INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23/57

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

VARIATIONAL INEQUALITIES

G. Stampacchia
Scuola Normale Superiore
Pisa, Italy

These are preliminary lecture notes intended for participants only. Copies are available outside the Publications Office (T-floor) or from room 112.

Stampacchia

— Reprinted from: *Theory and Applications of Monotone Operators.* Proceedings of a NATO Advanced Study Institute held in Venice, Italy, June 17-30, 1968.

VARIATIONAL INEQUALITIES (*)

GUIDO STAMPACCHIA, ROMA (ITALY)

PART I: — EXISTENCE OF SOLUTIONS.

§ 1. Introduction.

1. Before getting involved with the theoretical aspects of variational inequalities, it will be helpful to consider a few examples where such problems arise.

EXAMPLE 1: Let $f \in C^1$ with $f : [a, b] \to \mathbb{R}^1$. We wish to determine those points x_0 for which

$$f(x_0) = \min_{a \leq x \leq b} f(x).$$

Clearly there exists at least one such point x_0.

The following cases can occur.

If $a < x_0 < b$, then $f'(x_0) = 0$.

If $x_0 = a$, then $f'(x_0) \geq 0$.

If $x_0 = b$, then $f'(x_0) \leq 0$.

It is clear from this that for any such x_0 we have,

$$f'(x_0) \cdot (x - x_0) \geq 0 \text{ for all } x \text{ in } [a, b].$$

Such an inequality will be referred to as a variational inequality.

(*) The Author has used for the redaction some notes written by N. Eklund while he was lecturing at the University of Minneapolis during the spring quarter in 1968. The Author takes this opportunity for thanking the University of Minneapolis. He would like to thank also Prof. Aldo Ghizzetti for in

EXAMPLE 2: Let $\mathbb{K}$ be a closed, convex set in $\mathbb{R}^N$ and let

$$f : \mathbb{K} \to \mathbb{R}^1, \text{ with } f \in C^1(\mathbb{K}).$$

Let $x_0 \in \mathbb{K}$ be such that $f(x_0) = \underset{\mathbb{K}}{\text{Min}}\, f(x)$. Since $\mathbb{K}$ is convex we have for each $y \in \mathbb{K}$ that

$$\lambda x_0 + (1-\lambda) y \in \mathbb{K} \text{ for } 0 \leq \lambda \leq 1.$$

Define

$$F : [0,1] \to \mathbb{R}^1$$

by

$$F(\lambda) = f(\lambda x_0 + (1-\lambda)\, y).$$

Then $F(1) = \underset{\mathbb{K}}{\text{Min}}\, f$. It follows from example 1 that $F'(1)(\lambda - 1) \geq 0$ for all $0 \leq \lambda \leq 1$; but this is equivalent to $F'(1) \leq 0$. Therefore, for $x_0 \in \mathbb{K}$

$$[\text{grad } f(x_0)] \cdot (y - x_0) \geq 0 \text{ for all } y \in \mathbb{K}$$

becomes the variational inequality for this example.

Note that, if $\mathbb{K}$ is also bounded, the existence of such an x_0 follows immediately from the Weierstrass theorem.

EXAMPLE 3: Let $u \in C^1([a,b])$. We wish to consider expressions of the form $\int_a^b |u'(x)|^2\, dx$. Let us define

$$\mathbb{K} = \{u \in C^1([a,b]) \mid u(a) = u(b) = 0,\ h_1(x) \leq u(x) \leq h_2(x)\}.$$

Here h_1 and h_2 are two ~~a priori~~ given functions. Clearly we have to assume

$$h_1(a) \leq 0 \leq h_2(a)$$

and

$$h_1(b) \leq 0 \leq h_2(b).$$

We wish to look for

(1.1)
$$\underset{u \in \mathbb{K}}{\text{Min}} \int_a^b |u'(x)|^2\, dx.$$

Note that $\mathbb{K}$ is a convex set. Let $u_0 \in \mathbb{K}$ be a function where the minimum (1.1) is attained. Since $\mathbb{K}$ is convex, $u = \lambda u_0 + (1-\lambda)$ $v \in \mathbb{K}$ for all $v \in \mathbb{K}$ and for all λ with $0 \leq \lambda \leq 1$. Let us now define

$$F(\lambda) = \int_a^b |[\lambda u_0 + (1-\lambda) v]'|^2\, dx.$$

Then $F : [0,1] \to \mathbb{R}^1$ with $F(1) = \underset{0 \leq \lambda \leq 1}{\text{Min}} \int_a^b |u'(x)|^2\, dx$. $F(\lambda)$

It follows from example 1 that $F'(1)(\lambda - 1) \geq 0$. But this is equivalent to $F'(1) \leq 0$. Taking into account the definition of F we have

$$F(\lambda) = \int_a^b [\lambda u_0' + (1-\lambda) v']^2\, dx$$

and, therefore

$$F'(\lambda) = \int_a^b 2[\lambda u_0' + (1-\lambda) v'][u_0' - v']\, dx.$$

Therefore, our variational inequality becomes

$$2 \int_a^b u_0' (u_0' - v')\, dx \leq 0,$$

or

$$u_0 \in \mathbb{K}, \int_a^b u_0'(x)[v'(x) - u_0'(x)]\, dx \geq 0 \text{ for all } v \in \mathbb{K}.$$

EXAMPLE 4: Let Ω be an open set of $\mathbb{R}^N$ and let $u : \overline{\Omega} \to \mathbb{R}^1$ with $u \in C^1(\Omega)$.

We wish to look for

$$\underset{\mathbb{K}}{\text{Min}} \int_\Omega |\text{grad } u|^2\, dx.$$

Let

$$\mathbb{K} = \{u \in C^1(\Omega) \mid u(x) = 0 \text{ for } x \in \partial\Omega,\ \psi_1(x) \leq u(x) \leq \psi_2(x)\}$$

where $\psi_i \in C^1$ with $\psi_i : \overline{\Omega} \to \mathbb{R}^1$; $i = 1, 2$. As in example 3 let us define, for $0 \leq \lambda \leq 1$,

$$F(\lambda) = \int_\Omega |\text{grad } (\lambda u_0 + (1-\lambda) v)|^2\, dx$$

where u_0 is the minimizing element of $\mathbb{K}$ and $v \in \mathbb{K}$ is arbitrary. As in example 3 we get as our variational inequality for $u_0 \in \mathbb{K}$

$$\int_\Omega \operatorname{grad} u_0 \cdot \operatorname{grad}(v - u_0)\, dx \geq 0 \quad \text{for all} \quad v \in \mathbb{K}.$$

EXAMPLE 5: As in example 4, let $u : \overline{\Omega} \to \mathbb{R}^1$ with $u \in C^1(\Omega)$. This time we look for $\operatorname{Min}_{\mathbb{K}} \int_\Omega \sqrt{1 + |\operatorname{grad} u|^2}\, dx$. For this we define

$$\mathbb{K} = \{u \in C^1(\Omega) \mid u(x) = 0 \text{ on } \partial\Omega,\ \psi_1(x) \leq u(x) \leq \psi_2(x)\}$$

where ψ_i are a priori given with $\psi_i : \overline{\Omega} \to \mathbb{R}^1$; $i = 1, 2$.

As in example 4 we obtain the variational inequality for $u_0 \in \mathbb{K}$

$$\int_\Omega \sum_{i=1}^{N} \frac{u_{0x_i}(v - u_0)_{x_i}}{\sqrt{1 + |\operatorname{grad} u_0|^2}}\, dx \geq 0 \quad \text{for all } v \in \mathbb{K}.$$

EXAMPLE 6: *Projection on a convex set of a Hilbert space.*

Now let V be a real Hilbert space and let $\mathbb{K}$ be a closed, convex subset of V. Let $f \in V$. If $u_0 \in \mathbb{K}$ such that

$$\|u_0 - f\| = \operatorname{Min}_{\mathbb{K}} \|u - f\|,$$

we will say $u_0 = Pr_{\mathbb{K}} f$ and call u_0 the projection of f onto $\mathbb{K}$. Now clearly

$$\|u_0 - f\| \leq \|v - f\| \quad \text{for all } v \in \mathbb{K}.$$

Let us define

$$F(\lambda) = (\lambda u_0 + (1 - \lambda) v - f, \lambda u_0 + (1 - \lambda) v - f)$$

$$= \|\lambda u_0 + (1 - \lambda) v - f\|_V^2,$$

where $(\,,)$ denotes the inner product on V.

Then $F : [0, 1] \to \mathbb{R}^1$. From example 1 we have $F'(1) \leq 0$. Traslating this into our notation we get the variational inequality,

(1.2) $$u_0 \in \mathbb{K},\ (u_0, v - u_0) \geq (f, v - u_0) \quad \text{for all } v \in \mathbb{K}.$$

This can be equivalently written as $(u_0 - f, v - u_0) \geq 0$.

Now we claim that the variational inequality (1.2) implies $u_0 = Pr_{\mathbb{K}} f$. In fact, $0 \leq (u_0 - f, v - u_0) = (u_0 - f, v - f - (u_0 - f))$ if and only if $(u_0 - f, v - f) \geq (u_0 - f, u_0 - f)$.

But this implies $\|u_0 - f\|^2 \leq (u_0 - f, v - f) \leq \|u_0 - f\| \cdot \|v - f\|$. Therefore, $\|u_0 - f\| \leq \|v - f\|$ for all $v \in \mathbb{K}$. It follows, therefore, that $u_0 = Pr_{\mathbb{K}} f$ by the definition of projection. The existence of u_0 is known, but it will be shown in theorem 2.1, case a.

2. The variational inequality obtained above motivates the following situation:

Let V be a real Hilbert space.

Let $(\,,)$ denote the inner product on V.

Let $\langle\,,\rangle$ denote the pairing between V and V', this taken as $\langle f, v\rangle$ where $f \in V'$ and $v \in V$.

Let $f \in V'$ be given (fixed for the present).

Let $\mathbb{K}$ be a closed convex set in V.

Let $a(u, v)$ be a continuous bilinear form on V; that is, there is a constant C_0 such that

$$|a(u, v)| \leq C_0 \|u\| \|v\|.$$

We wish to solve.

PROBLEM 1: *Find* $u \in \mathbb{K}$ *such that*

(1.3) $$a(u, v - u) \geq \langle f, v - u\rangle \quad \textit{for all } v \in \mathbb{K}.$$

Let $\mathbb{K}_u$ defined by

$$\mathbb{K}_u = \{w = \varepsilon(v - u) \in V \mid \varepsilon > 0,\ v \in \mathbb{K}\}$$

where $v \in \mathbb{K}$. Then problem 1 is equivalent to

PROBLEM 1′: *Find* $u \in \mathbb{K}$ *such that*

(1.3′) $$a(u, w) \geq \langle f, w\rangle \quad \textit{for all } w \in \overline{\mathbb{K}}_u.$$

For, if $w \in \mathbb{K}_u$, then there exists a sequence $\{w_n\}$ with $w_n \in \mathbb{K}_u$ such that $w_n \to w$ and

$$a(u, w_n) = a(u, \varepsilon_n(v_n - u)) = \varepsilon_n a(u, v_n - u) \geq \varepsilon_n \langle f, v_n - u\rangle = \langle f, w_n\rangle$$

since $v_n \in \mathbb{K}$. Going to the limit

$$a(u, w) \geq \langle f, w \rangle.$$

Viceversa, if u satisfies (1.3), then for all $\varepsilon > 0$

$$a(u, \varepsilon(v - u)) \geq \langle f, \varepsilon(v - u) \rangle \quad \text{for all } v \in \mathbb{K}$$

i. e.: $a(u, w) \geq \langle f, w \rangle$ for all $w \in \mathbb{K}_u$ and therefore **(1.3′)** holds for all $w \in \overline{\mathbb{K}}_u$.

Note that if u is an interior point of $\mathbb{K}$, $\mathbb{K}_u \equiv V$ and (1.3) or (1.3′) becomes actually

(1.4) $$a(u, w) = \langle f, w \rangle \quad \text{for all } w \in V$$

since the inequality has to hold for w and $-w$.

This situation occurs when $\mathbb{K} \equiv V$. Thus for $\mathbb{K} \equiv V$, (1.4) holds and this problem has been solved by Lax-Milgram.

For convex $\mathbb{K}$ shown below we obtain the corresponding $\mathbb{K}_u$.

