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TOPICS IN LINEAR ELASTICITY

Chapter V - Elastostatics

Chapter VI - Regularity

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V - ELASTOSTATICS

1) - We first collect the fundamental system of field equ.s - These are:

$$E = \operatorname{sym} \nabla u \qquad \text{(the strain-displacement relation)}; \tag{1}$$

$$S = \mathbb{C} E \qquad \text{(the stress-strain rel.)}; \tag{2}$$

$$\operatorname{div} S + b = 0 \qquad \text{(the equilibrium rel.)}. \tag{3}$$

Given a body force field b and an elasticity field $\mathbb{C}$, we say that the triplet $[u, E, S]$ is a strong elastic state on $\overline{\mathcal{B}}$ corresponding to b and $\mathbb{C}$ if

(i) $u \in C^2(\mathcal{B}) \cap C^1(\overline{\mathcal{B}})$;

(ii) $E, S \in C^1(\mathcal{B}) \cap C^0(\overline{\mathcal{B}})$, symmetric;

(iii) $[u, E, S]$ satisfies eq.s (1), (2) and (3).

Once a s.e.s. is given, we define at each regular point of $\partial\mathcal{B}$ a surface traction field by

$$s(x) = S(x)\, n(x) \qquad (n(x) \equiv \text{outward unit normal at } x \in \partial\mathcal{B}), \tag{4}$$

and call $[b, s]$ the external force system corresponding to $[u, E, S]$; it follows from the above regularity requirements on elastic states that

(iv) $b \in C^0(\overline{\mathcal{B}})$;

(v) $s \in C^0(\partial\mathcal{B})$.

Moreover, consistency of (2), (3) and (ii) is guaranteed if

(vi) $\mathbb{C} \in C^1(\overline{\mathcal{B}})$.

The notion of a strong elastic state, as contrasted with the more inclusive notion of weak elastic state that will be introduced later, underlies the classic theory of linear elasticity, and is as simple as useful.

Theorem 1 - Let $[u, E, S]$ be a s.e.s. corresponding to the e.f.s. $[b, s]$. Then,

$$2\,\Sigma(E) = \int_{\mathcal{B}} b \cdot u \, dV + \int_{\partial\mathcal{B}} s \cdot u \, dS, \tag{5}$$

where

$$\Sigma(E) = \frac{1}{2}\int_{\mathcal{B}} E \cdot \mathbb{C} E \, dV. \tag{6}$$

Pf. Follows immediately from Cor. 1, Chap. III. ∎

It is clear from (ii) of Thm 2, Chap. IV that whenever a stored e.f. $\sigma(E)$ exists

$$\Sigma(E) = \int_{\mathcal{B}} \sigma(E)\, dV \quad . \tag{7}$$

Corollary 1 - Let $\mathbb{C}$ have the property (p). Then, for each s.e.s. fixed the work done by the corresp. e.f.s. is non-negative, and vanishes only when the displacement field is rigid.

Pf. If $\mathbb{C}$ is positive, $\Sigma(E) \geq 0$, and the first part of the proof is accomplished due to (5). Further, if $\Sigma(E) = 0$, $E \cdot \mathbb{C}E = 0$ on $\mathcal{B}$, or rather $E(u) \equiv 0$ on $\mathcal{B}$, so that u is an i. rigid d.f. by Thm 1, Chap. II. ∎

(b) - Throughout this Section, we assume that $\mathcal{B}$ is homogeneous, i.e., the elasticity field is independent of x.

