INTERNATIONAL ATOMIC ENERGY AGENCY
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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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THEORY OF PLATES
Part XI - Bifurcation theory

Claude Do
University of Nantes (E.N.S.M.)
France

XI - Bifurcation theory.

1 - Bilateral situation.

We begin by studying the bifurcation phenomena in the bilateral case; for unilateral situations we will have to consider bilateral problems like comparison ones.

Actually, the problem is

(1) $$(I-\lambda L)y + Cy = 0 \quad ,$$

with assumptions of section IX (p. 5). We put

(2) μ is the first characteristic value of L $(\mu > 0)$.

The main result is the following

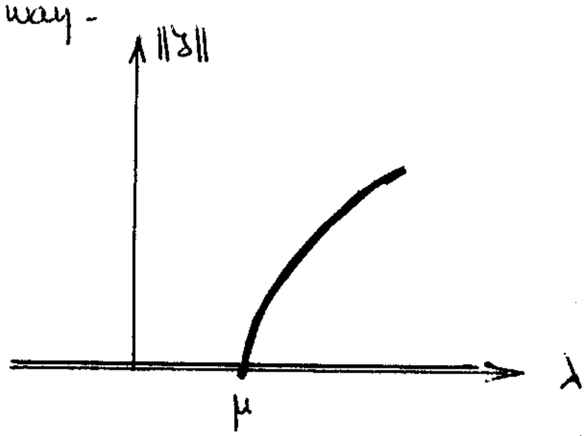
Theorem 1 (M.S. Berger & P. Fife).

μ is a bifurcation point from the trivial solution. More precisely:

Let (y_r, λ_r) the solution obtained via the theorem 1, § IX; then

$$\begin{cases} y_r \to 0 & \text{when } r \to \infty \quad \text{(already known)}; \\ \lambda_r \to \mu & \text{" " "} . \end{cases}$$

The following picture describes our results in a qualitative way.



Principle of the proof. By (1), we have

$$\frac{1}{\lambda} = \frac{(Ly,y)}{\|y\|^2 + (Cy,y)}$$

and, using the definition of y $(= y_r)$ we obtain:

(3) $$\frac{1}{\lambda} = \frac{1}{\|y\|^2 + (Cy,y)} \sup_{\partial A_r} (Lz,z)$$

Any way, we have

$$\frac{1}{\mu} = \sup_{Z} \frac{(Lz,z)}{\|z\|^2} = \frac{(Lu,u)}{\|u\|^2}$$

and u can be replaced by αu, $\alpha \in \mathbb{R}$; we put, for each $r > 0$:

(4) $$\partial \Sigma_r = \left\{ z \in Z \;\middle/\; \tfrac{1}{2}\|z\|^2 = r \right\} ;$$

then

(5) $$\frac{1}{\mu} = \frac{1}{2r} \sup_{\partial \Sigma_r} (Lz,z)$$

We have to compare $\frac{1}{\mu}$ and $\frac{1}{\lambda}$ (with $\lambda = \lambda_r$!), or

$$\sup_{\partial A_r} (Lz,z) \quad \text{and} \quad \sup_{\partial \Sigma_r} (Lz,z)$$

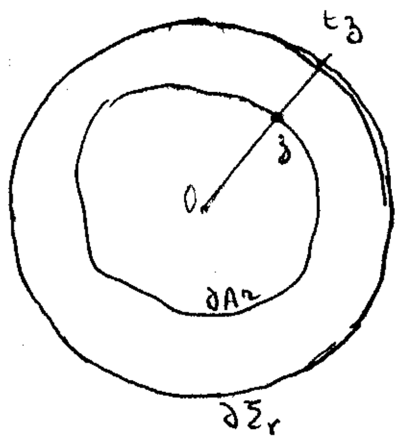
By a central projection from ∂A_r into $\partial \Sigma_r$, we can do the study on $\partial \Sigma_r$:

Let z be on ∂A_r and $tz \in \partial \Sigma_r$:

$$\begin{cases} \|z\|^2 + \frac{1}{2}(Cz,z) = 2r \\ t^2 \|z\|^2 = 2r \end{cases}$$

We deduce that t is given by

(6) $$\frac{1}{t^2} = 1 - \frac{1}{4r}(Cz,z)$$



Proof of theorem 1.

