rangle < \text{const}$, we can extract a subsequence which converges for the norm $\langle BY_n, Y_n \rangle$

The proof is not trivial and uses some lemmas.

* First, we consider the operator of elasticity C such that:

$$(64) \quad \langle Cu, v \rangle = \sum_{ij} \int_{\Omega_2} \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, d\Omega \qquad \forall u, v \in \tilde{V}$$

lemma 2-17 : C^{-1} is compact on $(L^2(\Omega_2))^3 / R$

* Now, we write $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with

$$\mathcal{H}_1 = \{ y \in \mathcal{H} \; ; \; \Delta\psi = 0 \text{ in } \Omega_2 \}$$

$$\mathcal{H}_2 = \{ y \in \mathcal{H} \; ; \; \frac{\partial \psi}{\partial n}\Big/_{\Gamma_2} = 0 \}$$

Define C_1 on $\mathcal{H}_1$ by.

$$(65) \qquad \langle C_1 x, y \rangle = \int_{\Gamma_2} \frac{\partial \varphi}{\partial n} \psi \, d\Gamma = \int_{\Omega_2} \nabla\varphi \, \nabla\psi \, d\Omega \quad , \forall x, y \in \mathcal{H}_1$$

$C_1^{-1} : L^2(\Gamma_2) \to \mathcal{H}_1$ is the Neumann's operator for the problem $\Delta\varphi = 0$ in Ω_2, $\frac{\partial \varphi}{\partial n} \in L^2(\Gamma_2)$

Then C_1^{-1} is a compact operator [15]

<u>lemma 2.18</u>

$C_1^{-1} : L^2(\Gamma_2) \to \mathcal{H}_1$ is compact.

($\mathcal{H}_1$ is equipped by the topology of $H^1(\Omega_2)$)

* Let C_2 be the linear continuous operator defined on $\mathcal{H}_2$ by:

$$(66) \qquad \langle C_2 x, y \rangle = \frac{1}{\rho_0 c^2} \int_{\Omega_2} p \, q \, d\Omega \qquad \forall x, y \in \mathcal{H}_2$$

C_2 defines exactly the problem of compressible fluids contained in rigid tanks. That is the following acoustics problem

$$(67) \qquad \begin{cases} \rho_0 c^2 \Delta\varphi + p = 0 & \text{in } \Omega_2 \\ p - \rho_0 \nu \varphi = 0 & \text{in } \Omega_2 \\ \frac{\partial \varphi}{\partial n} & \end{cases}$$

(67) has a pure point spectrum.
So by theorem 1.4 we have the:

<u>lemma 2.19</u>.

$C_2^{-1} : \dot{L}^2(\Omega_2) \to \mathcal{H}_2$ is compact
($\mathcal{H}_2$ is equipped with the topology of $\dot{H}^1(\Omega_2)$).

Then, by the 3 last lemmas and by the compactness definition, theorem 2.9 holds.

By theorem 1.5 we get now:

<u>Theorem 2.10</u>

Problem (60) has a pure point spectrum of eigenvalues
$0 < \nu_1 \leq \nu_2 \ldots\ldots \leq \nu_n \to \infty$
The corresponding eigenfunctions $Y_n = (\varphi_n, q_n, v_n)$ form a B-orthogonal and total sequence on $\mathcal{H}$:
$$\langle A Y_n, Y_m \rangle = \nu_n \langle B Y_n, Y_m \rangle = \nu_n \delta_{nm} \qquad \forall m$$

6 – <u>consequences</u>

The initial problem is completely caracterized from the mechanical point of view by
the set $\{\omega_n\}$ of eigenfrequencies $(\nu_n = \omega_n^2)$
the set Y_n of vibration shapes.

For the numerical analysis of the problem we approximate equations (55), (56), (57) by the finite element method.

Some numerical results are shown in

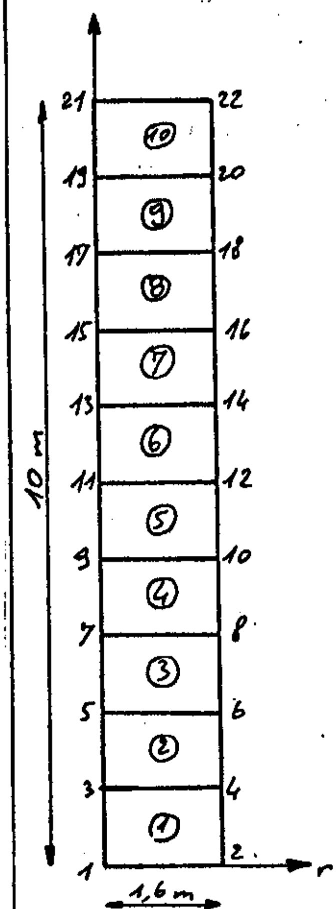
fig 3 acoustic pipe (rigid case)

fig. 4 : acoustic problem for a cylindrical tank.