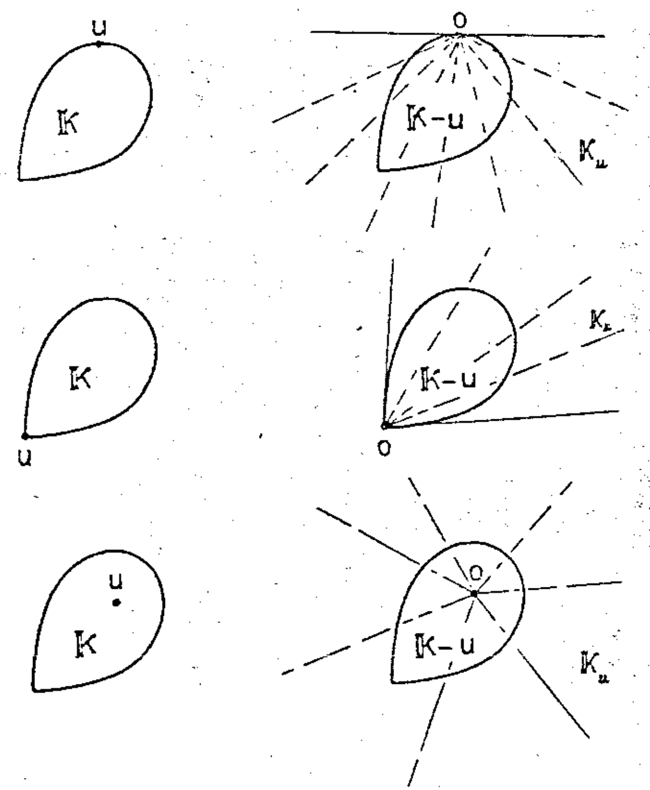

Fig 1

§ 2. Existence theorems for variational inequalities.

1. We are going to consider the existence of solutions of Problem 1 and of some related problems. Let us begin with Problem 1.

Let us now consider the special case where $a(u, v)$ is *coercive* on V; that is, there is an $\alpha > 0$ such that

(2.1) $$a(v, v) \geq \alpha \| v \|^2 \qquad \text{for all } v \in V.$$

In this situation there is at most one solution to the variational inequality of problem 1. For, suppose $f_1 . f_2 \in V'$ and let u_1, u_2 be two corresponding possible solutions of the variational inequality. Then, for u_1; $a(u_1, v - u_1) \geq \langle f_1, v - u_1 \rangle$ for $v \in \mathbb{K}$, and for u_2; $a(u_2, v - u_2) \geq \langle f_2, v - u_2 \rangle$ for $v \in \mathbb{K}$.

Setting $v = u_2$ in the expression for u_1 and $v = u_1$ in the expression for u_2 we get

$$a(u_1, u_2 - u_1) \geq \langle f_1, u_2 - u_1 \rangle;$$

$$-a(u_2, u_2 - u_1) = a(u_2, u_1 - u_2) \geq \langle f_2, u_1 - u_2 \rangle = -\langle f_2, u_2 - u_1 \rangle.$$

Adding we get

$$a(u_1 - u_2, u_2 - u_1) \geq \langle f_1 - f_2, u_2 - u_1 \rangle,$$

or

$$a(u_1 - u_2, u_1 - u_2) \leq \langle f_1 - f_2, u_1 - u_2 \rangle.$$

Using coerciveness, we get

$$\alpha \| u_1 - u_2 \|_V^2 \leq \| f_1 - f_2 \|_{V'} \cdot \| u_1 - u_2 \|_V,$$

or,

(2.2) $$\| u_1 - u_2 \|_V \leq \frac{1}{\alpha} \| f_1 - f_2 \|_{V'}.$$

This gives the uniqueness of the solution. Moreover, it also says that the map $f \to u$ from V' to V is a continuous (generally nonlinear) map.

THEOREM 2.1 *Let* $a(u, v)$ *be a bilinear form on the real Hilbert space* V *and assume that* $a(u, v)$ *is coercive on* V. *Let* $\mathbb{K}$ *be a closed convex set of* V *and let* $f \in V'$. *Then there exists a solution to the variational inequality*

(2.3) $$u \in \mathbb{K}, \ a(u, v - u) \geq \langle f, v - u \rangle \qquad \textit{for all } v \in \mathbb{K}.$$

Such a solution is unique. The map $f \to u$ *satisfies* (2.2).

Let us define the canonical isomorphism

$$\Lambda : V' \to V$$

by

$$\langle f, v \rangle = (\!(\Lambda f, v)\!) \qquad \text{for all } v \in V.$$

It is clear that

$$\| \Lambda \|_{\mathcal{L}(V', V)} = \| \Lambda^{-1} \|_{\mathcal{L}(V, V')} = 1.$$

Note that (2.3) can be written:

(2.3′) $\quad u_0 \in \mathbb{K}, \; a(u_0, v - u_0) \geq (\!(\Lambda f, v - u_0)\!)$ for all $v \in \mathbb{K}$.

We shall give two different proofs of theorem 2.1.

CASE *a*. Suppose $a(u, v) = (u, v)$. Then, to show that u_0 exists, is to show that $u_0 \equiv Pr_{\mathbb{K}}(\Lambda f)$ exists. Set $\Lambda f = \bar{f}$. But this means there is a u_0 for which

$$\| u_0 - \bar{f} \| = \min_{u \in \mathbb{K}} \| u - \bar{f} \|.$$

Let $d = \min_{u \in \mathbb{K}} \| u - \bar{f} \|$. We wish to show there is a $u_0 \in \mathbb{K}$ for which $\| u_0 - \bar{f} \| = d$. Clearly there is a sequence $\{u_n\} \subset \mathbb{K}$ for which $d \leq \| u_n - f \| \leq d + \frac{1}{n}$.

We wish to show that such a sequence is actually a Cauchy sequence. But, by the parallelogram law,

$$\frac{1}{2} \| u_n - u_m \|^2 = \| u_n - \bar{f} \|^2 + \| u_m - \bar{f} \|^2 - 2 \left\| \frac{u_n + u_m}{2} - \bar{f} \right\|^2$$

$$\leq \left(d + \frac{1}{n} \right)^2 + \left(d + \frac{1}{m} \right)^2 - 2d^2 = \frac{2d}{n} + \frac{2d}{m} + \frac{1}{n^2} + \frac{1}{m^2}$$

or $\| u_n - u_m \|^2 \leq 4d \left(\frac{1}{n} + \frac{1}{m} \right) + 2 \left(\frac{1}{n^2} + \frac{1}{m^2} \right)$.

Hence $\{u_n\}$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K}$ is closed, there is a $u_0 \in \mathbb{K}$ such that $u_n \to u_0$. Therefore, such a u_0 exists.

Taking account of the reasoning at the beginning of § 2, we show that the projection on a convex set does **not** increase distances.

CASE *b*. Suppose $a(u, v)$ is symmetric. Then clearly $((u, v)) \equiv a(u, v)$ defines an inner-product on V.

Moreover if $|||\cdot|||$ denotes the norm of this Hilbert space H under the inner-product $((\cdot,\cdot))$, then, by coercivity, we have

$$||| v |||^2 = ((v, v)) = a(v, v) \geq \alpha \| v \|^2.$$

Moreover, since $|a(u, v)| \leq C_0 \| u \| \| v \|$, we have

$$||| v |||^2 = ((v, v)) = a(v, v) \leq C \| v \|^2.$$

Therefore, H and V are isomorphic and we can use the argument of « case *a* » to get the existence of a u_0 for which

$$||| u_0 - \Lambda f ||| = \min_{u \in \mathbb{K}} ||| u - \Lambda f |||.$$

Hence, from example 6 of § 1, we have shown that there is an $u_0 \in \mathbb{K}$ for which

$$a(u_0, v - u_0) \equiv ((u_0, v - u_0)) \geq (\!(\Lambda f, v - u_0)\!), \text{ for all } v \in \mathbb{K}.$$

CASE *c*. Suppose $a(u, v)$ is not symmetric; that is,

$$a(u, v) \neq a(v, u).$$

Then define

$$a_0(u, v) = \frac{1}{2} [a(u, v) + a(v, u)],$$

$$\beta(u, v) = \frac{1}{2} [a(u, v) - a(v, u)],$$

to be the symmetric and anti-symmetric parts of $a(u, v)$.

Then

$$a(u, v) = a_0(u, v) + \beta(u, v).$$

For t, $0 \leq t \leq 1$, let us define

$$a_t(u, v) = a_0(u, v) + t\beta(u, v).$$

Then, for each t, a_t is a bilinear form which is continuous and coercive, with a constant indipendent of t; in fact,

$$| a_t(v, v) | \geq \alpha \| v \|^2$$

since $\beta(v, v) = 0$ and $a_0(v, v) = a(v, v)$.

Consider the variational inequality

$(2.3_t) \quad a_0(u, v - u) \geq \langle f, v - u \rangle - t\beta(u, v - u)$ for all $v \in \mathbb{K}$.

That is, (2.3_t) is the variational inequality

$$a_t(u, v - u) \geq \langle f, v - u \rangle.$$

Note that for $t = 0$ we have that $a_0(u, v)$ is symmetric and, therefore, the solution u of the variational inequality exists by case *b*.

Now let $w \in V$ be an arbitrarily fixed element and form the variational inequality $u \in \mathbb{K}$; $a_0(u, v-u) \geq \langle f, v-u \rangle - t\beta(w, v-u)$, for all $v \in \mathbb{K}$.

Denoting $\langle f, \cdot \rangle - t\beta(w, \cdot)$ by F we see that $F \in V'$ and we wish to consider

$$u \in \mathbb{K},\ a_0(u, v-u) \geq \langle F, v-u \rangle \qquad \text{for all } v \in \mathbb{K}.$$

But since a_0 is symmetric, we know that such a solution u exists from case *b*. Let T denote the correspondence between the given element w of V with the solution u.

Then

$$u = Tw : V \to \mathbb{K}.$$

Let w_1, and w_2 be two elements of V and u_1, u_2 be their corresponding solutions. Since we keep f fixed we have, as in a previous calculation,

$$a_0(u_1, u_2 - u_1) \geq \langle f, u_2 - u_1 \rangle - t\beta(w_1, u_2 - u_1)$$

$$a_0(u_2, u_1 - u_2) \geq \langle f, u_1 - u_2 \rangle - t\beta(w_2, u_1 - u_2).$$

Adding we get,

$$-a_0(u_1 - u_2, u_1 - u_2) \geq t\beta(w_1 - w_2, u_1 - u_2).$$

Therefore, using the coerciveness and the bilinearity of $a(u, v)$,

$$\alpha \| u_1 - u_2 \|^2 \leq t \mid \beta(w_1 - w_2, u_1 - u_2) \mid$$

$$\leq t \cdot c_0 \| w_1 - w_2 \| \| u_1 - u_2 \|.$$

Therefore,

$$\| u_1 - u_2 \| \leq \frac{t \cdot c_0}{\alpha} \| w_1 - w_2 \|.$$

Let t_0 be fixed with $t_0 < \frac{\alpha}{c_0}$. Then the map $u = Tw$ is, for $0 \leq t \leq t_0$, a contraction and, hence, there is a unique fixed point u corresponding to T; that is, $w = u$. Moreover, since $t_0 < \frac{\alpha}{c_0}$, the variational inequality has a solution for $0 \leq t \leq t_0$. Now, for t in $t_0 \leq t \leq 2t_0$ we can carry out the same calculation for the variational inequality,

$$u \in \mathbb{K},\ a_{t_0}(u, v-u) \geq \langle f, v-u \rangle - (t - t_0)\beta(w, v-u), \quad \text{for all } v \in \mathbb{K}.$$

As t did not enter into the calculations other than by being carried along, we obtain

$$\| u_1 - u_2 \| \leq \frac{(t - t_0) c_0}{\alpha} \| w_1 - w_2 \|$$

with $t_0 \leq t \leq 2t_0$. We see that if $0 \leq t - t_0 \leq t_0$.

Then the map: $w \to u$ from V into $\mathbb{K}$ is a contraction and therefore there is a unique fixed point.

Since we can step off an interval of length t_0 in each step, it takes only finitely many such steps to cover all the interval $0 \leq t \leq 1$. Thus, the variational inequality (2.3_t) has a unique solution u for all t, $0 \leq t \leq 1$. Hence, we have a unique solution u of

$$u \in \mathbb{K},\quad a_0(u, v-u) \geq \langle f, v-u \rangle - \beta(u, v-u),$$

or

$$u \in \mathbb{K},\quad a(u, v-u) \geq \langle f, v-u \rangle \qquad \text{for all } v \in \mathbb{K}.$$

This completes the first proof of theorem 1.

