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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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SPECTRAL ANALYSIS IN VIBRATION MECHANICS

(Part III)

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Part III – Numerical Analysis of vibration problems

III-1- Preliminary.

1- We recall first, the hypothesis of the problems (see II-1)

- V and H are two separable Hilbert spaces such that $V \subset H \subset V'$, and the mapping from V into H is moreover compact
- $A \in \mathcal{L}(V; V')$ is a symmetric operator (V and H are supposed real) such that the associated form:

(1) $$a(u, v) = \langle Au, v\rangle \qquad \forall u, v \in V$$

is a bilinear, continuous, symmetric and V-elliptic form.

- The general spectral problem is to find $\lambda \in R$ and $u \in V$ solution of:

(2) $$a(u, v) = \lambda (u, v) \qquad \forall v \in V$$

$(\cdot, \cdot)$ denotes the inner product in H.

2- In order to solve (2) in a numerical point of view, we begin to approximate V by finite dimensional spaces V_h ($h \in \mathcal{H}$).
Space approximation theory studies in what sense

Then we solve the finite dimensional problem $a_h(u_h, v_h) = \lambda_h (u_h, v_h)$ on the space V_h. For computing, some eigenvalue algorithms are then necessary.

Finally it remains to study how λ_h converges to λ, and u_h converges to u.

III.2. An elementary example:

Before giving general results, we propose to study a one dimensional problem.

1. The problem

p and q are two given positive scalars. We study the following equations

$$(3) \quad \begin{cases} -p \dfrac{d^2 u}{dx^2} + q u = \lambda u & \text{with } 0 < x < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

$u : x \to u(x)$ is a unknown function

(3) is equivalent to the variational spectral problem (2) with:

$\Omega =]0, \pi[$; $H = L^2(\Omega)$; $V = H_0^1(\Omega)$

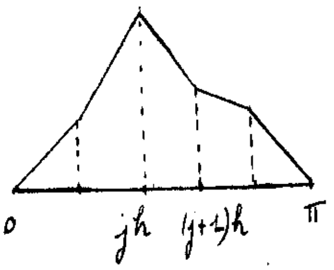
$a(u, v) = p \left(\frac{du}{dx}, \frac{dv}{dx} \right) + q (u, v)$ $(\cdot,\cdot)$ inner product in $L^2(\Omega)$

It is easy to find the true solution of the spectral problem. We obtain the following set of eigenvalues λ_k and associated eigenfunctions u_k :

$$(4)\quad \begin{cases} \lambda_k = p\,k^2 + q & k \in \mathbb{N} \\ u_k(x) = \sqrt{\frac{2}{\pi}}\ \sin kx \end{cases}$$

But we want to introduce, about this elementary example, the basis of numerical approximation.

2 – The finite element method in $H_0^1(\Omega)$



For the step $h = \frac{\pi}{N}$, the nodes are the points jh with $j = 0, 1, \ldots N$.
Let V_h be the space of continuous functions $u_h : [0, \pi] \longrightarrow \mathbb{R}$, such that u_h is linear on each interval $]jh, (j+1)h[$,
$u_h(0) = u_h(\pi) = 0$

<u>lemma 3.1.</u>
V_h is a subspace of $H_0^1(\Omega)$: $V_h \subset H_0^1(\Omega)$.

Of course V_h is a subspace of the vectorial space $H_0^1(\Omega)$. Moreover we have $u_h(0) = u_h(\pi) = 0$.
u_h belongs to $C^0(\bar{\Omega})$, so $u_h \in L^2(\Omega)$.
As u_h is continuous, the derivative $\frac{du_h}{dx}$ (considered as a distribution) is bounded on Ω.
Then $\frac{du_h}{dx} \in L^2(\Omega)$. and the lemma is proved –

Let $w_{hj}(x)$ be the roof functions such that:

