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UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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SMR/23 /55

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

THEORY OF PLATES

Part IX - Buckling problems

Part X - Galerkin process for buckling problems for bilateral situations

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IX. Buckling problems -

In a buckling problem, we have not, in general, the uniqueness.

In the following, we assume that

Hyp. $y = 0$ is solution for all real λ.

This assumption introduced in the variational problem

$$((I - \lambda L)y + Cy, z - y) + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z,$$

implies:

$$\psi(z) - \psi(0) \geq 0.$$

and, we can assume the following, which is equivalent to the previous hypothesis:

$$(1) \qquad \boxed{\begin{array}{ll} \psi \geq 0 & \text{on } Z \\ \psi(0) = 0 & \end{array}}$$

For problems 1, 2 or 3, described in ch. V, this hypothesis corresponds respectively to

$j \geq 0$, $j(0) = 0$

or $k \geq 0$, $k(0) = 0$, and $f = 0$ in Ω

or $l \geq 0$, $l(0) = 0$, and $f = 0$ in Ω

In the case of ~~the~~ example of problem 1 (plate placed on a rigid support) we must have the exterior (given) forces $p(x) \leq 0$ in Ω.

In addition, we assume that

$$\begin{cases} j(\tau\xi) \leq \tau^q j(\xi) & \forall \xi \in \mathbb{R} \text{ and } \tau \geq 1 \\ \text{and the same for } k \text{ and } l. \end{cases}$$

From that, we deduce for ψ:

$$(2) \qquad \psi(\tau z) \leq \tau^q \psi(z) \quad \forall z \in Z \text{ and } \tau \geq 1$$

This last hypothesis implies that the domain of ψ, defined by $\mathrm{dom}\ \psi = \{ z \in Z_1 / \psi(z) < +\infty \}$ is a cone in Z_1.

The assumption about the positivity of L, make in the previous chapter, holds always:
$(Lz, z) \geq 0$; $(Lz, z) > 0$ if $z \neq 0$.
Then (because L is compact), the characteristic (1) values of L, $\lambda_1, \lambda_2, \ldots$, are positive and $\lambda_n \to \infty$ if $n \to \infty$ (to see Lectures Notes of M. J. Boujot).

The problem is:

(3) **Problem 1.** To find $y \in Z$ such that
$$((I - \lambda L) y + Cy, z - y) + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z_1 .$$

It has been observed that this problem possesses always (for all real λ) the trivial solution $y = 0$. The question is to find other solutions for suitable values of λ; an existence theorem for non trivial solutions of (3) describes the phenomenon of buckling (at least in a qualitative way).

The main result about this problem is the following

$\boxed{\text{Theorem 1}}$ (M. S. Berger & P. Fife, for bilateral situations - 1968.
Cl. Do, for unilateral situations - 1974)

(i) $y = 0$ is the unique solution if $\lambda \leq \lambda_1$.
(ii) There exists a family of non trivial solutions $(\lambda_r, y_r, r > 0)$, indexed by all positive real numbers.
More precisely, for each $r > 0$, there exists a solution (λ_r, y_r) such that:

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(1) λ is a characteristic value of L if $\frac{1}{\lambda}$ is an eigenvalue.

$$(4)\quad \begin{cases} (Ly_r, y_r) = \sup\limits_{\partial A_r} (Lz, z) \\ \text{with} \\ \partial A_r = \left\{ z \in Z_1 \;/\; \frac{1}{2}\|z\|^2 + \frac{1}{4}(Cz, z) + \psi(z) = r \right\} \end{cases}$$

In addition : $y_r \to 0$ in Z_1 when $r \to 0$,
$\|y_r\| \to \infty$, when $r \to \infty$.

About this statement, we are doing the following comments.

1- By (4), we can obtain some nontrivial solutions of our problem, solving a conditional extremum problem ; that consists to consider the parameter λ like a Lagrange multiplier which is introduced for such a problem. This is a classical idea in Mechanics.

2- Because $y_r \to 0$ when $r \to 0$, the bifurcation phenomenon from the trivial solution, must be studied in the domain $r \to 0$.

3- This theorem gives only partial results : the set of λ such that there exists a non trivial solution is not specified.