1°) We know that $\lambda(=\lambda_r) > \mu$ (by IX, thm 1); therefore $0 \leq \frac{1}{\mu} - \frac{1}{\lambda}$. Then, we have to estimate, when $r \to 0$.

$$\frac{1}{\mu} - \frac{1}{\lambda} = \sup_{Z_1} \frac{(Lz,z)}{\|z\|^2} - \frac{(Ly,y)}{\|y\|^2+(Cy,y)}.$$

From $y \in \partial A_r$, we obtain

$$\|y\|^2 + (Cy,y) = 2r + \frac{1}{2}(Cy,y)$$

and we write

$$\frac{1}{\|y\|^2+(Cy,y)} = \frac{1}{2r} \frac{1}{1+\frac{1}{4r}(Cy,y)}$$

$$\geqslant \frac{1}{2r}\left\{1 - \frac{1}{4r}(Cy,y)\right\}$$

(we use $\frac{1}{1+\xi} \geqslant 1-\xi$ if $\xi > -1$)

Then

$$-\frac{(Ly,y)}{\|y\|^2+(Cy,y)} \leqslant -\frac{1}{2r}(Ly,y) + \underline{\frac{1}{8r^2}(Ly,y)(Cy,y)}$$

For the underlined term, we have

$$(Ly,y) \leq c_1\|y\|^2 \leqslant c_2 r,$$

$$(Cy,y) \leq c_3\|y\|^4 \leqslant c_4 r^2 \qquad (\text{by } (H.8)),$$

and this term is bounded by cr and tends to zero with r

From this, it results that

$$-\frac{(Ly,y)}{\|y\|^2+(Cy,y)} \leqslant \frac{-1}{2r}(Ly,y) + O(r). \tag{7}$$

2°) Now, we write

$$\frac{1}{\mu} = \sup_{Z} \frac{(Lz,z)}{\|z\|^2}$$

(8) $$\frac{1}{\mu} = \sup_{\partial\Sigma_r} \frac{(Lz,z)}{\|z\|^2} = \frac{1}{2r} \sup_{\partial\Sigma_r} (Lz,z)$$

From (7) & (3), we have

(9) $$0 \leqslant \frac{1}{\mu} - \frac{1}{\lambda} \leqslant \frac{1}{2r}\left\{ \sup_{\partial\Sigma_r} (Lz,z) - \sup_{\partial A_r} (Lz,z) \right\} + O(r)$$

and we have to estimate the bracket.

3°) We put

(10) $$\begin{cases} M(r) = \sup_{\partial A_r} (Lz,z) \\ m(r) = \sup_{\partial\Sigma_r} (Lz,z) \end{cases}$$

By the central projection $z \mapsto tz$, where t is given by (6), we transform ∂A_r into $\partial\Sigma_r$; we can write

$$M(r) = \sup_{\partial\Sigma_r} \frac{1}{t^2}(Lz,z)$$

and the comparison between $M(r)$ and $m(r)$ is now more easy: we have

$$|M(r) - m(r)| = \left| \sup_{\partial\Sigma_r} \frac{1}{t^2}(Lz,z) - \sup_{\partial\Sigma_r} (Lz,z) \right|$$

$$\leqslant \sup_{\partial\Sigma_r} \left| \left(\frac{1}{t^2} - 1\right)(Lz,z) \right|$$

Using (6), we deduce from this, that

$$\left| M(r) - m(r) \right| \le \frac{1}{4r} \sup_{\partial \Sigma_r} (L\mathfrak{z},\mathfrak{z})(C\mathfrak{z},\mathfrak{z})$$

Because

$$(L\mathfrak{z},\mathfrak{z}) \le c_1 \|\mathfrak{z}\|^2 \le c_2 r$$

$$(C\mathfrak{z},\mathfrak{z}) \le c_3 \|\mathfrak{z}\|^4 \le c_4 r^2 ,$$

We deduce

$$\left| M(r) - m(r) \right| \le c\, r^2$$

From this last relation, and using (9) and (10), we deduce that

$$0 \le \frac{1}{\mu} - \frac{1}{\lambda} \le c\, r \longrightarrow 0 \quad \text{with } r$$

and the therem 1 is prooved.