Theorem 2. Let $[u, E, S]$ be an elastic state corresponding to $[b, s]$. Then, the mean stress $\bar{S}(\mathcal{B})$ depends only on $u|_{\partial\mathcal{B}}$:

$$\bar{S}(\mathcal{B}) = \frac{1}{vol(\mathcal{B})}\, \mathbb{C}\Big[\int_{\partial\mathcal{B}} u \otimes n\, dS\Big] .$$

Pf. Follows from the def. of mean stress, the homogeneity of $\mathcal{B}$, the definition of mean strain, and Thm 3, Chap. II. ∎

Theorem 3 - Let $[u, E, S]$ be an e.s. corresp. to $[b, s]$ - Let further $\mathbb{C}$ be invertible, with inverse denoted by $\mathbb{C}^{-1}$. Then, the mean strain $\bar{E}(\mathcal{B})$ depends only on $[b, s]$:

$$\bar{E}(\mathcal{B}) = \frac{1}{vol(\mathcal{B})}\, \mathbb{C}^{-1}\Big[\int_{\mathcal{B}} (x-x_0) \otimes b\, dV + \int_{\partial\mathcal{B}} (x-x_0) \otimes s\, dS\Big] .$$

Pf. Same ingredients as in the pf. of Thm 2 above, with Cor. 2, Chap. III in place of Thm 3, Chap. II. ∎

Remark 1 - The thms on mean stress and mean strain have especially simple forms when $\mathcal{B}$ is isotropic - We leave it to the conscentious reader to work out these forms.

:) We now give the <u>strong formulation of the general boundary-value problem</u> of elastostatics.

Let first introduce the notion of an <u>orthogonal projection tensor</u>, i.e., a member P of Sym, s.th. $PP = P$; it is easy to show that there exist only 4 types of orth. proj.s, namely, $\underline{0}$, 1, and, for any $v \in \mathcal{S}(1)$, $v \otimes v$ and $(1 - v \otimes v)$.

(Insert here from page 23 bis)

(<u>General B-V Pb. of Elastostatics</u>)

Given : an elasticity field $\mathbb{C}(x)$ and a body force field $\hat{b}(x)$ on $\overline{\mathcal{B}}$;

: a surfacial displacement f. $\hat{u}(x)$, a surface traction f. $\hat{s}(x)$, and an orth. proj. f. $P(x)$ on $\partial\mathcal{B}$,

Find a strong elastic state $[u, E, S]$ corresponding to $\hat{b}$ and $\mathbb{C}$ such that

$$Pu = \hat{u} \quad , (1-P)\,S\,n = \hat{s} \qquad \text{on } \partial\mathcal{B}. \tag{8}$$

<u>Remark 2</u>. $\hat{u}$, $\hat{s}$ and P must obey the fllwg <u>compatibility cond.ns</u>

$$(1-P)\,\hat{u} = 0 \quad , \quad P\hat{s} = 0 \qquad \text{on } \partial\mathcal{B}. \tag{9}$$

<u>Remark 3</u>. According to various possible prescriptions of P, the general pb. reduces to the fllwg 4 cases of interest in classic elastostatics:

(i) <u>$P(x) \equiv 1$ on $\partial\mathcal{B}$</u> (<u>Dirichlet pb.</u>). Here no restriction is placed on $s(x)$ for $x \in \partial\mathcal{B}$, and :

$$u(x) = \hat{u}(x) \qquad \text{for } x \in \partial\mathcal{B}. \tag{10}$$

(ii) <u>$P(x) \equiv \underline{0}$ on $\partial\mathcal{B}$</u> (<u>Neumann pb.</u>). Conversely, no restriction is placed on $u(x)$ on the boundary of $\mathcal{B}$, but in fact

$$s(x) = S(x)\,n(x) = \hat{s}(x) \qquad \text{on } \partial\mathcal{B}. \tag{11}$$

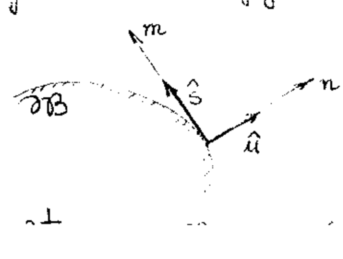
(iii) $P(x) = 1$ on $\partial_1\mathcal{B}$, $= \underline{0}$ on $\partial_2\mathcal{B}$, with $\partial_1\mathcal{B} \cup \partial_2\mathcal{B} = \partial\mathcal{B}$ & $\mathring{\partial_1\mathcal{B}} \cap \mathring{\partial_2\mathcal{B}} = \emptyset$ (<u>mixed pb.</u>)