About the problem, we recall that we have not the coercivity when $\lambda > \lambda_1$. Is it possible to use a Galerkin process ? A positive answer at this question has been recently given by A. Cimetière (1976) for the bilateral situation; the study of unilateral cases is in progress.

In the bilateral case, the problem (3) is reduced to

$$(5)\qquad (I - \lambda L)\, y + C y = 0$$

Let $z_1, z_2, \ldots, z_m, \ldots$ be a basis in Z_1 and $Z_{1m} = \overline{[z_1, \ldots z_m]}$

finite dimension is :

(6) $\begin{cases} \text{to find } y_m \in Z_m \text{ such that} \\ ((I-\lambda L)y_m + Cy_m , z) = 0 \quad \forall z \in Z_m \end{cases}$

and the result is

Theorem 2.

(i) For each $\lambda > \lambda_1$, there exists $y_m \in Z_m$, solution of (6); in addition, y_m is solution of the minimization problem :

$$P(y_m) = \inf_{z \in Z_m} P(z)$$

where $P(z) = \|z\|^2 - \lambda(Lz, z) + \frac{1}{2}(Cz, z)$.

(ii) There exists a subsequence y_m such that
$y_m \to y$ in Z_1 weak (and also in Z_1 strong) ; y is a non zero solution of the problem.

(Therefore, there exists a non trivial solution for all $\lambda > \lambda_1$)

Assumptions and properties - Recapitulation

H 1 $\quad$ L is compact, self adjoint, linear operator;

H 2 $\quad$ $(Lz, z) \geq 0$; $(Lz, z) > 0$ if $z \neq 0$;

H 3 $\quad$ C is bounded continuous non linear operator on Z_1 ;

H 4 $\quad$ Cz is the Frechet differential of $\frac{1}{4}(Cz, z)$;

H 5 $\quad$ $C0 = 0$;

H 6 $\quad$ $(Cz, z) \geq 0$; $(Cz, z) > 0$ if $z \neq 0$;

H 7 $\quad$ C is compact ;

H 8 $\quad$ $\| Cz \| \leq c \|z\|^3$;

H 9 $\quad$ ψ is convex lower semi continuous function on Z_1 ;

H 10 $\quad$ $\psi(0) = 0$;

H 11 $\quad$ $\psi \geq 0$;

H 12 $\quad$ $0 \in \partial\psi(0)$;

H 13 $\quad$ $\operatorname{dom} \psi = \{ z \in Z_1 \,/\, \psi(z) < +\infty \} \neq \{0\}$;

H 14 $\quad$ $\exists\, q > 0$ such that $\psi(\tau z) \leq \tau^q \psi(z)$ $\forall z \in Z_1$ and $\tau \geq 1$;

H 15 $\quad$ $\forall B$, bounded in Z_1, ψ is bounded in $B \cap \operatorname{dom}\psi$.

Recall the following (to see lecture Notes of M^{s} J. Boujot)

Let $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ be the characteristic values of L ; $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ the related eigenfunctions ; then, we have

$$(H.16) \quad \begin{cases} \dfrac{1}{\lambda_1} = \sup\limits_{z \in Z} \dfrac{(Lz, z)}{\|z\|^2} \\[2ex] \dfrac{1}{\lambda_p} = \sup\limits_{\{z \in Z} \dfrac{(Lz, z)}{\|z\|^2} \end{cases}$$

Because the property (4), announced in the statement, it is natural to introduce some notations, and a new problem: We put:

(7) $$F(z) = \frac{1}{2}\|z\|^2 + \frac{1}{4}(C_z, z) + \psi(z),$$

and, for each $r>0$,

(8) $$\begin{cases} A_r = \{z \in Z \,/\, F(z) \leq r\}, \\ \partial A_r = \{z \in Z \,/\, F(z) = r\}. \end{cases}$$

(9) <u>Problem 2.</u> Let $r>0$ be fixed. To find $y_r \in \partial A_r$ such that
$$(Ly_r, y_r) = \sup_{\partial A_r} (Lz, z).$$

The existence of a non trivial solution announced in the statement of the theorem 1 will be obtained by solving the problem 2.