$\blacksquare$

<u>Remark</u>. The linearized problem (from (1)) is

$$(I - \lambda L)\, \mathfrak{I} = 0 .$$

This problem cannot describe the buckling phenomenon. But we observe that we can find the bifurcation point μ, studying only this problem:

$\left\{ \begin{array}{l} \mu \text{ is the critical load ; its is fixed by} \\ \text{the linearized problem.} \end{array} \right.$

This remark is very general in the case of bilateral situations

| to see <u>B. Budiansky</u>, Theory of buckling and post-buckling behavior of elastic structures, Advance in applied Mechanics, 14,

Remark   Stability of solutions (bilateral case)

The energy functional is

$$E(z) = \frac{1}{2}\|z\|^2 - \frac{1}{2}\lambda(Lz,z) + \frac{1}{4}(Cz,z).$$

We have $E(0) = 0$.

Now, if y is a non trivial solution, then

$$\|y\|^2 - \lambda(Ly,y) + (Cy,y) = 0 \; ;$$

from this, we deduce that

$$E(y) = -\frac{1}{2}(Cy,y) < 0 = E(0)$$

Therefore, by virtue of the principle of least potential energy, the plate always buckles out of the plane when $\lambda > \lambda_1$.

From an energy point of view, a non trivial solution is more stable that the trivial one.

2. Unilateral situations. (Cl. Do).

The abstract problem (or functional problem) of buckling has been formulated like an eigenvalue problem for a variational inequality on an Hilbert space Z_1:

(11) $$((I-\lambda L)y + C(y), z-y) + \psi(z) - \psi(y) \geq 0 \quad \forall z \in Z$$

Existence of non trivial solutions has been proved with suitable assumptions. In the following, the same hypothesis (H1), ..., H 16) (§ IX, p5) are valid; in addition, we assume:

(12) $$\psi(tz) = t\,\psi(z) \qquad \forall t \geq 0, \ \forall z \in Z_1$$

(in place of (H. 14) for instance) and

(13) $$\begin{cases} \psi(z) = 0 \iff z \in Z_0, \text{ where} \\ Z_0 \text{ is a closed subspace of } Z_1. \end{cases}$$

We begin by giving the abstract results (thm 2 & 3); after, we are studying some applications for the plate theory, and specially for examples used in § V. Lastly, we prove theorems 2 and 3.

Abstract results are as following

First case : $Z_0 = \{0\}$.

Theorem 2 We assume $Z_0 = \{0\}$. Let $(y_n, \lambda_n, n \geq 0)$, $y_n \neq 0$ a sequence of solutions of (11), such that

$$y_n \to 0 \quad \text{in } Z_1.$$

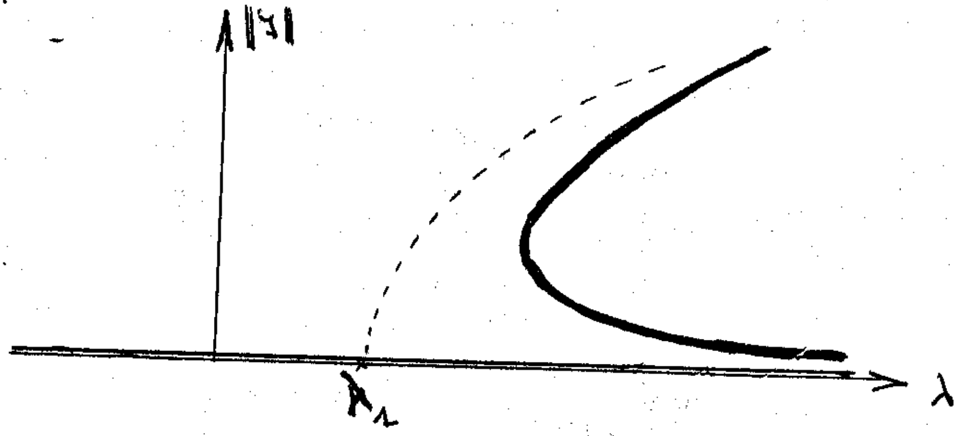
Then

$$\lambda_n \to \infty$$

In other words, only $\lambda = +\infty$ is a point of bifurcation from the trivial solution $\mathcal{Y} = 0$.