Here of course

$$u(x) = \hat{u}(x) \quad \text{on } \partial_1\mathcal{B} \quad , \quad s(x) = \hat{s}(x) \quad \text{on } \partial_2\mathcal{B}. \tag{12}$$

(iv) <u>$P(x) \equiv n(x) \otimes n(x)$ on $\partial\mathcal{B}$</u> (<u>contact pb.</u>). Here, compatibility cond.ns (9) imply

$$\hat{u} = (n \cdot \hat{u})\, n \quad , \quad \hat{s} \cdot n = 0$$



so that we can only prescribe the <u>normal displacement</u> and the <u>tangential surface traction</u>.

(To be inserted at page 23)

Let P be an orthogonal projection. Then, it is clear from the above that P and (1-P) are complementary orthogonal projections in the sense that the range of P and the range of (1-P) are one the orth. complement of the other, and their direct sum is all of $\mathcal{V}$, i.e.,

(i) for any given $v \in \mathcal{V}$, $\exists!$ decomp. $v = v^{\perp} + v^{\parallel}$, with $v^{\perp} = Pv$ and $v^{\parallel} = (1-P)v$, s.th. $v^{\perp} \cdot v^{\parallel} = 0$;

(ii) for any $v, w \in \mathcal{V}$, $v \cdot Pw = Pv \cdot Pw$, $v \cdot (1-P)w = (1-P)v \cdot (1-P)w$.

(Insert at p. 23 under (iii))

Remark 4. Most frequently, in order to get rid of rigid motions, the following restriction is appended:

$$\text{meas}\,(\partial_1 \mathcal{B}) > 0 \quad .$$

Remark 5 - The adjective "general" will perhaps appear a bit risky, if one considers that, although case (iv) is in fact new with respect to the classical list (i) ÷ (iii) of pbms, some other important cases are not included in this formulation - As an example, take the so-called Signorini's problem[1], i.e., the case when the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is in ambiguous contact with a rigid, frictionless wall in all of a region $\partial_c\mathcal{B}$ (whose extension is a priori unknown) so that on $\partial_c\mathcal{B}$ $s\times n=0$ and

either $\quad u\cdot n=0 \quad \& \quad s\cdot n\le 0$

or $\quad u\cdot n<0 \quad \& \quad s\cdot n=0$.

Theorem 4. Let $\mathbb{C}$ have the property (p). Then the general b-v pb. has unique sln. modulo an i.rigid displ. f.

Pf. Let $[u,E,S]_1$ and $[u,E,S]_2$ be two different slns. Then

$$[u,E,S]=[\quad]_1-[\quad]_2$$

clearly is a s. el. state corresponding to $\hat{b}=0$ and $\mathbb{C}$. Moreover,

$$Pu=0\ ,\ (1-P)Sn=0 \ \text{ on } \partial\mathcal{B}\ , \tag{$*$}$$

and $(*)$ implies

$$u\cdot s = u\cdot Ps = s\cdot Pu = 0 \quad \text{on } \partial\mathcal{B}.$$

Therefore, on recalling (5) of Thm. 1, we have

$$\Sigma(E)=0$$

and, by Cor. 1,

$$u \text{ is i. rigid }, \ E=S=0 \quad . \ ■$$

Let further a triplet $[u,E,S]$ be called a kinematically admissible state if appropriate smoothness properties are possessed by u, E and S (say, (i) & (ii) under ⓐ), and

$$E=\operatorname{sym}\nabla u\ , \quad S=\mathbb{C}E\ ; \quad Pu=\hat{u} \ \text{ on } \partial\mathcal{B}.$$

Let $\mathcal{A}$ be the collection of such triplets, and consider the functional defined on $\mathcal{A}$

$$\Phi\{[u,E,S]\}=\frac{1}{2}\int_{\mathcal{B}}E\cdot\mathbb{C}E\,dV-\int_{\mathcal{B}}\hat{b}\cdot u\,dV-\int_{\partial\mathcal{B}}\hat{s}\cdot u\,dS\ ; \tag{14}$$

(this funal is called the potential energy).