We have the

<u>Lemma 1.</u> For each $r>0$,

(i) A_r is bounded;

(ii) $0 \in \partial A_r$

(iii) $\forall z \in \mathrm{dom}\,\psi$, $z \neq 0$, $\exists t>0$ such that $tz \in \partial A_r$ (and $t \geq 1$ if $z \in A_r$)

<u>Proof of Lemma 1</u>

(i) by (H6) and (H11)

(ii) by (H6) and (H10)

(iii) because the function $t \mapsto F(tz)$ is continuous, strictly increasing, and goes to infinity with t.

Proof of theorem 1.

Like in the bending problem, the theorem is proved :

first : in regular case (ψ is assumed to be Frechet-differentiable)

second ; ψ is not differentiable and regularization techniques are used.

It can be observed that the estimates are very different in bending problems and buckling ones.

(i) Uniqueness. We choose $z=0$ in (3) ; because (H. 10), (H. 11), (H. 6), we obtain $\|y\|^2 < \lambda(Ly,y)$ if y is a non trivial solution; then, first $\lambda > 0$, and after (via (H. 17)):

$$\frac{1}{\lambda} < \frac{(Ly,y)}{\|y\|^2} \leq \sup \frac{(Lz,z)}{\|z\|^2} = \frac{1}{\lambda_1} .$$

Existence.

First part. In this part, we assume that

$$\begin{cases} \psi \text{ is Frechet-differentiable ;} \\ \text{its derivative } z \mapsto \psi'(z) \text{ is Lipschitz-continuous} \end{cases}$$

remember that this derivative is monotone -

Because ψ is differentiable, it is easy to obtain the

Lemma 2. If y_r is solution of problem 2, then y_r is non trivial solution of problem 1(i), with a suitable $\lambda = \lambda_r$.

Now, existence theorem comes from

Lemma 3. For each $r>0$, there exists a solution for the problem 2. In addition, $y_r \to 0$ in Z_1, with r, and $\|y_r\| \to \infty$ with r.

Proof of lemma 3. 1°) (Lz,z) is bounded on ∂A_r (which is a bounded part of Z_1). Let $\{z_n, n \geq 0\}$ be a maximizing sequence; because it is bounded, we can assume that this sequence is weakly convergent in Z_1, with weak limit z.

By compactness (H7), $(C_{z_n}, z_n) \longrightarrow (C_z, z)$.
By lower semi continuity of the norm and convex function ψ, with respect to the weak topology, we obtain for the limit z:

$$F(z) \leq r,$$

and therefore $z \in \partial A_r$. But $z \neq 0$ (because z_n is a maximizing sequence). Then, there exists $t \geqslant 1$ such that $tz \in \partial A_r$. But $(L(tz), tz) = t^2 (Lz, z)$, and the extremum condition implies $t = 1$. Therefore $z \in \partial A_r$ and

$z = y_r$ is solution of the problem 2.

2°) We have $y_r \in \partial A_r$, hence $\|y_r\|^2 < 2r$; we deduce of this that

$$\lim y_r = 0 \quad \text{in } Z, \quad \text{when } r \to 0.$$

Furthermore, $\|y_r\| \to \infty$ with r because F is a sum of bounded functions.

The theorem 1 is proved in the case of ψ is differentiable.

<u>Second part</u>. Now ψ is not differentiable. We use the notations introduced in [ch VIII : (3), ..., (7)].

We consider ψ_ε, with $\varepsilon > 0$, defined like in VIII, (3); with ψ_ε, we associate F_ε, A_r^ε, ∂A_r^ε as previously, replacing ψ by ψ_ε. Because ψ_ε is Frechet-differentiable, there exists $y_r^\varepsilon \in \partial A_r^\varepsilon$, λ_r^ε, solution of the problem associated with ψ_ε (1). In the following, r is fixed, and we omit this indice : we put

$$y_r^\varepsilon = y^\varepsilon \quad ; \quad \lambda_r^\varepsilon = \lambda^\varepsilon,$$

and we have

(10) $$\left((I - \lambda^\varepsilon L) y^\varepsilon + C y^\varepsilon, z - y^\varepsilon\right) + \psi_\varepsilon(z) - \psi_\varepsilon(y_\varepsilon) \geqslant 0 \quad \forall z \in Z$$

(1) It is sufficient to observe that assumptions (H.10) --- (H.15) are

with

(11) $$y^\varepsilon \in \partial A_r^\varepsilon .$$

<u>A priori-estimates</u> (r is fixed, >0).