$w_{hj} \in V_h$ and $\begin{cases} w_{hj}(jh) = 1 \\ \ldots \quad 0 \le k \le N \quad k \ne j \end{cases}$

lemma 3-2

V_h is a finite dimensional space with the basis w_{hj}, $1 \leq j \leq (N-1)$

Proof: First, we prove that every $u_h \in V_h$ is equal to

$$(5) \qquad u_h(x) = \sum_{j=1}^{N-1} u_h(jh)\, w_{hj}(x)$$

For each $x \in \Omega$, there exists j such that $x \in \,]jh, (j+1)h[$ Then

$$u_h(x) = u_h(jh)\, w_{hj}(x) + u_h((j+1)h)\, w_{h,j+1}(x) \qquad \forall x \in \,]jh,(j+1)h[$$

with:

$$w_{hj}(x) = \frac{(j+1)h - x}{h}$$

$$w_{hj+1}(x) = \frac{x - jh}{h}$$

$$w_{hj}(x) + w_{h,j+1}(x) = 1$$

Then (5) holds.

moreover, the functions w_{hj} are linearly independent (by $\sum_{j=1}^{N-1} \lambda_j w_{hj}(x) = 0$, we get $\lambda_j = 0$ in every point $x = jh$, $\forall j = 1, \dots, (N-1)$)

The reader can refer to [5], [3] in order to study general properties of normed space approximation

Then the following lemma can be proved

lemma 3-3-

The so called linear finite element method is a stable and convergent approximation of the space $H_0^1(\Omega)$

3- The finite dimensional discrete problem-

We define the approximate solution of (3) by:

(15) $$u_h(x) = \sum_{j=1}^{N-1} Q_j^h w_{hj}(x)$$

Q_j^h are the unknown nodal parameters ($Q_j^h = u_h(jh)$)

The discrete problem associated to (1) is:

(17) $$a(u_h, w_{hk}) = \lambda_h (u_h, w_{hk}) \qquad k = 1, 2 \ldots (N-1)$$

(17) is a linear system- A slighly longer computation gives:

$$\int_\Omega w_{hj}^2\, dx = \frac{2h}{3} \quad ; \quad \int_\Omega w_{hj} w_{h,j+1}\, dx = h/6$$

$$\int_\Omega w_{hj}'^2\, dx = \frac{2}{h} \quad ; \quad \int_\Omega w'_{hj} w'_{h,j+1}\, dx = -\frac{1}{h}$$

(17) leads to the equations:

$$\frac{p}{h}\left(2 Q_k^h - Q_{k+1}^h - Q_{k-1}^h\right) + \frac{qh}{6}\left(4 Q_k^h + Q_{k+1}^h + Q_{k-1}^h\right)$$
$$= \lambda_h \frac{h}{6}\left(4 Q_k^h + Q_{k+1}^h + Q_{k-1}^h\right) \qquad 1 \le k \le N-1$$

The eigenvalue problem is:

$$(8) \quad \boxed{K_h \, Q_h = \lambda_h \, M_h \, Q_h}$$

K_h (stiffness matrix), Q_h (nodal parameters matrix) and M_h (mass matrix) are equal to:

$$K_h = p \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ 0 & & & -1 & 2 \end{bmatrix} + q \frac{h}{6} \begin{bmatrix} 4 & & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & 1 & 4 \end{bmatrix}$$

$$M_h = \frac{h}{6} \begin{bmatrix} 4 & 1 & & \\ 1 & 4 & 1 & \\ & \ddots & & \\ & & & 1 \\ & & 1 & 4 \end{bmatrix} \qquad Q_h = \begin{bmatrix} Q_1^h \\ \vdots \\ Q_k^h \\ \\ Q_{N-1}^h \end{bmatrix}$$

The unknown are λ_h and Q_h

Remarks 3.1

- (8) is easy to compute. But in more general cases we have to solve a matrix eigenvalue problem $AX = \lambda BX$ which is not trivial and is scarcely discussed in classical [7] linear algebra. (We shall give some references for eigenvalue algorithms)
- We can compute as many eigenvalues as we have nodes.

solution of (8).