The fllg thm, whose proof we skip for brevity, is of importance for our present purposes, as it provides a link, in a sense, between the strong and weak formulations of elastostatics.

Theorem 5 - Let $\mathbb{C}$ have both property (m.s) and (p). Let further $[u, E, S]_*$ be a sln of the g. b.-v. pb. of elastostatics. Then

$$\Phi\{[\ ]_*\} \le \Phi\{[\ ]\} \quad \text{for each } [\ ] \in \mathcal{A},$$

and the equality sign holds only if $[\ ] \sim [\ ]_*$ modulo a rigid displacement.

An even better link is provided by the following theorem. We call kinematically admissible a displacement field $u \in C^2(\mathcal{B}) \cap C^1(\bar{\mathcal{B}})$, s.th. $Pu = \hat{u}$ on $\partial\mathcal{B}$. Clearly, if the triplet $[u, E, S]$ is a kin. adm. state, then u is a k.a. displacement f.; conversely, if u is a k.a.d.f., then $[u, E, S]$ is a k.a. state if E and S are constructed via (1) and (2), respectively. We denote by $\mathcal{U}_0$ the collection of k.a. displ. fields s.th. $Pu = 0$ on $\partial\mathcal{B}$.

Let u_{**} be a k.a.d.f. and let u_* be any sln of the g. b-v pb. - We then set

$$u = u_* - u_{**}$$

and deal with the pb. of finding a strong elastic state $[u, E, S]$ on $\bar{\mathcal{B}}$ corresponding to $(\hat{b} + \operatorname{div} \mathbb{C}\nabla u_{**})$ and $\mathbb{C}$ such that

$$u \in \mathcal{U}_0 \quad ; \quad (1 - P) S n = \hat{s} \quad \text{on } \partial\mathcal{B}. \tag{15}$$

From now on, by "general b-v pb of elastostatics" we shall mean the pb formally stated at p. 23 with (8) replaced by (15).

Theorem 6 - $u \in \mathcal{U}_0$ solves the g. b-v pb of eltics if and only if

$$\int_{\mathcal{B}} \nabla v \cdot \mathbb{C}\nabla u \, dV = \int_{\mathcal{B}} \hat{b} \cdot v \, dV + \int_{\partial\mathcal{B}} \hat{s} \cdot v \, dS, \quad \forall v \in \mathcal{U}_0. \tag{16}$$

Pf.- To prove the direct statement, we take the inner product of (3) and any $v \in \mathcal{U}_0$, integrate over $\mathcal{B}$, and use divergence thm to get

$$-\int_{\mathcal{B}} \nabla v \cdot \mathbb{C}\nabla u + \int_{\partial\mathcal{B}} v \cdot (\mathbb{C}\nabla u) n + \int_{\mathcal{B}} b \cdot v = 0.$$

$$v \cdot (\mathbb{C}\nabla u)n = v \cdot \hat{s} + v \cdot P((\mathbb{C}\nabla u)n) = \hat{s} \cdot v \ .$$

In order to confirm the converse assertion, we first apply div. thm. backwards to get

$$-\int_{\mathcal{B}} v \cdot (\operatorname{div}(\mathbb{C}\nabla u) + b)\, dV + \int_{\partial\mathcal{B}} v \cdot ((\mathbb{C}\nabla u)n - \hat{s})\, dS = 0 \ .$$

Further, we again use elementary properties of orthogonal projections and the fact that $u \in \mathcal{U}_0$ to show that

$$v \cdot ((\mathbb{C}\nabla u)n - \hat{s}) = v \cdot (1-P)((\mathbb{C}\nabla u)n - \hat{s}) =$$
$$= v \cdot ((1-P)(\mathbb{C}\nabla u)n - \hat{s})$$

where the last step follows from the compatibility condition $(9)_2$. The proof is completed by use of the appropriate version of DuBois-Raymond's Lemma. *

Remark 6. With a view towards appreciating the meaning of the weak formulat. of b-v pbs in elstics, we compare the fllng diffeent statements of balance of momentum.