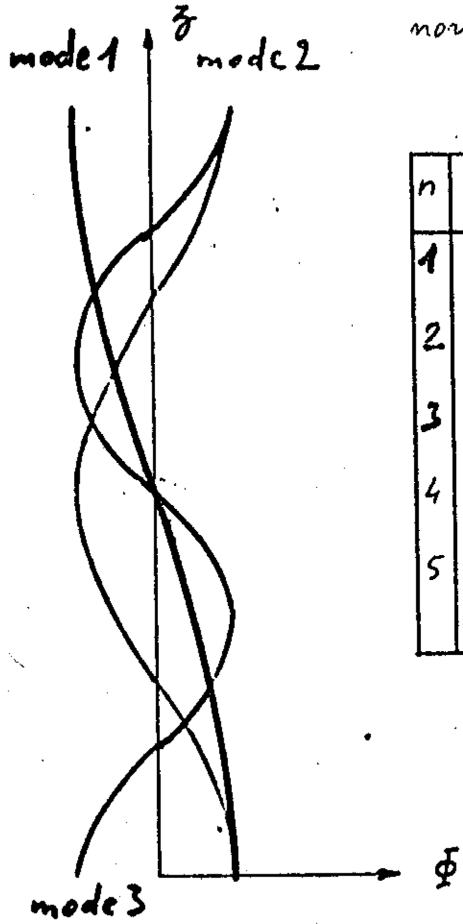
fig 5 : acoustic problem for a cylindrical tank with ellipsoidal bottom. Computations hold within the thin shell theory.

For the first two examples computed frequencies are in good agreement with the theorical frequencies.

Numerical results



model



velocity potential

normal modes

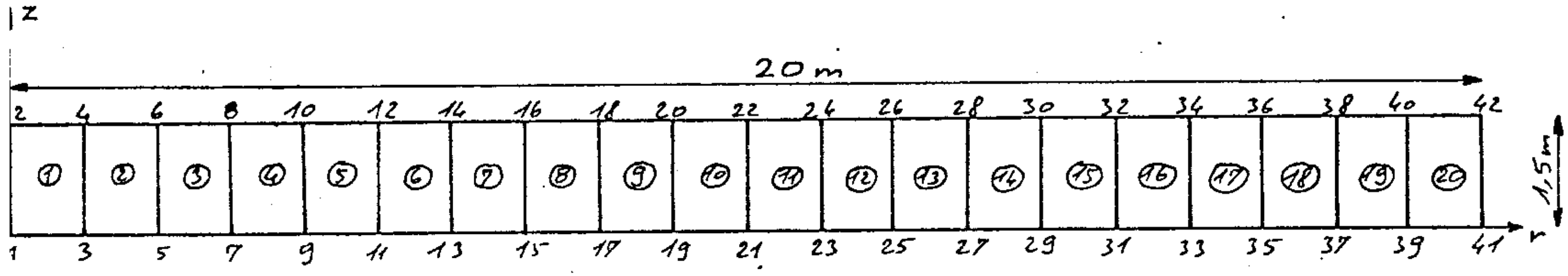
n	theorical frequency	computed frequency
1	15,81 Hz	15,87 Hz
2	31,62 Hz	32,14 Hz
3	47,43 Hz	49,20 Hz
4	63,24 Hz	67,43 Hz
5	79,05 Hz	87,17 Hz

10 éléments
22 nodes
22 degree of freedom

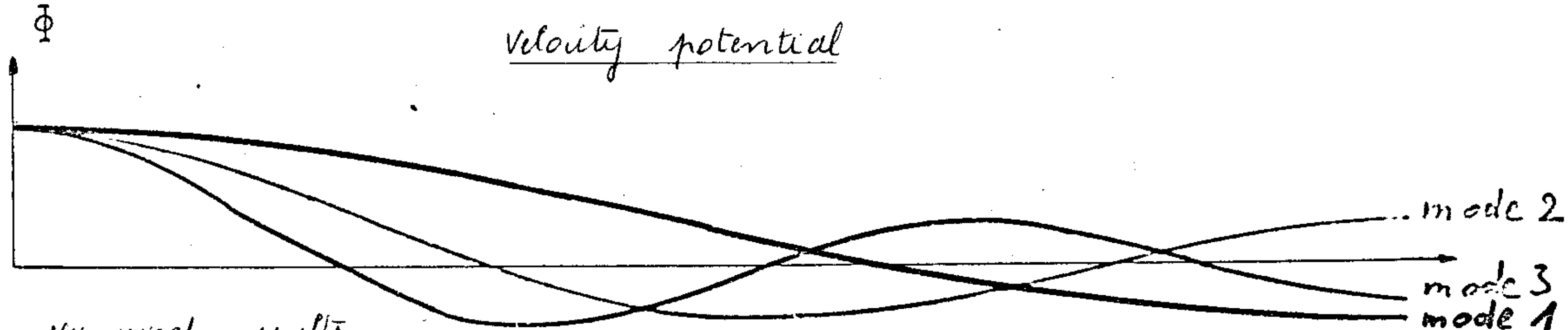
Fig. 3 - acoustic pipe

model : flat cylindrical tank.

20 éléments
42 nodes
42 degree of freedom



Velocity potential



Numerical results

Normal modes

n	1	2	3	4	5	6	7	8
theorical frequency (Hz)	9,64	17,65	25,60	33,52	41,44	49,36	57,27	65,18
computed frequency (Hz)	9,65	17,72	25,83	34,07	42,50	51,18	60,14	69,4

II - 9 - Bibliography.

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