The components of the r^{th} eigenvector are

(9) $$\left(Q_k^h\right)_r = \sqrt{\frac{2}{\pi}} \sin(r k h) \qquad k = 1, \dots (N-1)$$

The associated eigenvalue denoted by λ_r^h is:

(10) $$\lambda_r^h = p \frac{k_h(r)}{m_h(r)} + q$$

with:

$$k_h(r) = 2h^{-2}(1 - \cos r h)$$

$$m_h(r) = \frac{2 + \cos r h}{3}$$

so

(10) $$\lambda_r^h = \frac{p h^{-2} 6 [1 - \cos r h]}{2 + \cos r h} + q$$

4°- The question of convergence

In this special elementary problem, the eigen=
=functions given by:

(11) $$u_h^r(x) = \sum_{k=1}^{N-1} \left(Q_k^h\right)_r w_{hk}(x)$$

are equal to the interpolate of the r^{th} eigenfunction of problem (3). In fact by formula (4) we get:

$$u_h^r(kh) = u_r(kh) \qquad \text{for } k = 1, 2, \dots (N-1)$$

.1 — So, it is time now to recall some results about

<u>interpolation theory</u>.. To avoid the finite element method, we only recall simplified properties. For other approximation theorems, the reader can be referred to [1] and [2] [3].

Let Ω be a bounded open set of $\mathbb{R}^n$, and let u be a function of the Sobolev space $H^s(\Omega)$; $(s \in \mathbb{N})$. Suppose that u_I is the interpolate of u : u_I is a piecewise polynomial with degree equal to $(k-1)$ on each finite set K, included in Ω (in finite element method K belongs to an "admissible" triangulation $\mathcal{T}_h$ of the domain Ω, it assumes that $\bigcup_{K_h \in \mathcal{T}_h} K_h$ "approximate" Ω).

Denote by P_{k-1} the set of polynomials of degree $\leq (k-1)$. Suppose that the derivative D_j associated with the nodal parameters are all of order $|D_j| < (k - \frac{n}{2})$ and that k satisfies the property:

$$H^k(K) \subset H^s(K)$$

$$P^{k-1} \subset H^s(K)$$

Then for any function $u \in H^k(\Omega)$, there exists a constant c (depending on s) such that

(11) $$[u - u_I]_{s,K} \leq c\, h_K^{k-s}\, [u]_{k,K}$$

h_K denotes the diameter of K

$[\ ]_{s,K}$ denotes the semi norm on K:

$$[u]_{s,K} = \left(\sum_{|j|=s} \int_K |D^j u|^2 dx \right)^{1/2}$$

D^j is the derivative $\dfrac{\partial^{|j|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ $\quad |j| = \alpha_1 + \dots + \alpha_n$

-2 — Come back to problem (3). We define

$K_{hj} =]jh, (j+1)h[$.

Since the integral over Ω is the sum of the integrals over K_{hj} $(j = 0, \dots N-1)$ we obtain by (11) and $k = 2$:

$$[u_r - u_h^r]_{1,\Omega} \leq c_1\, h\, [u_r]_{2,\Omega}$$

$$[u_r - u_h^r]_{0,\Omega} \leq c_0\, h^2\, [u_r]_{2,\Omega}$$

Then:

(12): $$\| u_r - u_r^h \|_{H^1(\Omega)} \leq C\, h\, [u_r]_2$$

c_0, c_1, c denote constants

(12) leads directly to the rate of convergence of the method and we have:

lemma 3.4.

For $h \to 0$, each r^{th} approximate eigenfunction u_h^r converges strongly in $H_0^1(\Omega)$ to the r^{th} eigenfunction u_r of problem (3) –

As $ch[u_r]_2 = o(r^2 h)$ we remark that the error estimate $o(r^2 h)$ is the same as in an elliptic boundary value problem $Au = f$.

4.3 — eigenvalue approximation

Formula (10) leads to:

$$k_h(r) = 2h^{-2}(1 - \cos rh)$$
$$= r^2 - \frac{r^4 h^2}{12} + o(r^6 h^4)$$

$$m_h(r) = 1 - \frac{r^2 h^2}{6} + o(r^4 h^4)$$

$$\lambda_r^h = p\,\frac{k_h(r)}{m_h(r)} + q$$

$$\lambda_r^h = \underbrace{p\,r^2 + q}_{\lambda_r} + p\,\frac{h^2 r^4}{12} + o(r^6 h^4)$$

$$\lambda_r^h = \lambda_r + o(r^4 h^2)$$

(13) $$\lambda_r^h - \lambda_r \leq C^{te} \| u_r - u_h^r \|^2_{H^1(\Omega)}$$

lemma 3.5:

If $h \to 0$, then $\lambda_r^h \to \lambda_r$, the rate of convergence is given by (13)

Remarks 3.2

- The estimates (12) (13) are valid in general

- Another method (called lumping process) sometimes used to compute (8) is to replace the mass matrix M_h by the identity. (refer to [3])

In our example the eigenfunctions remain unchanged, the eigenvalues on the other hand are altered to

$$\tilde{\lambda}_r^h = p\, k_r(h) + q = \lambda_r - p\frac{r^4 h^2}{12} + O(r^6 h^4)$$

$$\lambda_r \geq \tilde{\lambda}_r^h$$

Thus $\tilde{\lambda}_r^h$ is a lower bound to λ_r and has the same $O(h^2)$ accuracy as λ_r^h

Lumping has a mechanical interpretation in terms of the stiffness of the system. The replacement of the mass matrix M_h by the identity matrix I has the effect of making the structure "softer" and hence reduces the magnitude of the approximate eigenvalues.

But sometimes, in other problems, the structure is made too soft and the accuracy is affected. So lumping process cannot become a standard method.

III - 3. Some general results in eigenvalue and eigenfunction approximation.

We consider the general elliptic eigenvalue element method.

A general theory of finite element method is necessary to prove approximation theorems. We only recall the main properties [3]

1- <u>the discrete problem.</u>

V denotes the Sobolev space $H^m(\Omega)$ $(m \in \mathbb{N})$ equipped with the norm:

$$\|u\|_m = \left(\sum_{0 \le j \le m} |D^j u|^2_{L^2(\Omega)} \right)^{1/2}$$

V_h denotes a conform finite element space (it means that $V_h \subset V$)

<u>Theorem 3.1.</u>

If V_h is a conform finite element space of degree $(k-1)$, then there exists a constant c_r such that the approximate eigenvalues λ_r^h are bounded for small h by:

(14) $$\lambda_r \le \lambda_r^h \le \lambda_r + c_r h^{2(k-m)}$$

The result is in agreement with the explicit bound (13) verified by linear elements in dimension one.

The constant c_r depends, in particular, on r, k, m. Strang has calculated it, and he shows that higher eigenvalues are progressively more difficult to compute, and this is verified [...] techniques. (see fig 3 and fig. 4)

As in part III.2- denote by u_r an eigenfunction associated to λ_r and by u_h^r the r^{th} approximate eigenfunction - Then we have:

Theorem 3-2

If V_h is a finite space of degree $(k-1)$ and λ_r is a distinct eigenvalue, then for small h;

$$\| u_r - u_h^r \|_0 \leq c_r \left(h^k + h^{2(k-m)} \right)$$

(15) $$a(u_r - u_h^r, u_r - u_h^r) \leq c_r' h^{2(k-m)}$$

If λ_r is a repeated eigenvalue, then the orthonormal eigenfunctions u_r can be choosen such that the estimates (15) still hold.

c_r and c_r' are two constants (depending on r, k, m) with respect to h.

The estimates (15) are consistent with the case (12) of linear elements.

2- About the computational problem -

The approximation leads to the problem

$$a(u_h, v_h) = \lambda_h (u_h, v_h) \quad \forall v_h \in V_h$$

equivalent to the matrix eigenproblem

(16) $$K_h U_h = \lambda_h M_h U_h$$

K_h and M_h respectively called stiffness matrix ... They are sparse matrices.

In classical liniar algebra it does not exist standard algorithms for sparse matrices, so general algorithms as QR (or others...) are often used. However when K and M are symmetric positive definite matrices, the inverse power method is very widely used [7]; but the inverse power methods are practical especially when only modest accuracy is required.
There exists another technique which has became a most highly recommended algorithm for the band eigenvalue problem. It is due to Peters and Wilkinson [8]. and is based on the bissection by the following theorem: the number of eigenvalues less than a given λ_0 can be determined just by counting the number of negative pivots when Gauss elimination is applied to the matrix

$K - \lambda_0 M$ -

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