(i) (global, or integral, form; cf. (1), Chap. III)

$$\int_{\mathcal{P}} \hat{b}\, dV + \int_{\partial\mathcal{P}} \hat{s}\, dS = 0 \ , \ \forall \mathcal{P} \subset \mathcal{B} \ . \tag{17}$$

(ii) (local, or differential, or strong, form; cf. (3)).

$$\hat{b} + \operatorname{div} S = 0 \ , \ \forall x \in \mathcal{B} \ ; \ \hat{s} - Sn = 0 \ , \forall x \in \partial\mathcal{B} \ . \tag{18}$$

(iii) (virtual work, or weak, form; cf. (16))

$$\int_{\mathcal{B}} \hat{b} \cdot v\, dV + \int_{\partial\mathcal{B}} \hat{s} \cdot v\, dS - \int_{\mathcal{B}} S \cdot \nabla v\, dV \ , \ \forall v \in \mathcal{U}_0 \ . \tag{19}$$

Clearly[2], eq.s (17) and (19) make sense for much more general functional classes than those we used up to now; it is not at all like so for eq.s(18) that are actually derived from (17), and shown equivalent to (19), under rather strong smoothness hypotheses. Thus, to start off with eq.s (18) is often unduly restrictive; not only that, the boundary conditions are most naturally

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(2) Although the substance of the present remark has long been well known to students of continuum mechanics we phrase our considerations in the way of S.S. ANTMAN, Arch. Rational Mech. An.

incorporated in eq. (19). Finally, eq.s (17) and (19) can be shown to be equivalent. To see this, let $\{e_k\}$ a fixed orthonormal set of vectors, and let $\chi(P)$ be the characteristic fn of the generic part $P \subset \mathcal{B}$. Then, we may choose v in (19) as to approximate $\chi(P)\, e_k$, and thus obtain (17) from (19). Conversely, we may write (17) as follows

$$e_k \cdot \Big(\int_{\mathcal{B}} \chi(P)\, \hat{b}\, dV + \int_{\partial\mathcal{B}} \chi(P)\, \hat{s}\, dS \Big) = 0 \qquad \forall P \subset \mathcal{B},\ \forall k \tag{17'}$$

and then obtain (19) by approximating v by finite linear combinations of $\chi(P)$, $P \subset \mathcal{B}$.

e - We will confine ourselves henceforth to study the Dirichlet problem, that we have described before, but in a fairly weaker sense. We shall repeatedly use the following <u>function spaces</u>.

- $(H^1(\mathcal{B}))^n$, the Hilbert space of $\square$ integrable (in the sense of Lebesgue) n-vector fields on $\mathcal{B}$ with $\square$ integrable 1st gradient. For brevity, we shall write H^1 in place of $(H^1(\mathcal{B}))^n$, and the dimension n will be clear from the context. If $u \in H^1$ we shall denote by

$$\|u\|_1 = \Big(\int_{\mathcal{B}} (u \cdot u + \nabla u \cdot \nabla u)\, dV \Big)^{1/2} = \big(\|u\|_0^2 + |u|_1^2 \big)^{1/2} \tag{20}$$

the norm affiliated with the inner product

$$(u|v)_1 = \int_{\mathcal{B}} (u \cdot v + \nabla u \cdot \nabla v)\, dV \quad, \forall v, w \in H^1 . \tag{21}$$

Of course, in (20) $\|u\|_0$ is the L^2 norm of u, and $|u|_1$ is the seminorm of the first partial derivatives.

- C_0^∞, the space of n-vector fields vanishing on $\partial\mathcal{B}$ with their gradients of any order
- H_0^1, the completion of C_0^∞ with respect to the norm of H^1.