1°) By (11), we obtain

(12) $$\begin{cases} \|y^\varepsilon\| \leq c_1 , \\ \psi_\varepsilon(y^\varepsilon) \leq c_2 , \end{cases}$$

and, using VIII,(3) (i.e. $\psi_\varepsilon(z) = \frac{1}{2\varepsilon}\|z - P_\varepsilon z\|^2 + \psi(P_\varepsilon z))$)

(13) $$\frac{1}{2\varepsilon}\|y_\varepsilon - P_\varepsilon y_\varepsilon\| \leq c_3$$

2°). We have $(Ly_\varepsilon, y_\varepsilon) = \sup\limits_{\partial A_r^\varepsilon} (Lz, z) = \sup\limits_{A_r^\varepsilon} (Lz, z)$. But $A_r^\varepsilon \supset A_r$ by VIII,(4). Therefore

(14) $$(Ly_\varepsilon, y_\varepsilon) \geq \sup_{z \in \partial A_r} (Lz, z) = c > 0 :$$

c is positive by lemma 1.

Choosing $z = 2y_\varepsilon$ in (10), we obtain

$$\lambda_\varepsilon (Ly_\varepsilon, y_\varepsilon) \leq \|y_\varepsilon\|^2 + (Cy_\varepsilon, y_\varepsilon) + \psi_\varepsilon(2y_\varepsilon) - \psi_\varepsilon(y_\varepsilon) \;;$$

from this, by (12) and (H.14), we deduce

(15) $$\lambda_\varepsilon \text{ is bounded in } \mathbb{R}.$$

<u>Limiting process.</u> From (12) and (15), there exists a subsequence ε, going to 0 such that

$$y_\varepsilon \to y \text{ in } Z \text{ weak},$$
$$\lambda_\varepsilon \to \lambda \text{ in } \mathbb{R}.$$

In addition, by (13), we have

$$P_\varepsilon y_\varepsilon \longrightarrow y \quad \text{in } Z \text{ weak}$$

and, by compactness

$$\lambda_\varepsilon (Ly_\varepsilon, y_\varepsilon) \to \lambda (Ly, y)$$

Now, because $\psi(P_\varepsilon y_\varepsilon) \leq \psi_\varepsilon(y_\varepsilon)$ (by VIII, (3)) and $\psi_\varepsilon(z) \leq \psi(z)$ (by VIII, (4)), we deduce from (10) the inequality

$$(16) \qquad ((I-\lambda^\varepsilon L)y^\varepsilon + Cy^\varepsilon, z - y^\varepsilon) + \psi(z) - \psi(P_\varepsilon y_\varepsilon) \geq 0 \qquad \forall z \in Z.$$

Via lower semi continuity, we have from (16)

$$\begin{aligned}\|y\|^2 + \psi(y) &\leq \liminf \left(\|y_\varepsilon\|^2 + \psi(P_\varepsilon y_\varepsilon)\right) \\ &\leq \liminf \left\{(y^\varepsilon, z) - \lambda^\varepsilon(Ly^\varepsilon, z-y^\varepsilon) + (Cy^\varepsilon, z-y^\varepsilon) + \psi(z)\right\} \\ &\leq (y, z) - \lambda(Ly, z-y) + (Cy, z-y) + \psi(z),\end{aligned}$$

let

$$((I-\lambda L)y + Cy, z-y) + \psi(z) - \psi(y) \geq 0 \qquad \forall z \in Z,$$

and (λ, y) is solution of the problem; finally, $y \neq 0$ because $(Ly, y) = \lim (Ly_\varepsilon, y_\varepsilon) > 0$ by (14).

The existence in the theorem 1 is proved.

<u>Third part</u>.

<u>Lemma 4.</u> The solution exhibited in the previous part is such that $y \in \partial A_r$, $(Ly, y) = \sup_{\partial A_r} (Lz, z)$.