Remark. The previous result is valid for the solution $(\lambda_r, \mathcal{Y}_r, r > 0)$, when $r \to 0$, exhibited in the theorem 1 of § IX, because $\mathcal{Y}_r \to 0$ when $r \to 0$ -

This solution can be represented as on the diagram:



$\square$

Second case. $Z_0 \neq 0$; we assume that $Z_0 \neq Z_1$.

We introduce a new bilateral problem, associated with (11)

(14) $\quad \eta \in Z_0 \quad ((I - \sigma L)\eta + C\eta, \zeta) = 0 \quad \forall \zeta \in Z_0,$

or, it is the same

$\eta \in Z_0$, $\eta - \lambda PL\eta + PC\eta = 0$, where P is the orthogonal projection on Z_0. Then, the previous study can be used for this problem: we summarize the results:

Lemma 1. There exists a solution $(\sigma_r, \eta_r, r > 0)$, $\eta_r \neq 0$ for (14); in addition, we have (thm 1, § IX)

$$(L\eta_r, \eta_r) = \sup_{\partial A_r^0} (L\zeta, \zeta), \text{ with}$$

$$\partial A_r^0 = \{\zeta \in Z^0 \ / \ \tfrac{1}{2}\|\zeta\|^2 + \tfrac{1}{4}(C\zeta, \zeta) = r\}.$$

In addition, we have, if σ_0 is the first characteristic value of PL:

Observe that $\sigma_0 \geq \lambda_1$ (because $\frac{1}{\sigma_0} = \sup_{Z_0} \frac{(3.3)}{\|3\|^2} \leq \frac{1}{\lambda_1}$).

In addition, we have

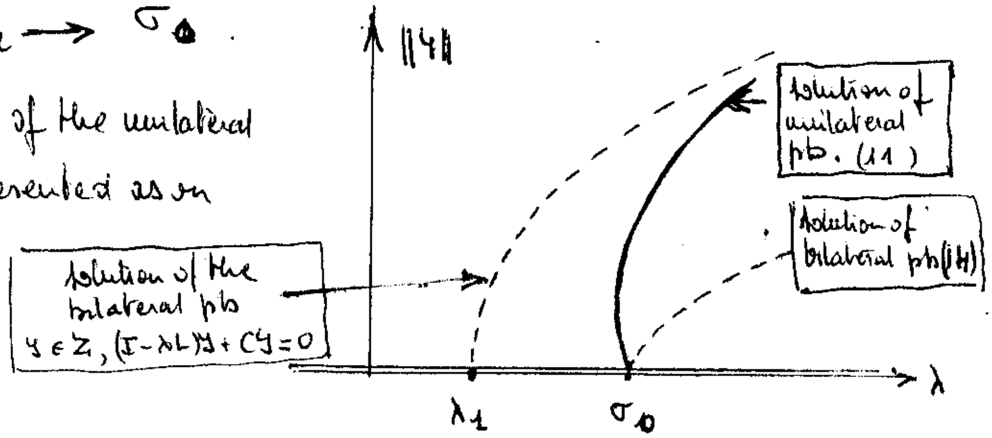
$$\lim_{r \to 0} \frac{(L\eta_r, \eta_r)}{2r} = \frac{1}{\sigma_0}$$

(this result is obtained in the proof of theorem 1 § XI: we have proved that $(M(r) - m(r))\frac{1}{2r} \to 0$, M and m defined by (10), and $\frac{1}{2r} m(r) \to \frac{1}{\sigma_0}$).