Here is a second proof of theorem 2.1, which differes only in the case *c*. First of all, we prove the following lemma,

LEMMA 2.1. *Let ϱ be such that $0 < \varrho < \frac{2\alpha}{c_0^2}$ where $c_0 = \| A \|_{\mathcal{L}(V,V')}$ and $A \in \mathcal{L}(V, V')$ is defined by $a(u, v) = \langle Au, v \rangle$. Then there is a ϑ, $0 < \vartheta < 1$, such that*

$$|(u, v) - \varrho\, a(u, v)| \leq \vartheta \| u \| \| v \|.$$

PROOF: As before we let $\Lambda : V' \to V$ denote the canonical isomorphism. Then $\| \Lambda \|_{\mathcal{L}(V',V)} = 1$. Therefore,

$$(u, v) - \varrho\, a(u, v) = (u, v) - \varrho \langle Au, v \rangle = (u, v) - \varrho (\Lambda A u, v)$$

implies

$$|(u, v) - \varrho\, a(u, v)| \leq \| u - \varrho \Lambda A u \| \| v \|.$$

But

$$\| u - \varrho \Lambda A u \|^2 = \| u \|^2 + \varrho^2 \| \Lambda A u \|^2 - 2\varrho (\Lambda A u, u)$$

$$\leq \| u \|^2 + \varrho^2 c_0^2 \| u \|^2 - 2\varrho\, a(u, u)$$

$$\leq \| u \|^2 [1 + \varrho^2 c_0^2 - 2\alpha\varrho].$$

Let $\vartheta^2 = 1 + \varrho^2 c_0^2 - 2\alpha\varrho$. Then, since $0 < \varrho < \frac{2\alpha}{c_0^2}$, we have, $\vartheta > 0$ and $\vartheta^2 < 1$.

Thus, $|(u, v) - \varrho\, a(u, v)| \leq \vartheta \|u\| \cdot \|v\|$ and the lemma is proved. For each fixed $w \in V$, define $\Phi(w) \in V'$ by

$$\langle \Phi(w), v \rangle = (w, v) - \varrho\, a(w, v) + \varrho \langle f, v \rangle .$$

Then, if $w_1, w_2 \in V$, we have

$$|\langle \Phi(w_1) - \Phi(w_2), v \rangle| = |(w_1 - w_2, v) - \varrho\, a(w_2 - w_2, v)|$$

$$\leq \vartheta \|w_1 - w_2\| \cdot \|v\|$$

for $0 < \varrho < \frac{2\alpha}{c_0^2}$ where $0 < \vartheta < 1$, by the previous lemma. Therefore,

$$\|\Phi(w_1) - \Phi(w_2)\|_{V'} \leq \vartheta \|w_1 - w_2\|_V .$$

Now for each $w \in V$ fixed, there is a unique $u \in \mathbb{K}$ such that

$$(u, v - u) \geq \langle \Phi(w), v - u \rangle \qquad \text{for all } v \in \mathbb{K}.$$

Let $T: V \to \mathbb{K}$ be defined by the above relation with $u = Tw$. From the case a we have that $u = Pr_{\mathbb{K}} \Lambda\Phi(w)$.

Since the projection map does not increase distances, we have

$$\|Tw_1 - Tw_2\|_V = \|u_1 - u_2\|_V = \|Pr_{\mathbb{K}} \Lambda\Phi w_1 - Pr_{\mathbb{K}} \Lambda\Phi w_2\|_V$$

$$\leq \|\Lambda\Phi w_1 - \Lambda\Phi w_2\|_V$$

$$\leq \|\Phi(w_1) - \Phi(w_2)\|_{V'}$$

$$\leq \vartheta \|w_1 - w_2\|_V .$$

Since $\|\Lambda\| = 1$. Therefore, T is a contraction. Hence, there is a unique fixed element $u = Tu$. Therefore, we have shown there is a unique u such that

$$u \in \mathbb{K}, (u, v - u) \leq \langle \Phi(u), v - u \rangle \qquad \text{for all } v \in \mathbb{K}.$$

Using our definition of Φ, we have

$$(u, v - u) \geq (u, v - u) - \varrho a(u, v - u) + \varrho \langle f, v - u \rangle$$

for all $v \in \mathbb{K}$. That is,

$$a(u, v - u) \geq \langle f, v - u \rangle \qquad \text{for all } v \in \mathbb{K}.$$

Let us now consider the special case where $V \equiv \mathbb{R}^N$. Then $V' \equiv \mathbb{R}_N$. Defining, as before, A by

$$a(u, v) \equiv \langle Au, v \rangle,$$

we can now say that for a given f we are looking for a vector $u \in \mathbb{K}$ for which $\langle Au, v - u \rangle \geq \langle f, v - u \rangle$, for all $v \in \mathbb{K}$.

This can be rewritten as

$$u \in \mathbb{K}, \langle Au - f, v - u \rangle \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

2. Therefore, we can generalize the question to the following: Let B be a continuous map from $\mathbb{K}^X$ to $\mathbb{R}^N$. We wish to exhibit a vector $u \in \mathbb{K} \subset \mathbb{R}^N$ for which $(Bu, v - u) \geq 0$ for all $v \in \mathbb{K}$.

In the special case where, for $F \in C^1$, we have B given by

$$B(u) = \operatorname{grad} F(u),$$

that is the vector field B is conservative, we have already solved this problem, when $\mathbb{K}$ is bounded (see example 2 of § 1).

In fact, for $\mathbb{K}$ a bounded, closed, convex set in $\mathbb{R}^N$, we showed that any vector u_0 minimizing F in $\mathbb{K}$ must satisfy

$$\operatorname{grad} F(u_0) \cdot (v - u_0) \geq 0 \qquad \text{for all } v \in \mathbb{K},$$

or, in our new notation,

$$B(u_0) \cdot (v - u_0) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

Let us now consider the general case where B is a continuous map of $\mathbb{K}$ into $\mathbb{R}^N$.

THEOREM 2.2. *Let $\mathbb{K}$ be a bounded, closed, convex set in $\mathbb{R}^N$. Let $B: \mathbb{K} \to \mathbb{R}^N$ be a continuous map. Then there is a $u_0 \in \mathbb{K}$ such that*

$$(2.4) \qquad (B u_0, v - u_0) \geq 0 \qquad \textit{for all } v \in \mathbb{K}.$$

Note that geometrically, if $u_0 \in \partial \mathbb{K}$, then this means that Bu_0 points interior to $\mathbb{K}$ from u_0:

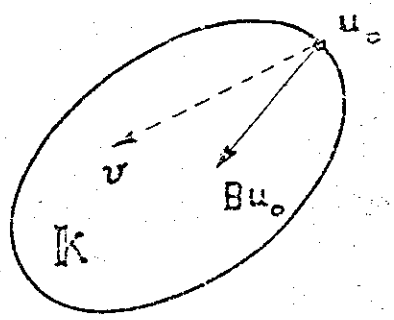

Fig. 2

If u_0 is an interior point of $\mathbb{K}$, reasoning as problem 1 of § 1, it follows that $Bu_0 = 0$.

PROOF OF THEOREM 2.2. We will show, equivalently, that

$$(u_0, v - u_0) \geq (u_0 - Bu_0, v - u_0) \qquad \text{for all } v \in \mathbb{K}.$$

For each $w \in \mathbb{K}$, there is a unique solution $u_0 \in \mathbb{K}$ of

$$(u_0, v - u_0) \geq (w - Bw, v - u_0) \qquad \text{for all } v \in \mathbb{K}$$

(example 6 of § 1).

Let us write $u_0 = Tw$. We have already shown that

$$u_0 = Pr_{\mathbb{K}}(w - Bw).$$

Now $I - B: \mathbb{K} \to \mathbb{R}^N$ is continuous and $Pr_{\mathbb{K}}$ is continuous from $\mathbb{R}^N$ to $\mathbb{K}$. Therefore, T is a continuous map of $\mathbb{K}$ into $\mathbb{K}$.

Since $\mathbb{K}$ is compact, the Brouwer fixed point theorem says there is a $u_0 \in \mathbb{K}$ such that $u_0 = Tu_0$; that is,

$$(u_0, v - u_0) \geq (u_0 - Bu_0, v - u_0) \qquad \text{for all } v \in \mathbb{K}.$$

We now wish to determine when in the previous theorem, such a u_0 is unique. Suppose that two solutions u_1 and u_2 exist. Then

$$(Bu_1, v - u_1) \geq 0 \qquad \text{for all } v \in \mathbb{K} \quad (\text{in particular, } v = u_2)$$

$$(Bu_2, v - u_2) \geq 0 \qquad \text{for all } v \in \mathbb{K} \quad (\text{in particular, } v = u_1).$$

Adding in the particular cases, we have

$$(Bu_1 - Bu_2, u_1 - u_2) \leq 0.$$

Therefore, if we assume

$$(Bu_1 - Bu_2, u_1 - u_2) \geq 0 \quad \text{for all } u_1, u_2 \in \mathbb{K},$$

then a unique solution exists if equality occurs whenever $u_1 = u_2$. This motivates the following definition:

DEFINITION 2.1. The map $B: \mathbb{K} \to \mathbb{R}^N$ will be called *monotone* if

$$(Bu - Bv, u - v) \geq 0 \quad \text{for all } u, v \in \mathbb{K}.$$

The map B will be called *strictly monotone* if

$$(Bu - Bv, u - v) = 0 \quad \text{if and only if } u = v.$$

3. We next wish to exstend these considerations from $\mathbb{R}^N$ to a general Banach space X. Consider the following assumptions:

i) Let X be a reflexive Banach space and X' be its dual.
ii) Let $(\cdot, \cdot)$ denote the pairing between X and X'.
iii) Let $\mathbb{K}$ be a closed convex set in X.

DEFINITION 2.1'. The map $A: X \supset D(A) \to X'$ is called *monotone* if

$$(Au - Av, u - v) \geq 0 \quad \text{for all } u, v \in D(A).$$

The map A will be called *strictly monotone* if moreover $(A(u) - A(v), u - v) = 0$ if and only if $u = v$.

DEFINITION 2.2. The map A whose domain $D(A)$ is convex is called *hemicontinuous* if, for all $u, v \in D(A)$ the function

$$t \in [0, 1] \to (A(tu + (1 - t)v, u - v) \text{ is continuous.}$$

We shall prove the following theorems.

THEOREM 2.3: *Assume* i)-iii). *Let* $A: X \to X'$ *be such that* A *is monotone and hemicontinuous. Then there is a* $u_0 \in \mathbb{K}$ *such that*

$$(Au_0, v - u_0) \geq 0 \quad \textit{for all } v \in \mathbb{K}.$$

THEOREM 2.3': *Assume* i)-iii). *Let* $A: \mathbb{K} \to X'$ *be monotone such that* A *is continuous on finite dimentional subspaces of* $\mathbb{K}$. *Then there is a* $u_0 \in \mathbb{K}$ *such that*

$$(Au_0, v - u_0) \geq 0 \quad \textit{for all } v \in \mathbb{K}.$$

Note that, if A is strictly monotone, u_0 is clearly unique. Moreover, we will show that *the set of all solutions of theorem* 2.3 (*or* 2.3′) *forms a closed convex subset of* $\mathbb{K}$. In the statement of theorem 2.3 we need only assume that the domain of A, $D(A) \supset \mathbb{K}$ where $D(A)$ is open.

We first prove the following lemma:

LEMMA 2.2. *Let* A *be a monotone hemicontinuous map from* $\mathbb{K}$ *to* X', *then* $u_0 \in \mathbb{K}$ *satisfy*

1 $$(Au_0, v - u_0) \geq 0 \quad \textit{for all } v \in \mathbb{K}$$

if and only if

2 $$(Av, v - u_0) \geq 0 \quad \textit{for all } v \in \mathbb{K}.$$

PROOF: **1 ⟹ 2.** By monotonicity we have

$$0 \leq (Av - Au_0, v - u_0) = (Av, v - u_0) - (Au_0, v - u_0).$$

Thus,

$$0 \leq (Au_0, v - u_0) \leq (Av, v - u_0) \quad \text{for all } v \in \mathbb{K}.$$

2 ⟹ 1. Let $w \in \mathbb{K}$ and define

$$v = tw + (1 - t)u_0 = u_0 + t(w - u_0) \quad \text{for } 0 \leq t \leq 1.$$

$\mathbb{K}$ being convex, $v \in \mathbb{K}$. Therefore, for $t > 0$ by 2

$$(A(u_0 + t(w - u_0)),\ t(w - u_0)) \geq 0,$$

or

$$(A(u_0 + t(w - u_0)),\ w - u_0) \geq 0.$$

Letting $t \to 0$ we get from the hemicontinuity of A that $(Au_0, w - u_0) \geq 0$. Since $w \in \mathbb{K}$ was arbitrary, this completes the proof.