We will also make use of another seminorm of $u \in \mathcal{H}^1$, namely the strain seminorm

$$|||u|||_1 = \Big(\int_{\mathcal{B}} E(u) \cdot E(u) \Big)^{1/2} .$$

It turns out that $\mathcal{H}_0^1$ is a fn space where it is reasonable to look for finding candidate to solve our pb., mostly because for fns in this space the fllng well-known inequalities are easily established.

$$\text{(POINCARÉ's ineq.)} \qquad |||u|||_1^2 \geq \varkappa_1 \, \|u\|_0^2 \quad , \ \varkappa_1 > 0 \ ; \tag{22}$$

$$\text{(KORN's ineq.)} \qquad |||u|||_1^2 \geq \varkappa_2 \, |u|_1^2 \quad , \ \varkappa_2 > 0 \ . \tag{23}$$

<u>Remark 7.</u> Both inequalities can be proved, at the expenses of using some more complicated analytical machinery[3], and of introducing some more stringent hypotheses on $\partial\mathcal{B}$ and u, also for fns that are members of subspaces of $\mathcal{H}^1$ larger than $\mathcal{H}_0^1$.

It might be an useful exercise to go through the proof of the fllng thm, where the ingredients, but for some direct Fourier transforms here and there, are the same ingredients used by M.me BOUJOT in her lectures.

<u>Theorem 7.</u> Let $u \in C^2(\mathcal{B}) \cap C_0^0(\overline{\mathcal{B}})$. Then Korn's inequality holds (with $\varkappa_2 = 2$, no matter how $\mathcal{B}$ looks like).

<u>Pf.</u> Let

$$W(u) = \operatorname{skw} \nabla u ,$$

and recall the direct decomposition $\mathrm{Lin} = \mathrm{Sym} \oplus \mathrm{Skw}$ and the defns of the symmetrizers $\operatorname{sym}(\cdot)$ and $\operatorname{skw}(\cdot)$ to get

$$|\nabla u|^2 = |E(u)|^2 + |W(u)|^2 \ ; \tag{24}$$

$$|E(u)|^2 = \frac{1}{2}\big(|\nabla u|^2 + \nabla u \cdot \nabla u^T \big) \tag{25}$$

$$|W(u)|^2 = \frac{1}{2}\big(|\nabla u|^2 - \nabla u \cdot \nabla u^T \big) . \tag{26}$$

Recall further the identity

Then by (25), (26) and (27) we get

$$\int_{\mathcal{B}} (|E|^2 - |W|^2)\, dV = \int_{\mathcal{B}} (\mathrm{div}(\qquad\qquad) + (1\cdot\nabla u)^2)\, dV ,$$

and further, by use of div. thm and the fact that $u|_{\partial\mathcal{B}} = 0$, we arrive at

$$\int_{\mathcal{B}} |W|^2\, dV \leq \int_{\mathcal{B}} |E|^2\, dV ,$$

so that, in view also of (24), we conclude with the desired inequality, with $K_2 = 2$. ∎

VI - REGULARITY

In this Chapter we give a quick formulation of the displacement problem of elastostatics in the by now popular weak form, and indicate how to get weak slns for this problem; we then show, by means of an example due to DE GIORGI, that many optimistic conjectures on the regularity of weak solutions are unsound and in fact false.

ⓐ - Assume the elasticity tensor field $\mathbb{C}(x)$ is measurable and bounded in $\overline{\mathcal{B}}$ and consider the energy form, i.e. the fllng 1st order, real, bilinear, integro-differential form

$$\alpha(\cdot,\cdot) : H_0^1 \times H_0^1 \longrightarrow \mathbb{R}, \quad \alpha(u,v) = \int_{\mathcal{B}} \nabla v \cdot \mathbb{C}\nabla u \, dV \quad , \forall u,v \in H_0^1 . \qquad (1)$$

It is easy to show that α is continuous, i.e. $\exists\, \varkappa_c > 0$ s.th.

$$|\alpha(u,v)| \leq \varkappa_c \|u\|_1 \|v\|_1 \quad , \forall\, u,v \in H_0^1 .$$

Moreover, the use of the inequalities of Poincaré and Korn allows us to establish the substantial equivalence in H_0^1 of the various norms and seminorms that we have introduced for H^1. In view of this equivalence, establishing the coerciveness of the form α, i.e. the existence of $\varkappa_{cc} > 0$ s.th.

$$\alpha(u,u) \geqslant \varkappa_{cc} \|u\|_1 \quad , \forall\, u \in H_0^1 , \qquad (2)$$

reduces to checking that the fllng inequality hold

$$\int_{\mathcal{B}} (\nabla u \cdot \mathbb{C}\nabla u - \varkappa_* |E(u)|^2)\, dV \geqslant 0 , \forall\, u \in H_0^1 \quad (\varkappa_* \equiv \text{a positive const.}). \qquad (3)$$

It is clear from eq. (19) and Thm. 6 of Chap. IV that ineq. (3) hold if we assume further that $\mathbb{C}$ has property (p).