The bifurcation result about the unilateral problem is:

<u>Theorem 3.</u> If $Z_0 \neq \{0\}$, we have ($\gamma_r \to 0$ and) $\lambda_r \to \sigma_0$.

The solution (λ_r, y_r) of the unilateral problem can be represented as on the diagram.



Before the proof of theorems 2 and 3, we give some applications; for this, we recall the examples used in the beginning (section V).

First, observe that the assumption (12) is always satisfied in our examples.

In the <u>first problem</u> (section V, p. 8), the plate is clamped at its boundary. Here, we assume that

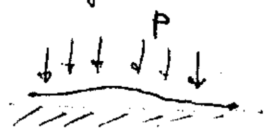
$j(\xi) \geq 0$, $j(0) = 0$ (gives H.10, H 11), and

(15) $j(\xi) > 0$, if $\xi \neq 0$

In the concrete example of this problem (plate placed on a rigid support), we have

$j(\xi) = \begin{cases} +\infty \text{ if } \xi < 0, \end{cases}$

and (15) gives $p < 0$ on Ω (we had already $p \leq 0$ by IX, (1)) : given exterior forces are everywhere < 0.



Then, in this first problem, we have (from (13)) $Z_0 = \{0\}$, and we can use the statement of theorem 2 : in particular, we can conclude that :

<u>The plate cannot bend by a quasistatic process.</u>

For the <u>second problem</u> (section V, p 8), we have $Z_1 = H^2(\Omega) \cap H^1_0(\Omega)$. We assume that

(16) $k(\xi) \geq 0$, $k(0) = 0$ (gives H 10 and H. 11) and
$k(\xi) > 0$ if $\xi \neq 0$.

From (13), we deduce

$$Z_0 = H^2_0(\Omega),$$

(because $\psi(z) = \begin{cases} \int_\Gamma k(\frac{\partial z}{\partial n})\,d\sigma & \text{if } k(\frac{\partial z}{\partial n}) \in L^1(\Gamma) \\ +\infty & \text{otherwise} \end{cases}$;

then $\psi(z) = 0$ is equivalent to $k(\frac{\partial z}{\partial n}) = 0$ a.e. on Γ and $\frac{\partial z}{\partial n} = 0$ on Γ)

Observe that (16) is true for the example illustrating this second problem (V, p 2, example 2). Here, we can use the previous theorem 3, which gives :

For this second problem, the solution y_r bifurcates from the trivial one at the point σ_0, which is <u>the critical load of the (bilateral) clamped plate</u>.

<u>In other words, concerning the bifurcation theory, the behavior of the plate is the same as if the plate was clamped.</u>

<u>The third problem</u> (section V, p 8) is very comparable ;
we assume that

(17) $\ell(\xi) \geq 0$, $\ell(0) = 0$
$\ell(\xi) > 0$ if $\xi \neq 0$ if $x \in \Gamma_1$

we have, from (13), $Z_0 = H^1_0(\Omega)$, and we can use the [illegible] theorem 2 to obtain :

Concerning the bifurcation theory, the behavior of the plate is the same as if the plate was simply supported.

Proof of theorem 2. We put $y^n = \|y^n\| z^n$, and we have $\|z^n\| = 1$ because $y^n \neq 0$. Choosing $z = 0$ in (11), we can write

$$\lambda^n \|y^n\|^2 (Lz^n, z^n) = \|y^n\|^2 + (Cy^n, y^n) + \psi(y^n)$$
$$\geq \|y^n\|^2 + \psi(y^n)$$

but $\psi(y^n) = \|y^n\| \psi(z^n)$, and we obtain:

(18) $$\lambda^n (Lz^n, z^n) \geq 1 + \frac{1}{\|y^n\|} \psi(z^n).$$

At least for a subsequence, we have $z^n \to z$ weak.

If $z \neq 0$, we have $\psi(z) > 0$ since Z_0 (defined by (13)) is $\{0\}$; and, via the lower semi continuity $\liminf \psi(z^n) \geq \psi(z) > 0$; using the compactness of L, then we deduce $\lambda^n \to +\infty$ ($(Lz,z) \neq 0$!)