We now proof theorem 2.3′.

Let M be any, fixed for the present, finite dimentional subspace of X with, say, $\dim M = m < \infty$. Assume, without loss of generality, $0 \in \mathbb{K}$. Let us define

$$j : M \to X \quad \text{to be the injection map,}$$

$$j^* : X' \to M' \quad \text{to be the map dual to } j.$$

Since M is finite dimentional, M' is isomorphic to M. Since the map $j^* A j$ is continuous, there is a solution $y_M \in \mathbb{K}_M$ such that

$$(Ay_M, z - y_M) \equiv (j^* A j\, y_M, z - y_M) \geq 0 \quad \text{for all } z \in \mathbb{K}_M.$$

Therefore, by lemma 2.2

$$(Az, z - y_M) \geq 0 \quad \text{for all } z \in \mathbb{K}_M.$$

In view of this, for each $v \in \mathbb{K}$ define

$$S(v) \equiv \{u \in \mathbb{K} \mid (Av, v - u) \geq 0\}.$$

Clearly, for each $v \in \mathbb{K}$, $S(v)$ is weakly closed. Moreover, since $\mathbb{K}$ is bounded, $\mathbb{K}$ is weakly compact. Since $\bigcap_{v \in \mathbb{K}} S(v)$ is weakly closed in $\mathbb{K}$, is weakly compact. Therefore, in order to show $\bigcap_{v \in \mathbb{K}} S(v) \neq \Phi$, we need only show that for each finite collection

$$\{v_1, \dots, v_m\} \subset \mathbb{K}$$

the set

$$S(v_1) \cap S(v_2) \cap \dots \cap S(v_m) \neq \Phi.$$

Let M be a finite dimentional space spanned by $\{v_1, v_2, \dots, v_m\}$ such that $\mathbb{K}_M = \mathbb{K} \cap M$ as $\mathbb{K}_M$ has been defined. By the finite dimentional case we know that there is an element $y_M \in \mathbb{K}_M$ such that

$$(Ay_M z - y_M) \geq 0, \quad \text{or} \quad (Az, z - y_M) \geq 0, \quad \text{for all } z \in \mathbb{K}_M.$$

Then, in particular,

$$(Av_i, v_i - y_M) \geq 0 \quad \text{for } i = 1, 2, \dots, m.$$

Therefore, $y_M \in S(v_1) \cap \dots \cap S(v_m)$ and, since M was arbitrary, we now have by the finite intersection property that there is a $u_0 \in \mathbb{K}$ such that

$$u_0 \in \bigcap_{v \in \mathbb{K}} S(v).$$

Therefore, $(Av, v - u_0) \geq 0$ for all $v \in \mathbb{K}$.

It follows from the lemma 2.2 that

It follows from the lemma 2.2 that the set of all solutions $\{v\}$ for which we have

$$(Au, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

forms a closed convex set. For, suppose u_1 and u_2 are any two solutions in $\mathbb{K}$. Then, by the lemma,

$$(Av, v - u_1) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

and

$$(Av, v - u_2) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

Multiplying the first inequality by λ and the second inequality by $1 - \lambda$, for $0 \leq \lambda \leq 1$, we get

$$0 \leq \lambda (Av, v - u_1) + (1 - \lambda)(Av, v - u_2)$$
$$= (Av, \lambda v - \lambda u_1) + (Av, (1 - \lambda) v - (1 - \lambda) u_2)$$
$$= (Av, \lambda v - \lambda u_1 + (1 - \lambda) v - (1 - \lambda) u_2)$$
$$= (Av, v - [\lambda u_1 + (1 - \lambda) u_2]) \qquad \text{for all } v \in \mathbb{K}.$$

Therefore, using the lemma 2.2 again, we have that $\lambda u_1 + (1 - \lambda) u_2$ is a solution. Hence, the set of all solutions forms a convex subset of $\mathbb{K}$. It is obviously closed.

THEOREM 2.4: *Let A be a monotone operator with domain X. Then, if A is hemicontinuous, A is continuous on finite dimentional subspaces of X.*

Note: Since we are concerned only with finite dimentional subspaces, we need only consider the case where X is finite dimentional.

PROOF: *a*) To show: A maps bounded sets into bounded sets. Suppose this is not true. Then there is a sequence $\{v_n\}$ such that $v_n \to v$ and for which $\| Av_n \| \to +\infty$. Since A is monotone, we have

$$(Av_n - Au, v_n - u) \geq 0 \qquad \text{for all } u \in X.$$

Setting $y_n = \dfrac{Av_n}{\| Av_n \|}$, then we have

$$\left(y_n - \frac{Au}{\| Av_n \|}, v_n - u\right) \geq 0 \qquad \text{for all } u \in X.$$

Since X is finite dimentional and since: $\| y_n \| = 1$, there is a subsequence $\{y_{n'}\}$ such that $y_{n'} \to y$. Clearly we have $\| y \| = 1$. However, since $\| Av_n \| \to +\infty$ and $v_n \to v$, we have that

$$(y, v - u) \geq 0 \qquad \text{for all } u \in X.$$

Since this true for all $u \in X$, we have that it is true for $u = v \pm w$ for any $w \in X$. But this implies $(y, w) = 0$ for all $w \in X$. Therefore, $y = 0$, contradicting the fact that $\| y \| = 1$. Hence, A maps bounded sets into bounded sets.

b) Suppose $v_n \to v$ and $Av_n \to w$ for some w. Then $w = Av$. Since

$$(Av_n - Au, v_n - u) \geq 0 \qquad \text{for all } u \in X,$$
$$(w - Au, v - u) \geq 0 \qquad \text{for all } u \in X.$$

From lemma 2.2 it follows that

$$(w - Av, v - u) \geq 0 \qquad \text{for all } u \in X.$$

Here again, since this it true for all $u \in X$, we have that

$$(w - Av, v - u) = 0 \qquad \text{for all } u \in X.$$

Therefore it follows that $w = Av$. Note that the whole sequence $\{Av_n\}$ converges since any subsequence converges to the same element Av.

Note that this theorem holds if $D(A)$ is open since we only need consider a neighborhood of v.

Theorem 2.4, together with theorem 2.3′, gives theorem 2.3. We have considered so far the variational problems with the following conditions:

a) Let V be a Hilbert space with $a: V \times V \to R$ a bilinear form.

Define A by $(Au, v) = a(u, v)$ where $a(\cdot, \cdot)$ is coercive.

Let $\mathbb{K}$ be a closed, convex set in V.

b) Let X be a reflexive Banach space and let $\mathbb{K}$ be a closed, bounded, convex set in X. Let A be an operator form X into X'. We next wish to generalize the problem *b*) by not restricting $\mathbb{K}$ to be bounded.

Consider the bounded closed convex set

$$\mathbb{K}_R = \mathbb{K} \cap \Sigma_R$$

where Σ_R denotes a closed ball about 0 and radius R. If R is large enough, the set $\mathbb{K}_R$ is non empty.

We show the following.

THEOREM 2.5: *Let X be a reflexive Banach space and let $\mathbb{K}$ be a closed, convex set in X. Let $A : \mathbb{K} \to X'$ be monotone, continuous on finite dimensional subspace of $\mathbb{K}$. [Alternatively $A : X \to X'$ be monotone, hemicontinuous]. Then necessary and sufficient condition in order that a solution of the variational inequality*

$$(2.5) \qquad u \in \mathbb{K}, (Au, v - u) \geq 0 \qquad \textit{for all } v \in \mathbb{K}$$

exists is that there exists a constant R such that at least a solution of the variational inequality

$$(2.5') \qquad u_R \in \mathbb{K}_R, (Au_R, v - u_R) \geq 0 \qquad \textit{for all } v \in \mathbb{K}_R$$

satisfies the inequality

$$(2.6) \qquad \| u_R \| < R.$$

In fact if there exists a solution u of (2.5), choosing R such that: $\| u \| < R$ then u satisfy (2.5′).

On the other side if there exists R such that a solution u_R of (2.5′) satisfies (2.6), then u_R satisfies (2.5). In fact $u_R \in \mathbb{K}$ and for any $v \in \mathbb{K}$ there exists $w \in \mathbb{K}_R$ such that

$$w - u_R = \varepsilon (v - u_R)$$

for a suitable $\varepsilon > 0$. Therefore from (2.5′) it follows

$$0 \leq (Au_R, w - u_R) = \varepsilon (Au_R, v - u_R),$$

and thus u_R satisfies (2.5).

From theorem 2.5 follow several conditions which assure the existence of solutions of variational inequalities in the case of un-

a) Assume that there is $\varphi_0 \in \mathbb{K}$ and $R > \| \varphi_0 \|$ such that

$$(2.7) \qquad (Av, \varphi_0 - v) < 0$$

for all v in $\mathbb{K}$ with $\| v \| = R$.

Then the condition (2.6) of Theorem 2.5 is satisfied. In fact, denoting by u_R the solution of (2.5′), it must be

$$\| u_R \| < R,$$

otherwise $(Au_R, \varphi_0 - u_R) \geq 0$ in contradiction to (2.7).

b) Assume that there is a $\varphi_0 \in \mathbb{K}$ for which

$$(2.8) \qquad (Av - A\varphi_0, v - \varphi_0)/\| v - \varphi_0 \| \to +\infty \qquad \text{as } \| v \| \to +\infty$$

for $v \in \mathbb{K}$.

In this assumption (2.7) is verified.

In fact fix $H > \| A\varphi_0 \|$ and R large enough in such a way that $R > \| \varphi_0 \|$ and

$$(Av - A\varphi_0, v - \varphi_0) \geq H \| v - \varphi_0 \| \qquad \text{for } \| v \| > R.$$

Therefore

$$(Av, v - \varphi_0) \geq H \| v - \varphi_0 \| + (A\varphi_0, v - \varphi_0)$$

$$\geq H \| v - \varphi_0 \| - \| A\varphi_0 \| \cdot \| v - \varphi_0 \| \geq$$

$$\geq (H - \| A\varphi_0 \|) \| v - \varphi_0 \| \geq (H - \| A\varphi_0 \|) (\| v \| - \| \varphi_0 \|)$$

and thus, (2.7) is satisfied.

c) In some case, instead of (2.8), it is enough to verify the condition

$$(2.9) \qquad (Av, v)/\| v \| \to +\infty \quad \text{as} \quad \| v \| \to +\infty \quad \text{for} \quad v \in \mathbb{K}.$$

If $0 \in \mathbb{K}$, it is easy to prove that (2.9) implies (2.8).

If $0 \notin \mathbb{K}$, we assume that there exists in $\mathbb{K}$ a point φ_0 together with the point $(1 + \vartheta) \varphi_0$ where $\vartheta > 0$.

First of all, remark that, if $\varphi \in \mathbb{K}$, then, because of the *monotonicity* of $\tilde{A}v = Av - A\varphi_0$, from it easily follows

$$\varliminf_{\|v\|\to+\infty} \inf(\widetilde{A}v, v-\varphi)/\|v\| > -\infty.$$

Next, suppose that for a sequence $\{v_n\} \subset \mathfrak{K}$ such that $\|v_n\| \to +\infty$ as $n \to +\infty$, we have

$$\lim_{n\to+\infty} (\widetilde{A}v_n, v_n - \varphi_0)/\|v_n\| < +\infty.$$

Since

$$(\widetilde{A}v_n, v_n)/\|v_n\| \to +\infty \quad \text{as} \quad n \to +\infty,$$

then

$$\lim_{n\to+\infty} (\widetilde{A}v_n, \varphi_0)/\|v_n\| = +\infty.$$

At last end, we have

$$-\infty < \lim_{n\to+\infty} \inf(\widetilde{A}v_n, v_n - (1+\vartheta)\varphi_0)/\|v_n\| =$$

$$= \lim_{n\to+\infty} (\widetilde{A}v_n, v_n - \varphi_0)/\|v_n\| - \vartheta \lim_{n\to\infty} (\widetilde{A}v_n, \varphi_0)/\|v_n\| = -\infty.$$

and this is a contradiction which proves the statement.