<u>Proof of the lemma 4.</u> First, we prove $y \in A_r$: y is weak limit of y^ε, with $y^\varepsilon \in \partial A_r^\varepsilon$:

$$\frac{1}{2}\|y_\varepsilon\|^2 + \frac{1}{4}(Cy_\varepsilon, y_\varepsilon) + \frac{1}{2\varepsilon}\|y_\varepsilon - P_\varepsilon y_\varepsilon\|^2 + \psi(P_\varepsilon y_\varepsilon) = r$$

therefore, by lower semi continuity

$$\begin{aligned}\frac{1}{2}\|y\|^2 + \psi(y) &\leq \liminf \left\{\frac{1}{2}\|y_\varepsilon\|^2 + \psi(P_\varepsilon y_\varepsilon)\right\} \\ &\leq r - \frac{1}{4}(Cy, y)\end{aligned}$$

from this, it results that $y \in A_r$

But $(Ly, y) = \sup_{A_r} (Lz, z)$; if $y \notin \partial A_r$, there exists $t > 1$ such that $ty \in \partial A_r$ and we have $(L(ty), ty) = t^2(Ly, y) > (Ly, y)$ and the extremum property is false. That is impossible and

Finally, $y_r \in \partial A_r$ implies $\|y_r\| < 2r$, and $y_r \to \infty$ (in Z_1) with r.

In addition, F is bounded on $\operatorname{dom} \varphi \cap B$ for all bounded subset B of Z_1. Therefore, if $r \to +\infty$, we have $\|y_r\| \to \infty$ since $F(y_r) \to \infty$.

$\blacksquare$

X. Galerkin process for buckling problems in bilateral situations.

Now, we are going to study the Galerkin process for bilateral problem:

To find $y \in Z_1$ such that

(1) $$(I - \lambda L) y + C y = 0 ,$$

and to prove the theorem 2 of the previous section IX.

It is natural to consider equation (1) as the Euler equation associated with the problem

To find critical points $\neq 0$ of the functional

$$\Phi(z) = \|z\|^2 - \lambda (Lz, z) + \tfrac{1}{2}(Cz, z) , \quad z \in Z_1 .$$

The coerciveness sets some difficulties: it has been observed that $((I - \lambda L) z, z)$ is not coercive as soon as λ is greater than λ_1. In addition, it is not possible to hope the coerciveness of the problem via (Cz, z): an inequality such that

(2) $$(Cz, z) \geq \gamma \|z\|^4$$

is not possible on Z_1, because C is compact.

Meantime, if (2) is false on Z_1, it is true on subsets which are not "too distant" from finite dimension subspace: this property (lemma 1) plays an important part in the following since it is proved (lemma 2) that the solutions of (1) are not "too distants" from proper subspaces of L (which are finite dimension subspace; to see lecture notes of Mr J. Boujot)

(3) **Lemma 1.** Let E be a finite dimension subspace of Z_1, and $F = E^{\perp}$; for given $\alpha, \beta > 0$, we put

$$K = K(\alpha,\beta) = \{ z \in Z_1 \,/\, z = e + f,\ e \in E,\ f \in F,\ \alpha \|f\| \leq \beta \|e\| \}.$$

Then, there exists $\gamma = \gamma(\alpha,\beta)$ such that

$$(C_{z}, z) \geq \gamma \|z\|^4 \qquad \forall z \in K.$$

The two following lemmas permit to localize the solutions of (1).

We use the notations:

$$\begin{cases} E_j \text{ is the proper subspace associated with the characteristic value } \lambda_j \\ F_j = E_1 \oplus \cdots \oplus E_j. \end{cases}$$

Lemma 2. Let λ be fixed, with $\lambda_1 < \lambda < \lambda_{K+1}$. If

$$y = e + f, \quad e \in F_K, \quad f \in F_K^{\perp},$$

is solution of (1), then

$$\left(1 - \frac{\lambda}{\lambda_{K+1}}\right) \|f\|^2 \leq \left(\frac{\lambda}{\lambda_1} - 1\right) \|e\|^2.$$