Remark 1 - An interesting pb. is to determine a set of reasonable restrictions on $\mathbb{C}$, milder than assuming (p), to insure coerciveness. Some results of this kind are already available. We say that $\mathbb{C}$ is strongly elliptic whenever $\mathbb{C}$ has property

$$\text{(s.e) (strong ellipticity)} \quad A \cdot \mathbb{C}A > 0 \quad , \forall\, A \in \mathrm{Dya} - \{0\} . \qquad (4)$$

VAN HOVE has shown that (3) holds whenever $\mathbb{C}$ is constant in $\mathcal{B}$ & has property (s.e). FICHERA has got a much better result: he was able to prove (3) whenever $\mathbb{C}$ has property (s.e) & a sufficiently small (in a sense he make precise)

Now, let $(H^1_0)'$ be the space of linear continuous fnals on H^1_0, and denote by $\langle \varphi, v \rangle = \varphi|_v$ the pairing between an element $\varphi \in (H^1_0)'$ and an element $v \in H^1_0$. Assume that

$$\langle \varphi, v \rangle = \int_{\mathcal{B}} \hat{b} \cdot v \, dV \tag{5}$$

is an element of $(H^1_0)'$.

We are now in a position to give the displacement pb of els.tics the fllng weak formulation: given $\mathbb{C}$ and $\hat{b}$ as above, find $u \in H^1_0$ s.th.

$$\alpha(u, v) - \langle \varphi, v \rangle = 0 \quad \forall v \in H^1_0 \text{ (or, equivalently, } \forall v \in C^\infty_0). \tag{6}$$

As soon as α is continuous and coercive, the well-known Lax & Milgram argument applies, and one proves the existence of a unique solution of eq. (6) whose norm is dominated by the norm of the datum φ.

Likewise, it is well known that if one further assumes that $\mathbb{C}$ has the property (ms), than (6) can be interpreted as the <u>weak Euler equation</u> associated with the extremum pb. for the quadratic fnal on H^1_0

$$\Phi\{u\} = \frac{1}{2} \alpha(u, u) - \langle \varphi, v \rangle \; ; \tag{7}$$

to solve eq. (6) is then shown to be equivalent to finding the minimum of this fnal on H^1_0.

(b) - The former results do not meet completely the demands of applications In order to show that the displacement pb. admits of a sln in the traditional strong sense, one should be able to move a number of steps towards a satisfactory <u>regularization</u> of the weak sln provided by L & M lemma. The fact that these steps cannot in general be too many (so that one should not expect to obtain in general, say, hölder-continuity of slns) was rendered patent by DE GIORGI[1] by means of a counterexample.

De Giorgi considers the bilinear form (1) with

$$\left.\begin{aligned} \mathbb{C} &= \mathbb{1} + (n-2)^2\, \mathrm{B}\otimes\mathrm{B}\,, \\ \mathrm{B} &= 1 + \frac{n}{n-2}\,\mathrm{P}\,, \quad \mathrm{P} = |p|^{-2}\, p\otimes p \\ n &= \dim \mathcal{E} \geq 3 \end{aligned}\right\} \tag{8}$$

and calls u an extremal of the integral

$$\Phi\{u\} = \alpha(u, u) \tag{9}$$

if $u \in \mathcal{H}^1$ and

$$\alpha(u, v) = 0 \quad , \quad \forall v \in C_0^\infty \quad . \tag{10}$$

He then shows that

$$\check{u} = |p|^\alpha\, p \tag{11}$$

is an extremal of the integral (9) if

$$\alpha = -\frac{n}{2}\left(1 - \frac{1}{\sqrt{1 + 4(n-1)^2}}\right) \quad . \tag{12}$$

Therefore, regularity of extremals depends somewhat surprisingly on the dimension of the underlying space: it is easy to see that $\check{u}$ is neither continuous nor bounded at the origin, although the bilinear form constructed with $\mathbb{C}$ and B as in (8) has measurable and bounded coefficients.