REMARKS: Let us now assume $\mathfrak{K} \equiv X$; that is, we look for $u_0 \in X$ which satisfy $Au_0 = 0$.

i) Let $\Sigma_R = \{v \in X / \|v\| \leq R\}$. We know there is a $u_R \in \Sigma_R$ such that

$$(Au_R, v - u_R) \geq 0 \qquad \text{for all} \quad v \in \Sigma_R.$$

A necessary and sufficient condition that there exist a $u_0 \in X$ for which $Au_0 = 0$ is that there exist an R for which $\|u_R\| < R$. This is a necessary condition since, if $u_0 \in X$ exists, we choose $R > \|u_0\|$. It is also a sufficient condition. For suppose there is a $u_R \in \Sigma_R$ with $\|u_R\| < R$ such that

$$(Au_R, v - u_R) \geq 0 \qquad \text{for all} \quad v \in \Sigma_R.$$

Then, since $(\Sigma_R)_{u_R} \equiv X$, (see (1.3), (1.3')) we have

$$(Au_R, w) \geq 0 \qquad \text{for all} \quad w \in X$$

and this implies $Au_R = 0$. Thus, take $u_0 \equiv u_R$.

ii) A sufficient condition for existence of u_0 such that $Au_0 = 0$ is that there exist an R and a $v_1 \in \Sigma_R$ such that

$$(Au, v_1 - u) < 0 \quad \text{for all } u \text{ with} \quad \|u\| = R.$$

Suppose there is such an R. Then there is a $u_R \in \Sigma_R$ such that

$$(Au_R, v - u_R) \geq 0 \qquad \text{for all} \quad v \in \Sigma_R.$$

In particular, $(Au_R, v_1 - u_R) \geq 0$. Therefore, from the given condition we have $\|u_R\| < R$. From theorem 2.5 we have that there is a $u_0 \in X$ for which $Au_0 = 0$.

c) If X is an uniformly convex Banach space; that is, $u, v \in X$ with $\|u\|, \|v\| = 1$, $u \neq v$ implies $\|tu + (1-t)v\| < 1$ for $0 < t < 1$, and if there exist two different solutions for the some R of

$$(Au_R, v - u_R) \geq 0 \qquad \text{for all} \quad v \in \Sigma_R$$

where $u_R \in \Sigma_R$, then there is a solution u_0 of $Au_0 = 0$. For, if one of the solutions u or v satisfies $\|u\| < R$ or $\|v\| < R$, then we are done by a). If no such solutions exist then $\|u\| = R$, $\|v\| = R$ and this implies $\|tu + (1-t)v\| < R$ for $0 < t < 1$. But the set of all solutions is closed and convex. Thus, taking $t = \frac{1}{2}$ we have $\frac{u+v}{2} \in \mathfrak{K} \equiv X$ and $\left\|\frac{u+v}{2}\right\| < R$ and we are now back to case a).

§ 3. Perturbations of variational inequalities.

1. Let us now return, for the moment, to the case where $X \equiv V$ is a Hilbert space, where $(\cdot,\cdot)$ denotes the inner product in V and $\langle \cdot,\cdot \rangle$ denotes the pairing between V and V'. Let $\mathfrak{K}$ be a closed, convex set in V.

Let $a(\cdot,\cdot)$ be a bilinear form an $V \times V$ such that $a(v, v) \geq 0$. Define the operator $A: V \to V'$ by

$$\langle Au, v \rangle = a(u, v), \qquad \text{for all } v \in V.$$

We have shown that, if $\mathbb{K}$ is bounded or in the case when $\mathbb{K}$ is unbounded but theorem 2.5 holds, there is a solution $u_0 \in \mathbb{K}$ of the variational inequality

$$\langle A u_0, v - u_0 \rangle \geq \langle f, v - u_0 \rangle \qquad \text{for all } v \in \mathbb{K},$$

where $f \in V'$ is given. However, if neither of these conditions is satisfied, such a solution $u_0 \in \mathbb{K}$ may not exist. However, we can show that the set of all such solutions $v_0 \in \mathbb{K}$ satisfies certain properties.

Let us define

$$\mathcal{X} = \{u_0 \in \mathbb{K} \mid \langle A u_0, v - u_0 \rangle \geq \langle f, v - u_0 \rangle \quad \text{for all } v \in \mathbb{K}\}$$

Using lemma 2.2 it is clear that $\mathcal{X}$ is a closed, convex set. It should be noted, however, that $\mathcal{X}$ may be empty.

Now let $\beta(u, v)$ be a bilinear, continuous, coercive form on $V \times V$ and let $g \in V'$. For each $\varepsilon > 0$ we can consider the form

$$a(u, v) + \varepsilon \beta(u, v).$$

Clearly this form is bilinear and coercive on $V \times V$. We have shown that there is a unique solution $u_\varepsilon \in \mathbb{K}$ such that

$$(3.1_\varepsilon) \qquad a(u_\varepsilon, v - u_\varepsilon) + \varepsilon\beta(u_\varepsilon, v - u_\varepsilon) \geq \langle f + \varepsilon g, v - u_\varepsilon \rangle$$

$$\text{for all } v \in \mathbb{K},$$

THEOREM 3.1: *Using the notation given above, and assuming, $\mathcal{X} \neq \Phi$ we have $u_\varepsilon \to u_0$ strongly as $\varepsilon \to 0$, where $u_0 \in \mathcal{X}$ and*

$$\beta(u_0, v - u_0) \geq \langle g, v - u_0 \rangle \qquad \textit{for all } v \in \mathcal{X}.$$

Note: The solutions u_0 obtained above depend on the bilinear form β and the function g. Also, if we take $\beta(u, v) = (u, v)$ and $g = 0$, then the corresponding solution u_0 in $\mathbb{K}$ is that which minimizes $\|u\|$ for $u \in \mathcal{X}$.

PROOF: *a*) Since $\Phi \neq \mathcal{X} \subset \mathbb{K}$ and $\mathcal{X}$ is closed and convex subset of $\mathbb{K} \subset V$, if we consider $\beta(\cdot,\cdot)$ as a continuous, bilinear, coercive form on V, and $g \in V'$, there is a unique solution $u_0 \in \mathcal{X}$ of

Moreover, since $u_0 \in \mathcal{X}$, we have

$$(3.3) \qquad a(u_0, v - u_0) \geq \langle f, v - u_0 \rangle \qquad \text{for all } v \in \mathbb{K}.$$

Setting $v = u_\varepsilon$ in (3.3) and $v = u_0$ in (3.1_ε) and then add, we get

$$a(u_0, u_\varepsilon - u_0) + a(u_\varepsilon, u_0 - u_\varepsilon) + \varepsilon\beta(u_\varepsilon, u_0 - u_\varepsilon) \geq \varepsilon \langle g, u_0 - u_\varepsilon \rangle.$$

But

$$a(u_0, u_\varepsilon - u_0) + a(u_\varepsilon, u_0 - u_\varepsilon) = -a(u_\varepsilon - u_0, u_\varepsilon - u_0) \leq 0.$$

Therefore,

$$(3.4) \qquad \beta(u_\varepsilon, u_0 - u_\varepsilon) \geq \langle g, u_0 - u_\varepsilon \rangle$$

and, by the coerciveness or $\beta(\cdot,\cdot)$,

$$\begin{aligned} \beta_0 \|u_\varepsilon\|^2 &\leq \beta(u_\varepsilon, u_\varepsilon) \leq \beta(u_\varepsilon, u_0) + \langle g, u_\varepsilon - u_0 \rangle \\ &\leq C_1 \|u_\varepsilon\| \|u_0\| + C_2(\|u_\varepsilon\| + \|u_0\|) \\ &\leq C(1 + \|u_\varepsilon\|) \end{aligned}$$

where $C = C(\|u_0\|, g, \beta)$. Hence, since $C\|u_\varepsilon\| \leq \frac{\beta_0}{2}\|u_\varepsilon\|^2 + \frac{C^2}{2\beta_0}$.

$$\|u_\varepsilon\| \leq L = 1 + \frac{C}{\beta_0}$$

where L is independent of ε.

b) Since $\|u_\varepsilon\| \leq L$, there is a subsequence $\{u_\eta\}$ of $\{u_\varepsilon\}$ such that $u_\eta \to w$ weakly in V.

We wish to show that

i) $\qquad w \in \mathcal{X}$,

ii) $\qquad \beta(w, u_0 - w) \geq \langle g, u_0 - w \rangle$.

Now it follows from the lemma 2.2 that we have

$$a(v, v - u_\eta) + \eta\beta(v, v - u_\eta) \geq \langle f, v - u_\eta \rangle + \eta \langle g, v - u_\eta \rangle.$$

Letting $\eta \to 0$, since $u_\eta \to w$ weakly, we have

Using the lemma 2.2 again we get

$$a(w, v-w) \geq \langle f, v-w \rangle$$

and, therefore, $w \in \mathfrak{X}$.

In order to show ii), we use the fact that we have

$$\beta(u_\varepsilon, u_0 - u_\varepsilon) \geq \langle g, u_0 - u_\varepsilon \rangle \tag{3.4}$$

to prove ii). To show this we prove the following lemma.

LEMMA 3.1. *Let $\gamma(u, v)$ be a bilinear, continuous form such that $\gamma(v, v) \geq 0$. Then the function $v \rightarrow \gamma(v, v)$ is lower-semicontinuous, with respect to the weak topology.*

PROOF: From the bilinearity we have for all $u, v \in V$,

$$\gamma(v, v) = \gamma(u, u) + \{\gamma(u, v-u) + \gamma(v-u, u)\} + \gamma(u-v, u-v)$$

$$\geq \gamma(u, u) + \{\gamma(u, v-u) + \gamma(v-u, u)\}$$

since $\gamma(w, w) \geq 0$. Now let $v \rightarrow u$ weakly. Then from the continuity of γ, $\gamma(u, v-u) \rightarrow 0$ and $\gamma(v-u, u) \rightarrow 0$. Thus,

$$\liminf_{v \to u} \gamma(v, v) \geq \gamma(u, u)$$

and, hence, the map $v \rightarrow \gamma(v, v)$ is lower-semicontinuous. We now have $u_\eta \rightarrow w$ weakly in V. Since

$$\beta(u_\varepsilon, u_0 - u_\varepsilon) \geq \langle g, u_0 - u_\varepsilon \rangle$$

we have

$$\beta(u_\eta, u_\eta) \leq \beta(u_\eta, u_0) + \langle g, u_\eta - u_0 \rangle.$$

Hence, from the lemma just proved,

$$\beta(w, w) \leq \liminf_{\eta \to 0} \beta(u_\eta, u_\eta)$$

$$\leq \liminf_{\eta \to 0} [\beta(u_\eta, u_0) + \langle g, u_\eta - u_0 \rangle]$$

$$= \beta(w, u_0) + \langle g, w - u_0 \rangle.$$

Hence,

$$-\langle g, w - u_0 \rangle \leq \beta(w, u_0 - w)$$

or

$$\beta(w, u_0 - w) \geq \langle g, u_0 - w \rangle. \tag{3.5}$$

c) We next show $u_\varepsilon \rightarrow u_0$ strongly by proving

I) $w = u_0$

II) $u_\varepsilon \rightarrow u_0$ weakly in V,

III) $u_\varepsilon \rightarrow u_0$ strongly in V.

Now $\beta(u_0, v - u_0) \geq \langle g, v - u_0 \rangle$, from (3.2), for all $v \in \mathfrak{X}$.
In particular we have $w \in \mathfrak{X}$. Therefore,

$$\beta(u_0, w - u_0) \geq \langle g, w - u_0 \rangle.$$

From (3.5) we have

$$\beta(w, u_0 - w) \geq \langle g, u_0 - w \rangle.$$

Adding we get

$$-\beta(w - u_0, w - u) \geq 0.$$

Hence it follows, from the coerciveness of β, that

$$-\beta_0 \| w - u_0 \| \geq 0.$$

Hence, $w = u_0$.

Since u_0 is the unique limit of any weakly convergent subsequence of $\{u_\varepsilon\}$, we have that $u_\varepsilon \rightarrow u_0$ weakly in V.