Therefore, for fixed λ, the set of solutions of (1) is contained in a subset like (3) and we can use lemma 1 on this subset. We put, for $\lambda < \lambda_{K+1}$

(4) $$K(\lambda) = \left\{ z = e + f \,/\, e \in F_K,\ f \in F_K^{\perp},\ \left(1 - \frac{\lambda}{\lambda_{K+1}}\right) \|f\|^2 \leq \left(\frac{\lambda}{\lambda_1} - 1\right) \|e\|^2 \right\}$$

and we have, via lemma 1:

(5) $$\begin{cases} \text{there exists a constant } \gamma(\lambda) > 0 \text{ such that} \\ (C_{z}, z) \geq \gamma(\lambda) \|z\|^4 \qquad \forall z \in K(\lambda) \end{cases}$$

Lemma 3. We have (for each solution of (1)):

$$\|y\|^2 \leq \frac{1}{\gamma(\lambda)} \left(\frac{\lambda}{\lambda_1} - 1\right)$$

where $\gamma(\lambda)$ is defined by (5).

<u>Remark</u> Estimates in lemma 2 & 3 are again valid for $\bar{y} \in \bar{Z}$, where $\bar{Z}$ is a closed subspace of Z, and $\bar{y}$ is solution of the problem

$$((I - \lambda L)\bar{y} + C\bar{y}, \bar{z}) = 0 \quad \forall\, \bar{z} \in \bar{Z} .$$

Especially, that is true (and will be used in the following) when $\bar{Z} = Z_m$ is a finite dimension subspace

<u>Proof of lemma 1.</u> By negation. Assume that there exists a sequence $z_n = e_n + f_n \in K(\alpha, \beta)$, $n \geq 0$, with

$$\begin{cases} \|z_n\|^2 = \|e_n\|^2 + \|f_n\|^2 = 1 \\ (Cz_n, z_n) \to 0 \end{cases}$$

Because $z_n \in K$, $\alpha^2 \|f_n\|^2 \leq \beta^2 \|e_n\|^2$ and therefore

$$1 \geq \|e_n\|^2 \geq \frac{\alpha^2}{\alpha^2 + \beta^2} .$$

Hence (for a subsequence at least),

$$e_n \to e \neq 0 \text{ in } E$$

(strong because E is a finite dimension space). But $\|f_n\| \leq 1$ and (for a subsequence)

$$f_n \rightharpoonup f \text{ in } Z \text{ weak}, \text{ with } f \in E^\perp$$

($E^\perp$ is closed and therefore weakly closed). Then

$$z_n \rightharpoonup e + f = z$$

and, by compactness $(Cz_n, z_n) \to (Cz, z)$.
But $\|z\|^2 = \|e\|^2 + \|f\|^2 \geq \|e\|^2 > 0$, and $(Cz, z) > 0$ by (H6).
Then, the lemma is proved. ▨

Proof of lemma 2. Let y be a non zero solution of (1), for $\lambda_1 < \lambda < \lambda_{K+1}$. We write $y = e + f$ with $e \in F_K$ and $f \in F_K^\perp$; then $(Ly, y) = (Le, e) + (Lf, f)$, and

$$\frac{(Le,e)}{\|e\|^2} \leq \sup_{Z} \frac{(Lz.z)}{\|z\|^2} = \frac{1}{\lambda_1},$$

$$\frac{(Lf,f)}{\|f\|^2} \leq \sup_{F_K^\perp} \frac{(Lz.z)}{\|z\|^2} = \frac{1}{\lambda_{K+1}}.$$

From (1), we obtain

$$\|y\|^2 = \|e\|^2 + \|f\|^2 \leq \lambda (Ly, y)$$

$$\leq \lambda \left\{ \frac{\|e\|^2}{\lambda_1} + \frac{\|f\|^2}{\lambda_{K+1}} \right\},$$

and the statement of lemma 2 results from that. ▨

Proof of lemma 3. Using $y \in K(\lambda)$ and $\frac{(Ly,y)}{\|y\|^2} \leq \frac{1}{\lambda_1}$, we deduce from the relation $\|y\|^2 - \lambda(Ly, y) + (Cy, y) = 0$ the following:

$$\left(1 - \frac{\lambda}{\lambda_1}\right) \|y\|^2 + \gamma(\lambda) \|y\|^4 \leq 0 .$$

▨

Galerkin process. Let $z_1, z_2, \ldots, z_m, \ldots$ be a basis in Z; $Z_m = \{z_1, \ldots, z_m\}$ the subspace generated by $(z_1, \ldots, z_m)$. Z_m is a finite dimension space, with dimension m.