At the contrary, if $z = 0$, then $(Lz^n, z^n) \to +\infty$ and $\lambda^n \geq \dfrac{1}{(Lz^n, z^n)} \to +\infty$. ▨

Proof of theorem 3. This proof consists to compare (when $r \to 0$) the behavior of y_r, solution of (11) (recall that y_r is furnished by theorem 1, section IX), and η_r, solution of (14) (and characterized by the previous lemma 1): we know the behavior of η_r by the first part of this section and the results are recalled in the statement of lemma 1.

1°) We have $\|y_r\|^2 \leq 2r$, because $y_r \in \partial A_r$. Any way, $\partial A_r^0 = \partial A_r \cap Z_0$ furnishes

(19) $$(Ly_r, y_r) \geq (L\eta_r, \eta_r)$$

Hence

$$\frac{(Ly_r, y_r)}{\|y_r\|^2} \geq \frac{(L\eta_r, \eta_r)}{2r}$$

and

2°) Choose $z = 2y_r$ in (11) and $z = \eta_r$ in (14), to obtain:

$$\lambda_r (Ly_r, y_r) \leq \|y_r\|^2 + (Cy_r, y_r) + \psi(y_r) \qquad \text{(because (12))}$$

$$\sigma_r (L\eta_r, \eta_r) = \|\eta_r\|^2 + (C\eta_r, \eta_r)$$

Writing y_r and η_r belong to ∂A_r, we deduce from this that

$$\lambda_r (Ly_r, y_r) \leq 2r + \frac{1}{2}(Cy_r, y_r) - \psi(y_r) \leq 2r + \frac{1}{2}(Cy_r, y_r)$$

$$\sigma_r (L\eta_r, \eta_r) = 2r + \frac{1}{2}(C\eta_r, \eta_r)$$

This last relation gives

$$2r \leq \sigma_r (L\eta_r, \eta_r)$$

which, reintroduced in the first furnishes:

$$\lambda_r (Ly_r, y_r) \leq \sigma_r (L\eta_r, \eta_r) + \frac{1}{2}(Cy_r, y_r)$$

it results, by (19), that (using (H.8), section IX, p 5)

$$\lambda_r \leq \sigma_r + c \, \frac{\|y_r\|^4}{(Ly_r, y_r)}$$

We deduce from this that

(21) $$\limsup \lambda_r \leq \sigma_0 ,$$

using (20) and the lemma 1.

3°) We put $y_r = \|y_r\| v_r$; $\|v_r\| = 1$ because $y_r \neq 0$, and, at least for a subsequence, $v_r \to v$ weak. Choosing $z = 0$ in (11), we have

(22) $$\lambda_r (Lv_r, v_r) \geq 1 + \frac{1}{\|y_r\|} \psi(v_r) \qquad \text{(by (12))}.$$

If $\psi(v) \neq 0$, then $v \neq 0$ and $\lambda_r \geq \dfrac{c}{\|y_r\|} \to +\infty$; that is impossible by (21).

If $v = 0$, $\lambda_r \geq \dfrac{1}{(Lv_r, v_r)} \to +\infty$, is impossible.

We can only have

$$\begin{cases} \psi(v) = 0 \\ v \neq 0 \end{cases}$$

r, by (13) :

$$v \in Z_0, \quad v \neq 0$$

and

$$\|v\| \leq 1 \qquad (\text{because } v \text{ is the weak limit of } v_r,\ \|v_r\| = 1)$$

Now, (22) implies

$$\frac{1}{\lambda_r} \leq (Lv_r, v_r)$$

and then

$$\limsup \frac{1}{\lambda_r} \leq (Lv, v) \leq \sup_{\substack{z \in Z_0 \\ \|z\| = 1}} (Lz, z) = \frac{1}{\sigma_0},$$

that is

$$\limsup \frac{1}{\lambda_r} \leq \frac{1}{\sigma_0}.$$

With (21), we deduce from this that

$$\lambda_r \to \sigma_0 \qquad \text{when } r \to 0$$

and the theorem 3 is proved. $\blacksquare$