Therefore, we now wish to show that the convergence is strong.
But

$$\beta_0 \| u_\varepsilon - u_0 \|^2 \leq \beta(u_\varepsilon - u_0, u_\varepsilon - u_0)$$

$$= \beta(u_\varepsilon, u_\varepsilon - u_0) - \beta(u_0, u_\varepsilon - u_0).$$

Now as $\varepsilon \rightarrow 0$, $\beta(u_0, u_\varepsilon - u_0) \rightarrow 0$ by the continuity of the form $\beta(\cdot, \cdot)$. But from (3.4)

$$\beta(u_\varepsilon, u_\varepsilon - u_0) \leq \langle g, u_\varepsilon - u_0 \rangle \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

we have

$$\beta_0 \| u_\varepsilon - u_0 \|^2 \rightarrow \lim_{\varepsilon \to 0} \langle g, u_\varepsilon - u_0 \rangle = 0$$

as $\varepsilon \rightarrow 0$. Hence, $u_\varepsilon \rightarrow u_0$ strongly in V.

Therefore, this proves the theorem.

COROLLARY OF THEOR. 3.1. $\mathcal{X} \neq \Phi \Longleftrightarrow \|u_\varepsilon\| \leq L$, L *independent* of ε.

PROOF. $\Longrightarrow$ Part a) of above proof

$\Longleftarrow$ Part b) of above proof.

2. Let us now return again to the case of a general Banach space. We set the following notation

X is a reflexive Banach space with dual X'

$A: X \to X'$ is a monotone, hemicontinuous operator

$\mathbb{K}$ is a closed, convex set in X.

We wish to look for those $u \in \mathbb{K}$ for which

$$(Au, v - u) \geq 0 \quad \text{for all } v \in \mathbb{K}.$$

Here, $(\cdot,\cdot)$ denotes the pairing between X and X'.

Let $\mathcal{X}$ be given by

$$\mathcal{X} = \{u \in \mathbb{K} \mid (Au, v - u) \geq 0 \quad \text{for all } v \in \mathbb{K}\}.$$

We have already shown that $\mathcal{X}$ is a closed, convex set. Moreover, we know that $\mathcal{X}$ is non-empty if

i) $\mathbb{K}$ is bounded, there is a $\varphi_0 \in \mathbb{K}$ for which

$$\|v - \varphi_0\|^{-1}(Av - A\varphi_0, v - \varphi_0) \to +\infty \quad \text{as } \|v\| \to +\infty$$

Assume that conditions i), ii) are not satisfied. Let us now consider another operator $B: X \to X'$ which is strictly monotone, hemicontinuous, and satisfies

iii) $\quad (Bu - Bv, u - v) \geq c(\|u - v\|)\cdot\|u - v\|$

where $c(r)$ is a monotone, continuous function on $[0, \infty]$ with

$$(c(0) = 0), \quad c(r) > 0 \quad \text{for } r > 0, \quad \text{and } c(r) \to +\infty \quad \text{as } r \to +\infty.$$

We note that this condition iii) implies the condition b) of the previous § 2 with A replaced by B and therefore, from theorem 2.5 there is a unique solution: $u \in \mathbb{K}$ of

$$(Bu, v - u) \geq 0 \quad \text{for all } v \in \mathbb{K}.$$

Let us now define for $\varepsilon > 0$,

$$T_\varepsilon \equiv A + \varepsilon B;$$

that is, let T_ε be a perturbation of the operator A. Clearly, since B is strictly monotone, T_ε is strictly monotone.

Moreover, T_ε is hemicontinuous and satisfies ii). Therefore, there is a unique $u_\varepsilon \in \mathbb{K}$ for which

$$(T_\varepsilon u_\varepsilon, v - u_\varepsilon) \geq 0 \quad \text{for all } v \in \mathbb{K}.$$

We will prove the following theorem

THEOREM 3.2: *Assume $\mathcal{X}$ is non-empty. Let u_ε be obtained as described above. Then $u_\varepsilon \to u_0$ strongly in X where $u_0 \in \mathcal{X}$, and*

$$(Bu_0, v - u_0) \geq 0 \quad \textit{for all } v \in \mathcal{X}.$$

First of all let us prove the following lemma.

LEMMA 3.2. *Suppose $u_\varepsilon \to w$ weakly in X. Let B be any monotone, hemicontinuous operator. Assume $(Bu_\varepsilon, u_\varepsilon - w) \leq 0$. Then*

$$(Bw, w - v) \leq \liminf\,(Bu_\varepsilon, u_\varepsilon - v)$$

for all $v \in D(B)$ where the domain of B, $D(B)$, is closed and convex.

PROOF OF THE LEMMA. From monotonicity we have

$$0 \geq (Bu_\varepsilon, u_\varepsilon - w) \geq (Bw, u_\varepsilon - w) \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, we have

$$\lim_{\varepsilon \to 0}\,(Bu_\varepsilon, u_\varepsilon - w) = 0. \tag{3.5}$$

Let $v \in D(B)$. Since $w \in D(B)$ we define

$$\zeta = (1 - t)w + tv \in D(B) \quad \text{for } 0 \leq t \leq 1.$$

Then from monotonicity

$$\begin{aligned} 0 &\leq (Bu_\varepsilon - B\zeta, u_\varepsilon - \zeta) \\ &= (Bu_\varepsilon, u_\varepsilon - w) + (Bu_\varepsilon, w - \zeta) \\ &\quad - (B\zeta, u_\varepsilon - w + w - \zeta). \end{aligned}$$

Since $w - \zeta = t(w - v)$, we have

$$0 \leq (Bu_\varepsilon, u_\varepsilon - w) + t(Bu_\varepsilon, w - v)$$
$$-(B\zeta, u_\varepsilon - w + t(w - v)).$$

Note that ζ depends only on t and not on ε. As $\varepsilon \to 0$ we have

$$(B\zeta, u_\varepsilon - w + t(w - v) \to (B\zeta, t(w - v).$$

Therefore, as $\varepsilon \to 0$, we get

$$t(B\zeta, w - v) \leq t \liminf_{\varepsilon \to 0} (Bu_\varepsilon, w - v)$$

or

$$(B\zeta, w - v) \leq \liminf_{\varepsilon \to 0} (Bu_\varepsilon, w - v).$$

But from the definition of ζ we have

$$(B\zeta, w - v) = (B((1 - t)w + tv), w - v).$$

Letting $t \to 0$, since B is hemicontinuous, we get

$$(3.5) \qquad (Bw, w - v) \leq \liminf_{\varepsilon \to 0} (Bu_\varepsilon, w - v)$$

$$\leq \liminf_{\varepsilon \to 0} (Bu_\varepsilon, u_\varepsilon - v) + \liminf_{\varepsilon \to 0} (Bu_\varepsilon, w - u_\varepsilon)$$

$$= \liminf_{\varepsilon \to 0} (Bu_\varepsilon, u_\varepsilon - v).$$

by (3.5).

PROOF OF THEOREM 3.2: Let u_1 be any element of $\mathfrak{X}$. Then $u_1 \in \mathbb{K}$ and

$$(3.6) \qquad (Au_1, v - u_1) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

Since T_ε satisfies the hypothesis of theorem 2.5 and moreover it is strictly monotone, there exists one $u_\varepsilon \in \mathbb{K}$ such that

$$(3.7) \qquad (T_\varepsilon u_\varepsilon, v - u_\varepsilon) = (Au_\varepsilon + \varepsilon Bu_\varepsilon, v - u_\varepsilon) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

Set $v = u_\varepsilon$ in (3.6) and $v = u_1$ in (3.7) and add. We get

$$-(Au_1 - Au_\varepsilon, u_1 - u_\varepsilon) + \varepsilon(Bu_\varepsilon, u_1 - u_\varepsilon) \geq 0.$$

Since $-(Au_1 - Au_\varepsilon, u_1 - u_\varepsilon) \leq 0$ this implies

$$(3.8) \qquad (Bu_\varepsilon, u_1 - u_\varepsilon) \geq 0 \qquad \text{for all } u_1 \in \mathfrak{X}.$$

Since B satisfies iii) of § 2, n. 3, and from (3.8) we have

$$c(\| u_\varepsilon - u \|) \| u_1 - u_\varepsilon \| \leq (Bu_\varepsilon - Bu_1, u_\varepsilon - u_1) \leq -(Bu_1, u_\varepsilon - u_1)$$
$$\leq \| Bu_1 \|_{X'} \| u_\varepsilon - u_1 \|_X$$

and this implies

$$c(\| u_\varepsilon - u_1 \|) \leq \| Bu_1 \|_{X'}.$$

Since the bound on the right is independent of ε, we see that there is on L such that $\| u_\varepsilon - u_1 \| \leq L$ and, hence

$$\| u_\varepsilon \|_X \leq \zeta.$$

Now since X is reflexive and the sequence $\{u_\varepsilon\}$ is uniformly bounded, there is a weakly convergent subsequence $\{u_\eta\}$ and $u_\eta \to w$ weakly where $w \in \mathfrak{X}$. For, by the lemma 2.2 we have

$$(Av + \eta Bv, v - u_\eta) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

which implies, since $u_\eta \to w$ weakly with $w \in \mathbb{K}$,

$$(Av, v - w) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

and, hence

$$(Aw, v - w) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

Therefore $w \in \mathfrak{X}$.

From (3.8) we have

$$(Bu_\eta, v - u_\eta) \geq 0 \qquad \text{for all } v \in \mathfrak{X}.$$

Hence,

$$(Bu_\eta, w - u_\eta) \geq 0.$$

Now since B is strictly monotone we know there is a $u_0 \in \mathfrak{X}$ such that

$$(3.9) \qquad (Bu_0, v - u_0) \geq 0 \qquad \text{for all } v \in \mathfrak{X}.$$

But by the lemma 3.2

$$(3.10) \qquad (Bw, w - v) \leq \liminf (Bu_\eta, u_\eta - v) \leq 0 \qquad \text{for all } v \in \mathfrak{X}.$$

Setting $v = w \in \mathfrak{X}$ in (3.9) and $v = u_0 \in \mathfrak{X}$ in (3.10) and adding we get

$$0 \leq (Bu_0 - Bw, w - u_0) = -(Bu_0 - Bw, u_0 - w).$$

Therefore, from the strict monotonicity of B we have $w = u_0$.

Therefore, $u_\eta \to u_0$ weakly for any subsequence $\{u_\eta\}$ of $\{u_\varepsilon\}$ and, hence,

$$u_\varepsilon \to u_0 \qquad \text{weakly.}$$

Since $w = u_0 \in \mathfrak{X}$, (3.10) gives us

$$0 \leq c(\| u_\varepsilon - u_0 \|) \| u_\varepsilon - u_0 \| \leq (Bu_\varepsilon - Bu_0, u_\varepsilon - u_0) =$$

$$= -(Bu_\varepsilon, u_0 - u_\varepsilon) - (Bu_0, u_\varepsilon - u_0) \leq -(Bu_0, u_\varepsilon - u_0)$$

as $\varepsilon \to 0$. Therefore, $\| u_\varepsilon - u_0 \| \to 0$ as $\varepsilon \to 0$ since

$$c(r) > 0 \quad \text{for } r > 0 \quad \text{and} \quad c(0) = 0.$$

3. Let us now consider an application of variational inequalities and of perturbations of monotone operators. Let V be a Hilbert space and let $U: V \to V$ be an non expansive operator that is, satisfying,

$$\| Uu - Uv \| \leq \| u - v \|.$$

Let $\mathbb{K}$ be a closed, bounded, convex set in U and assume $U(\mathbb{K}) \subset \mathbb{K}$. Then we claim that there is a closed, convex set of fixed points.

Let us define $T \equiv I - U$. Then T is monotone, for

$$(Tu - Tv, u - v) = (u - v - Uu + Uv, u - v)$$

$$= \| u - v \|^2 - (Uu - Uv, u - v) \geq 0.$$

Clearly T is hemicontinuous. Therefore, since $\mathbb{K}$ is a bounded, closed, convex set in V, there is a $u_0 \in \mathbb{K}$ such that

(3.11) $$(Tu_0, v - u_0) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

That is,

$$(u_0 - Uu_0, v - u_0) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

But $u_0 \in \mathbb{K}$ and $U(\mathbb{K}) \subset \mathbb{K}$ implies $Uu_0 \in \mathbb{K}$. Set $v = Uu_0$. Then

$$(u_0 - Uu_0, Uu_0 - u_0) = -(Uu_0 - u_0, Uu_0 - u_0) \geq 0,$$

and it follows that $Uu_0 = u_0$; that is, u_0 is a fixed point relative to the operator U. Since $(Tu_0, v - u_0) \geq 0$ for all $v \in \mathbb{K}$, the set of all fixed points relative to the operator U forms a closed, convex set in $\mathbb{K}$.