For each m, we consider the following problem, in finite dimension:

(6) || Problem. To find $y_m \in Z_m$ such that
$y_m \neq 0$, $((I - \lambda L) y_m + C y_m, z) = 0 \quad \forall z \in Z_m$

It is clear that is equivalent to

$$(7)\quad \begin{cases} y_m \in Z_m, \quad y_m \neq 0, \\ y_m \text{ is a critical point of } \Phi \text{ on } Z_m, \end{cases}$$

where Φ is defined by

$$(8)\qquad \Phi(z) = \|z\|^2 - \lambda(Lz, z) + \frac{1}{2}(Cz, z)\ .$$

First, we resolve the previous problem, proving that (7) has a solution, at least; more precisely, we prove that:

Lemma 4. If m is sufficiently large, and if $\lambda > \lambda_1$, there exists $y_m \in Z_m$ such that

$$(9)\qquad \Phi(y_m) = \inf_{Z_m} \Phi(z)\ ;$$

in addition $y_m \neq 0$.

Corollary If $\lambda > \lambda_1$, (6) possesses at least one solution, if m is sufficiently large.

Proof of lemma 4. Because Φ is continuous, the statement results from the two properties:

(i) $\{z \in Z_m \,/\, \Phi(z) \leq 0\}$ is compact;

(ii) $\exists\, z \in Z_m$ such that $\Phi(z) < 0$.

(i) It is sufficient to prove that $\{z \in Z_m / \Phi(z) \leq 0\}$ is bounded (this set is closed because Φ is continuous), since Z_m is a finite dimension space. Now

$$\Phi(z) \geq \|z\|^2\left(1 - \frac{\lambda}{\lambda_1}\right) + \gamma_m \|z\|^4$$

because $(Cz, z) \geq \gamma_m \|z\|^4$ on Z_m. $\Phi(z) \leq 0$ implies

$$\gamma_m \|z\|^2 \leq \left(\frac{\lambda}{\lambda_1} - 1\right).$$

(ii) Let λ_1^m be the first characteristic value of $L_m = P_m L$ where P_m is the orthogonal projection on Z_m, defined on Z;

Then
$$\frac{1}{\lambda_1^m} = \sup_{Z_m} \frac{(L\mathfrak{z},\mathfrak{z})}{\|\mathfrak{z}\|^2} \leq \frac{1}{\lambda_1};$$
in addition $\lambda_1^m \to \lambda_1$ when $m \to \infty$.

Let be $\lambda > \lambda_1$; there exists N_0 such that
$$\lambda_1^m < \lambda \qquad \text{if } m > N_0.$$
Now, let m be fixed, greater than N_0, and
$$a_m \in Z_m \text{ such that } \frac{(La_m, a_m)}{\|a_m\|^2} = \frac{1}{\lambda_1^m} \left(< \frac{1}{\lambda_1} \right);$$
we have $\|a_m\|^2 - \lambda(La_m, a_m) < 0$.

Consider the function
$$\xi \mapsto \varphi(\xi) = \Phi(\xi a_m) = \left\{ \|a_m\|^2 - \lambda(La_m, a_m) \right\} \xi^2 + \frac{1}{2} \xi^4 (Ca_m, a_m)$$
Because the bracket is negative, this function takes some negative values; therefore (ii) is proved. ▨

We can complete the proof of part (ii) of previous lemma to obtain a result which will be profitable in the following.

(10) **Lemma 5.** For m sufficiently large,
$$\inf_{Z_m} \varphi(\mathfrak{z}) \leq -\frac{1}{4} \frac{1}{(Ce_1, e_1)} \left(\frac{\lambda}{\lambda_1} - 1\right)^2,$$
where e_1 is a normalized eigenvector of L, associated with the characteristic value λ_1.