We now adapt slightly De Giorgi's example in order to cast it into the form of a result of non-regularity for the weak slns of a certain displacement problem in linear elastostatics.[2]

Consider a two-parameters family of mappings as $\mathbb{C}$ above, namely

$$\mathbb{C} = \mathbb{1} + \gamma\, \mathrm{B}\otimes\mathrm{B} \quad , \quad \mathrm{B} = 1 + \beta\, \mathrm{P} \tag{13}$$

where β, γ are constants which may depend on n, and $\gamma > 0$.

Prescription (13) of $\mathbb{C}$ actually defines a class of linearly elastic materials: $\mathbb{C}$ has both minor and major symmetries, is strongly elliptic, the associated symmetry group at $x \in \mathcal{B}$ is the group of those $Q \in \mathrm{Orth}$ s.that $Q\mathrm{P}(x) = \mathrm{P}(x)Q$, i.e. any rotation whose axis is the line spanned by $p(x)$. Thus, $\mathbb{C}$ defines a transversely isotropic elastic material with moduli β, γ.

Let $\mathcal{B}$ coincide with the unit ball centered at the origin O, comprised of the

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(2)

elastic material defined by (12). Let also body forces vanish. One can easily show that $\mathbb{C}$ is strongly elliptic, and that $\alpha(\cdot,\cdot)$ is symmetric, continuous and coercive. Therefore, Lax & Milgram's lemma allows one to conclude that there exists a unique sln $u \in H^1$ of the displacement problem: find $u \in H^1$ s. that

$$(u - p) \in H^1_0 \quad \& \quad \alpha(u,v) = 0 \quad , \forall v \in H^1_0 \,. \tag{14}$$

Moreover, $\check{u}$ as given by (11) is a sln of (14) provided that

$$n \geq 3 \;;\; \beta \geq \frac{1+\gamma}{\gamma(n-2)} \quad ; \; \alpha = -\frac{n}{2}\left(1 - \sqrt{1 - \frac{4\beta\gamma(n-1)(n+\beta)}{n^2(1+\gamma(1+\beta)^2)}}\right) \,. \tag{15}$$

By inspection of the behaviour of $\hat{u}$, we see that we have obtained a variational problem with finite energy, at first sight an honest & good problem, but with a displeasing physically meaningless sln.

<u>Remark 2</u>. Loss of physical meaning occurs (cf. (15)) when $n=3$ and the modulus of aelotropy $\beta > 0$. In fact, if $\beta = 0$ $\mathbb{C}$ would reduce to the familiar form of isotropic elasticity, and the pb. would admit only a smooth sln.

Now we consider a family of spherical shells, of unit outer radius, comprised of the material (13). Taking as a parameter the radius ε of the inner cavity, we construct the centrally symmetric sln u_ε of the flltg displacement pb:

$$u_\varepsilon = 0 \quad \text{for } |p(x)| = \varepsilon \quad , \quad u_\varepsilon = p \quad \text{for } |p(x)| = 1 \,. \tag{16}$$

We further extend u_ε to $\overline{B}$ as follows

$$\begin{aligned} u_\varepsilon &= 0 \qquad \text{for} \quad 0 \leq |p(x)| < \varepsilon \\ &= u_\varepsilon \qquad \text{for} \quad \varepsilon \leq |p(x)| \leq 1 \,. \end{aligned}$$

Then we show that $\{u_\varepsilon\} \longrightarrow \check{u}$ in energy, or, equivalently, in H^1. Therefore, De Giorgi's is the limit pb. of the sequence (16). However, as soon as $\varepsilon \leqslant \varepsilon_0(n;\beta,\gamma)$, u_ε looses its physical meaning.