Now let W be any operator on V (a contraction) for which $\| Wu - Wv \| \leq k \| u - v \|$ where $0 \leq k < 1$ and define U_ε by

$$U_\varepsilon \equiv (1 - \varepsilon) U + \varepsilon W.$$

Now U_ε is a contraction since, for $u, v \in V$

$$\| U_\varepsilon u - U_\varepsilon v \| \leq (1 - \varepsilon) \| Uu - Uv \| + \varepsilon \| Wu - Wv \|$$

$$\leq [(1 - \varepsilon) + \varepsilon k] \| u - v \| = [1 - \varepsilon(1 - k)] \| u - v \|.$$

Now define $T_\varepsilon \equiv I - U_\varepsilon = T + \varepsilon(U - W)$. T_ε is strictly monotone since, for $u, v \in V$,

$$(T_\varepsilon u - T_\varepsilon v, u - v) = (u - v, u - v) - (U_\varepsilon u - U_\varepsilon v, u - v)$$

$$\geq \| u - v \|^2 - [1 - \varepsilon(1 - k)] \| u - v \|^2 = \varepsilon(1 - k) \| u - v \|^2.$$

Therefore, there is for each $\varepsilon > 0$ a $u_\varepsilon \in \mathbb{K}$ such that

$$(T_\varepsilon u_\varepsilon, v - u_\varepsilon) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

From theorem 3.2 we have that $u_\varepsilon \to u_0$ strongly, where u_0 is a fixed point relative to the operator U.

Exactly, if we denote by $\mathfrak{X}$ the set of the fixed points relative to the operator U, then u_0 is the point of $\mathfrak{X}$ satisfying

$$u_0 \in \mathfrak{X}, (u_0 - Wu_0, v - u_0) \geq 0 \qquad \text{for all } v \in \mathfrak{X}.$$

The following question arises naturally.

Is u_ε a fixed point for U_ε? The answer is yes if $W(\mathbb{K}) \subset \mathbb{K}$. In fact, being $U(\mathbb{K}) \subset \mathbb{K}$ and $W(\mathbb{K}) \subset \mathbb{K}$, $U_\varepsilon(\mathbb{K}) \subset \mathbb{K}$ for $0 \leq \varepsilon \leq 1$.

Then, as before, the inequality

$$u_\varepsilon \in \mathbb{K}, (u_\varepsilon - U_\varepsilon u_\varepsilon, v - u_\varepsilon) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

implies $u_\varepsilon = U_\varepsilon u_\varepsilon$. Notice that u_ε can be construct by iterative method.

As special case, assume $W = v_0$, a fixed point of $\mathbb{K}$; then $u_\varepsilon \to u_0$ strongly where u_0 is the nearest fixed point for U to v_0.

§ 4. Generalization of theorems of § 2.

1. Let us return to the case where $X \equiv V$, a Hilbert space. Let $\|\cdot\|$ denote a norm on V which makes V a Hilbert space, and assume $\|\cdot\| \backsim p_0(\cdot) + p_1(\cdot)$ (« $\backsim$ »: is equivalent to)

where $p_0(\cdot)$ is a norm under which V is a pre-hilbert space

and $p_1(\cdot)$ is a semi-norm on V.

Let $\mathbb{K}$ be a closed, convex set in V and assume $0 \in \mathbb{K}$. Let

$$Y = \{v \in V \mid p_1(v) = 0\}$$

and assume Y is finite dimentional. Assume *there is a $c_1 > 0$ such that*

$$\text{(4.1)} \qquad \operatorname*{Inf}_{y \in Y} p_0(v - y) \leq c_1 p_1(v) \qquad \text{for all} \quad v \in V.$$

Assume $a(\cdot,\cdot)$ is a bilinear form on $V \times V$ such that

$$\text{(4.2)} \qquad a(v, v) \geq \alpha [p_1(v)]^2 \ \alpha > 0, \ v \in V.$$

In particular note that $a(v, v) = 0 \Longrightarrow v \in Y (\Leftarrow\!/\!\Rightarrow v = 0)$.

Let $f \in V'$ and assume we can write $f = f_0 + f_1$ where

$$\text{(4.3)} \qquad |\langle f_1, v\rangle| \leq c_2 p_1(v)$$

and

$$\text{(4.4)} \qquad \langle f_0, y\rangle < 0, \qquad \text{for} \quad y \in Y \cap \mathbb{K}, \ y \neq 0.$$

THEOREM 4.1: *Assume we have the above given. Then there is a $u \in \mathbb{K}$ such that*

$$a(u, v - u) \geq \langle f, v - u\rangle \qquad \textit{for all} \quad v \in \mathbb{K}.$$

PROOF. Let $\mathbb{K}_R = \mathbb{K} \cap \Sigma_R$. Since $\mathbb{K}_R$ is bounded, there is a u_R such that

$$\text{(4.5)} \qquad a(u_R, v - u_R) \geq \langle f, v - u_R\rangle \qquad \text{for all} \quad v \in \mathbb{K}_R.$$

From theorem 2.5 we know that $\|u_R\| < R$ for some R if and only if there is a $u \in \mathbb{K}$ such that

$$a(u, v - u) \geq \langle f, v - u\rangle \qquad \text{for all} \quad v \in \mathbb{K}.$$

Assume the theorem is not true. Then, for each $R > 1$, the solution u_R, obtained as above, satisfies $\|u_R\| = R$.

We will show that this leads to a contradiction. Let $w_R = \frac{1}{R} u_R$. Then $\|w_R\| = 1$. Moreover, since $0 \in \mathbb{K}$, $u_R \in \mathbb{K}$, and $\mathbb{K}$ is convex, we have $w_R \in \mathbb{K}$. Since $0 \in \mathbb{K}$ we can take $v = 0$ and get from (4.2) (4.5)

$$\text{(4.6)} \qquad \alpha [p_1(u_R)]^2 \leq a(u_R, u_R) \leq$$

$$\leq \langle f, u_R\rangle \leq \|f\|_{V'} \cdot \|u_R\| = R \|f\|_{V'}.$$

Therefore,

$$p_1(u_R) = 0(\sqrt{R}).$$

If follows that

$$p_1(w_R) = 0\left(\frac{1}{\sqrt{R}}\right)$$

and, therefore, $p_1(w_R) \to 0$ as $R \to \infty$.

Now since $\|w_R\| = 1$ for all R there is a subsequence, call it $\{w_R\}$ again, such that $w_R \to w$ weakly in V. Since $\mathbb{K}$ is closed we have $w \in \mathbb{K}$. Also we have that $p_1(w) = 0$. Therefore, $w \in Y \cap \mathbb{K}$.

Consider the projection $P: V \to Y$ given by $Pv = x$ i.e.,

$$p_0(v - x) = \operatorname*{Inf}_{y \in Y} p_0(v - y).$$

Here we use the assumption that Y is finite dimensional.

We are going to prove that, if $v_n \to v$ weakly in V, then $Pv_n \to Pv$ strongly in Y.

Since

$$\text{(4.7)} \qquad (Pv, \zeta) = (v, \zeta) \qquad \text{for all} \quad \zeta \in Y$$

and

(4.7$_n$) $$(Pv_n, \zeta) = (v_n, \zeta) \qquad \text{for all} \quad \zeta \in Y$$

we have

$$\| Pv_n \| \leq \| v_n \| \leq M < \infty \qquad \text{for all} \quad n.$$

But $\{Pv_n\} \subset Y$, finite dimentional space. Thus, for a subsequence of v_n still called v_n, it follows $Pv_n \to \vartheta$ strongly for some ϑ. But, since $v_n \to v$ weakly, we have, from (4.7$_n$) $(\vartheta, \zeta) = (v, \zeta)$ for all $\zeta \in Y$. Therefore, for all $\zeta \in Y$, from (4.7) $(\vartheta, \zeta) = (Pv, \zeta)$ and, since $\vartheta \in Y$, $\vartheta = Pv$.

Note that there is a constant $c_3 > 0$ such that $p_0(Pw_R) \geq c_3$! For, suppose this is not true. Then $p_0(Pw_R) \to 0$ as $R \to \infty$ at least for a sequence of values of R. But

$$p_0(w_R) \leq p_0(w_R - Pw_R) + p_0(Pw_R)$$
$$\leq c_1 p_1(w_R) + p_0(Pw_R) \to 0,$$

Since $p_1(w_R) = 0\left(\frac{1}{\sqrt{R}}\right)$. Thus $\| w_R \| \backsim p_0(w_R) + p_1(w_R) \to 0$ implies the desired contradiction of $\| w_R \| = 1$.

Now since $w \in Y$, we have $Pw = w$ and, hence,

$$p_0(w) \geq c_3 > 0.$$

Case. i) $Y \cap K = \{0\}$. This is impossible since $p_0(w) > 0$ implies $w \neq 0$ and since $w \in Y \cap \mathbb{K}$.

Case. ii) $Y \cap \mathbb{K} \neq \{0\}$. Now $\langle f_0, w \rangle < 0$ since $w \in Y \cap \mathbb{K}$ and $w \neq 0$. Thus, from (4.4),

$$-\langle f_0, w \rangle \equiv 2\beta > 0.$$

Since $Pw_R \to Pw$ strongly and since $Pw = w$, there is an R_0 such that

$$-\langle f_0, Pw_R \rangle > \beta > 0 \qquad \text{for all} \quad R \geq R_0.$$

Now, from (4.6)

$$\alpha [p_1(u_R)]^2 \leq \langle f, u_R \rangle$$
$$= \langle f_0, u_R - Pu_R \rangle + \langle f_0, Pu_R \rangle + \langle f_1, u_R \rangle$$

Therefore,

$$\alpha [p_1(u_R)]^2 - R \langle f_0, Pw_R \rangle \leq C [p_0(u_R - Pu_R) + p_1(u_R)]$$
$$\leq C' p_1(u_R).$$

And, since $\alpha [p_1(u_R)]^2 \geq 0$,

$$-R \langle f_0, Pw_R \rangle \leq C' p_1(u_R) = O(\sqrt{R})$$

But $-\langle f_0, Pw_R \rangle > \beta > 0$ for $R \geq R_0$. Thus, for $R \geq R_0$, $\beta R \leq O(\sqrt{R})$. This is impossible. This is the contradiction desired to complete the proof.

2. Now let X be a Banach space and let X' be its dual. Let $A : X \to X'$ be monotone and hemicontinuous. Let $\mathbb{K}$ be a closed, convex set in X.

We consider the following two problems

A) There is a $u \in \mathbb{K}$ such that

$$(Au, v - u) \geq 0 \qquad \text{for all} \quad v \in \mathbb{K}$$

B) Let $f : X \to [-\infty, +\infty]$ be arbitrarily given. There is a $u \in X$ such that

$$(Au, v - u) \geq f(u) - f(v) \qquad \text{for all} \quad v \in X.$$

Note that problem B) implies problem A) since we can take

$$f(v) = \delta_{\mathbb{K}}(v) = \begin{Bmatrix} 0 & \text{if } v \in \mathbb{K} \\ +\infty & \text{if } v \notin \mathbb{K} \end{Bmatrix}$$

where we assume $\mathbb{K} \neq \Phi$. Then, for all $v \in \mathbb{K}$, since $u \in X$,

$$f(u) \leq (Au, v - u) + f(v) = (Au, v - u) < \infty.$$

Therefore, $f(u) < \infty$ and, hence, $u \in \mathbb{K}$.

Hence, there is a $u \in \mathbb{K}$ such that

$$(Au, v - u) \geq 0 \qquad \text{for all} \quad v \in \mathbb{K}.$$

Viceversa, we will show that problem B) can be deduced from

DEFINITION 4.1 : *a*) The function $f: X \to [-\infty, +\infty]$ is *convex* if, for $0 \leq t \leq 1$ and for $u, v \in X$,

$$f(tu + (1-t)v) \leq t f(u) + (1-t) f(v).$$

b) the function $f: X \to [-\infty, +\infty]$ is *strictly convex* if, for

$$0 < t < 1 \quad \text{and} \quad u, v \in X,$$

$$f(tu + (1-t)v) < t f(u) + (1-t) f(v).$$

DEFINITION 4.2 : The function $f: X \to (-\infty, +\infty]$ is *lower semi-continuous* if

a) $$f(v) \leq \liminf_{\tilde{v} \to v} f(\tilde{v}) \qquad \text{for all } v \in X$$

or *b*) the epigraph of f is closed; that is, $\{(v, \beta) \in X \oplus \mathbb{R}^1 \mid f(v) \leq \beta\}$ is a closed set.

or *c*) for all $\mu \in \mathbb{R}^1$, the set $\{v \in X \mid f(v) \leq \mu\}$ is closed

Note: *a*) $\Longleftrightarrow$ *b*) $\Longleftrightarrow$ *c*).