Observe that the second member of (9) does not depend of m.

Proof. In the proof of the part (ii) of previous lemma, we can choose $a_m = P_m e_1$ (P_m = orthogonal projection on Z_m), because $a_m \to e_1$ when $m \to \infty$ (since $\mathfrak{z}_1, \ldots, \mathfrak{z}_m, \ldots$ is a basis in Z).

Now, we can observe that

$$M_m = \inf_{\xi \in \mathbb{R}} \varphi(\xi) = -\frac{1}{2} \frac{\|a_m\|^2 - \lambda (La_m, a_m)}{(Ca_m, a_m)}$$

and, when $m \to \infty$,

$$M_m \to -\frac{1}{2} \frac{[1 - \lambda (Le_1, e_1)]^2}{(Ce_1, e_1)} = -\frac{1}{2} \frac{\left(1 - \frac{\lambda}{\lambda_1}\right)^2}{(Ce_1, e_1)} .$$

Therefore, if m is sufficiently large,

$$\inf_{\xi \in \mathbb{R}} \varphi(\xi) \leq \frac{1}{2} M_\infty$$

(11) __Corollary__ If y_m is solution of (9), then

$$\|y_m\| \geq c_1 > 0$$

__Proof of corollary__. We have

$$\phi(z) \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|z\|^2 + (Cz, z) \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|z\|^2 .$$

By lemma 5, we have $\phi(y_m) \leq -c < 0$. From these two relations, we deduce

$$\|y_m\|^2 \geq \frac{c \lambda_1}{\lambda - \lambda_1} > 0 .$$

__Limiting process.__ By the previous study, (lemma 4), we know there exists a solution y_m for the problem (6), in finite dimension, when m is sufficiently large. Now, we must study the convergence of such a sequence when m tends to infinity.

More generally, we consider y_m any solution of

(12) $$y_m \in Z_m,\ y_m \neq 0,\ ((I - \lambda L) y_m + C y_m, z) = 0 \quad \forall z \in Z_m,$$

and no necessarily the solution exhibited by lemma 4-

We have the following

<u>Lemma 6.</u> (At least for a subsequence), $y_m \to y$ in Z_1 strong, and y is solution of the problem (1).

<u>Proof.</u> By lemma 3 (and remark which follows its) y_m is bounded in Z_1, and, at least for a subsequence

$$y_m \to y \quad \text{in } Z_1 \text{ weak}$$

Let P_m be the orthogonal projection on Z_{1m}; for each $z \in Z_1$, we can write (via (12))

$$((I - \lambda L) y_m + C y_m, P_m z) = 0 \qquad \forall z \in Z_1, \tag{13}$$

But $P_m z \to Pz$ strong, $\forall z \in Z$, and we have

$$\begin{cases} (y_m, P_m z) = (y_m, z) \to (y, z) \\ (C y_m, P_m z) \to (C y, z) \\ (L y_m, P_m z) \to (L y, z) \end{cases} \qquad \text{use (H.1) and (H.7)}$$

From (13), we deduce when $m \to \infty$:

$$((I - \lambda L) y + C y, z) = 0 \quad \forall z \in Z_1,$$

and y is solution for our problem.

Now, we choose $z = y_m$ in (13), and we have

$$\|y_m\|^2 = -(C y_m, y_m) + \lambda (L y_m, y_m)$$

from this, we obtain, via compactness of C and L:

$$\|y_m\|^2 \to -(C y, y) + \lambda (L y, y) = \|y\|^2$$

The two relations

$$\begin{cases} y_m \to y & \text{weak} \\ \|y_m\| \to \|y\| \end{cases}$$

give $y_m \to y$ in Z_1 strong ▨

<u>Corollary</u>. For all $\lambda > \lambda_1$, there exists a non trivial solution for the problem (1)

<u>Proof</u>. We use the previous lemma for the sequence γ_m exhibited by lemma 4. For this sequence, we have (via corollary of lemma 5)

$$\|\gamma_m\| \geq c > 0$$

Then $\|\gamma\| \geq c > 0$, because $\gamma_m \to \gamma$ strong. ▨

Then, the theorem 2 of section IX is proved.