THEOREM 4.2 Let $A: X \to X'$ *be a monotone, hemicontinuous operator. Let* $f: X \to (-\infty, +\infty]$ *be a convex function with* $f(0) = 0$ (*). *Assume* f *is lower semi-continuous. Also assume that there exists* R_0 *such that*

$$(Av, v) + f(v) > 0 \qquad \textit{for all } \|v\| \geq R_0$$

then, there exists u_0 such that

$$(Au_0, v - u_0) \geq f(u_0) - f(v) \qquad \textit{for all } v \in X.$$

Note: u_0 is unique if either

a) A is strictly monotone

or

b) f is strictly convex.

For, suppose A is strictly monotone. If there exist $u_1, u_2 \in X$ such that

$$(Au_1, v_1 - u_1) \geq f(u_1) - f(v_1) \qquad \text{for all } v_1 \in X$$

and

$$(Au_2, v_2 - u_2) \geq f(u_2) - f(v_2) \qquad \text{for all } v_2 \in X$$

(*) This is not a restriction if $f(0) < +\infty$.

then we can set $v_1 = u_2$ and $v_2 = u_1$ and add to obtain

$$-(Au_1 - Au_2, u_1 - u_2) \geq 0.$$

By the strict monotonicity of A we have $u_1 = u_2$.

Now suppose f is strictly convex. If there exist $u_1, u_2 \in X$ such that

$$(Au_1, v_1 - u_1) \geq f(u_1) - f(v_1) \qquad \text{for all } v_1 \in X$$

and

$$(Au_2, v_2 - u_2) \geq f(u_2) - f(v_2) \qquad \text{for all } v_2 \in X$$

then we can set $v_1 = v_2 = \frac{u_1 + u_2}{2}$ and obtain

$$\left(Au_1, \frac{u_2 - u_1}{2}\right) \geq f(u_1) - f\left(\frac{u_1 + u_2}{2}\right)$$

$$\left(Au_2, \frac{u_1 - u_2}{2}\right) \geq f(u_2) - f\left(\frac{u_1 + u_2}{2}\right).$$

Adding we get, using the monotonicity of A,

$$0 \geq -\left(Au_1 - Au_2, \frac{u_1 - u_2}{2}\right) \geq f(u_1) + f(u_2) - 2f\left(\frac{u_1 + u_2}{2}\right)$$

Therefore.

$$\frac{f(u_1) + f(u_2)}{2} \leq f\left(\frac{u_1 + u_2}{2}\right) < \frac{f(u_1)}{2} + \frac{f(u_2)}{2}.$$

This last strict inequality holds if $u_1 \neq u_2$. Since this strict inequality is impossible, we must have $u_1 = u_2$.

PROOF: Let $\bar{X} = X \oplus \mathbb{R}$ be given by $\{v, \beta\} \in \bar{X}$ if and only if $v \in X$ and $\beta \in \mathbb{R}$.

Let

$$\mathbb{K} = \{\{v, \beta\} \in \bar{X} \mid f(v) \leq \beta\}$$

that is, $\mathbb{K}$ is the epigraph of f. Then $\mathbb{K}$ is a closed, convex set in $\bar{X}$.

Define

$$\bar{A}: \bar{X} \to X'$$

Proof: Set $\tilde{X} = X \times \mathbb{R}$, $\tilde{v} = (v, \xi)$ $v \in X$, $\xi \in \mathbb{R}$

$\tilde{A}v = (Av, 1)$ $\tilde{\mathbb{K}}$ = epigraph of f

$\tilde{A}$ is monotone and hemicontinuous.

We claim that the inequality of the theorem is a consequence of the following variational inequality

$$(*) \qquad \tilde{u} \in \tilde{\mathbb{K}} : (\tilde{A}\tilde{u}, \tilde{v} - u) \geq 0 \qquad \forall \tilde{v} \in \tilde{\mathbb{K}}$$

In fact

$$\tilde{u} = (u, \alpha) \in \tilde{\mathbb{K}} = \{u \in X : \alpha \geq f(u)\}$$

and

$$(Au, v-u) + \xi - \alpha \geq 0 \qquad \forall \tilde{v} = (v, \xi) \in \tilde{\mathbb{K}} = \{v \in X : \xi \geq f(v)\}.$$

Therefore

$$(Au, v-u) + f(v) - \alpha \geq 0 \qquad \forall v \in X$$

choosing $v = u$, we get $f(u) \geq \alpha$ and consequently $f(u) = \alpha$.

Thus we have

$$(Au, v-u) \geq f(u) - f(v)$$

which is the desired inequality.

Therefore we have to prove that there exists a solution of $(*)$.

Set

$$\mathbb{K}_R = \{\tilde{v} = (v, \xi) \in \tilde{\mathbb{K}}, \|v\| + |\xi| \leq R\}.$$

From a previous theorem, since $\tilde{\mathbb{K}}_R$ is bounded, there exists $\tilde{u}_R \in \tilde{\mathbb{K}}_R$ such that

$$(A\tilde{u}_R, \tilde{v} - \tilde{u}_R) \geq 0 \qquad \forall v \in \tilde{\mathbb{K}}_R.$$

Let $\tilde{u}_R = (u_R, \alpha_R)$ with $\alpha_R \geq f(u_R)$; since $(0, f(0)) = (0,0) = \tilde{\mathbb{K}}_R$ we get

$$(Au_R, u_R) + \alpha_R \leq 0 \qquad , \quad \alpha_R \geq f(u_R)$$

and thus

$$(Au_R, u_R) + f(u_R) \leq 0.$$

By the assumptions of the theorem, it must be

$$\|u_R\| < R_0.$$

Moreover, using monotonicity of A

$$\alpha_R \leq -(Au_R, u_R) \leq |(A0, u_R)| \leq \text{const} \|u_R\| \leq \text{Const} \cdot R_0$$

On the other hand, from the convexity of f, we get

$$\alpha_R \geq f(u_R) \geq -c\,\|u_R\| \geq -cR_0$$

We have proved that for $\tilde{u}_R = (u_R, \alpha_R)$ we have an estimation of the type

$$\|u_R\| + |\alpha_R| < \text{Const. (independent of } R)$$

Therefore if we construct $\tilde{u}_R$ for an R larger than this constant we will get

$$\|u_R\| + |\alpha_R| < R$$

and this is sufficient in order to assure the existence of the solution of the variational inequality (*).

The theorem is so proved.

PART II. — REGULARITY OF SOLUTIONS

§ 5. Some problems of variational inequalities.

1. Let Ω be a domain of $\mathbb{R}^N$ which, for sake of semplicity, we assume to be bounded. We now define the spaces $H^1(\Omega)$, $H_0^1(\Omega)$, $H^1(\bar{\Omega})$. We first define

$$C^1(\Omega) = \{f \text{ defined on } \Omega \mid f \text{ continuously differentiable in } \Omega\},$$

$$C^1(\bar{\Omega}) = \{f \text{ defined on } \bar{\Omega} \mid f \text{ continuously differentiable in } \Omega \text{ with first derivatives continuous in } \bar{\Omega}\}$$

$$C_0^1(\Omega) = \{f \in C^1(\Omega) \mid f \text{ vanishes outside some compact set in } \Omega\}.$$

Next we define the norms

$$\| u \|_{H^1(\Omega)} = \| u \|_2 + \sum_{i=1}^{n} \| u_{x_i} \|_2 ,$$

$$\| u \|_{H_0^1(\Omega)} = \sum_{i=1}^{n} \| u_{x_i} \|_2$$

where $\| u \|_2 = \left\{ \int_\Omega | u |^2 \, dx \right\}^{1/2}$. We then define

$H^1(\Omega) = \overline{C^1(\Omega)}$ where closure is taken in $\| u \|_{H^1(\Omega)}$.

$H_0^1(\Omega) = \overline{C_0^1(\Omega)}$ » » » » » » .

But Poincaré has shown that, if $v \in H_0^1(\Omega)$, we have

$$\int_\Omega | v |^2 \, dx \leq \beta \int_\Omega | v_x |^2 \, dx$$

where β is independent of v. Therefore, we could equivalently norm $H_0^1(\Omega)$ by taking closure in $\| u \|_{H_0^1(\Omega)}$. This will be the understood

Note that, if we close $C^1(\bar{\Omega})$ with $\| u \|_{H^{1,p}(\Omega)} = \| u \|_{L^p} + \sum_{i=1}^{n} \| u_{x_i} \|_{L^p}$ we obtain $H^{1,p}(\Omega)$. If $p = 2$, the case which is state above, we just omit the p and write $H^{1,2}(\Omega) \equiv H^1(\Omega)$. In the sam way we can define $H_0^{1,p}(\Omega)$.

Let us now consider two examples which use these space Let Δ denote the Laplacian. These two examples are the Dirichl and the Neumann problems.

DIRICHLET PROBLEM: Formally, for smooth u, consider f given $f(x)$

$$\left. \begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u(x) &= 0 && x \in \partial\Omega \end{aligned} \right\}.$$

For any $\varphi \in C^1(\Omega)$ we can multiply both sides of $-\Delta u = f$ by and integrate over Ω:

$$(u, \varphi)_{H_0^1(\Omega)} \equiv \sum_{i=1}^{n} \int_\Omega u_{x_i} \varphi_{x_i} \, dx = \int_\Omega f \varphi \, dx.$$

We call $u \in H_0^1(\Omega)$ a weak solution of the Dirichlet problem, if

$$(u, \varphi)_{H_0^1(\Omega)} = \int_\Omega f \varphi \, dx$$

for all functions $\varphi \in H_0^1(\Omega)$.

NEUMANN PROBLEM: Again formally, for smooth u, conside for given $f(x)$

$$\left. \begin{aligned} -\Delta u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned} \right\}$$

where $\frac{\partial}{\partial n}$ denotes the « outward pointing » normal to $\partial\Omega$.

For any $\varphi \in C^1(\Omega)$ we have

We call any function $u \in H^1(\Omega)$ which satisfies

$$(u, \varphi)_{H_0^1(\Omega)} = \int_\Omega f \varphi \, dx$$

for all $\varphi \in H_0^1(\Omega)$ a weak solution of the Neumann problem.

Note that the Dirichlet problem is a coercive problem whereas the Neumann problem is not coercive. Also, we know that the variational boundary value problem

$$a(u, v - u) = \langle f, v - u \rangle \qquad \text{for all } v \in V,$$

V a Hilbert space, has a solution u if $a(u, v)$ is coercive. Therefore we consider the new operator associated with the Dirichlet and Neumann problems: $-\Delta + \lambda$. We wish to find conditions on λ to guarantee coerciveness for each of these problems.

DIRICHLET PROBLEM: $-\Delta u + \lambda u = f$, $u = 0$ on $\partial\Omega$. To test for coerciveness, we have

$$a(v, v) = \sum_{i=1}^{n} \int_\Omega v_{x_i} v_{x_i} \, dx + \lambda \int_\Omega v^2 \, dx.$$

CLAIM: For $\lambda > -\dfrac{1}{\beta}$ we have coerciveness (this β comes from the Poincaré inequality). For such λ, we have for $0 < t < 1$

$$\int_\Omega |v_x|^2 dx + \lambda \int_\Omega v^2 dx = t \int_\Omega |v_x|^2 dx + (1 - t) \int_\Omega |v_x|^2 dx + \lambda \int_\Omega v^2 dx =$$

$$\geq t \int_\Omega |v_x|^2 dx + \left(\frac{1}{\beta} + \lambda - \frac{t}{\beta}\right) \int_\Omega v^2 dx.$$

Choose $0 < t < 1$ so that $t < \beta\left(\dfrac{1}{\beta} + \lambda\right) = 1 + \lambda\beta > 0$. Then there is a constant $\alpha > 0$ such that

$$\int_\Omega |v_x|^2 dx + \lambda \int_\Omega v^2 dx \geq \alpha \| v \|^2_{H_0^1(\Omega)}.$$

NEUMANN PROBLEM: $-\Delta u + \lambda u = f$, $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$. T
the coerciveness we have

$$a(v, v) = \sum_{i=1}^{n} \int_\Omega v_{x_i}^2 \, dx + \lambda \int_\Omega v^2 \, dx.$$

Since v does not necessarily have compact support in Ω we
$\lambda > 0$ and take $\alpha = \text{Min}(1, \lambda)$ to get

$$\int_\Omega |v_x|^2 dx + \lambda \int_\Omega v^2 dx \geq \alpha \| v \|^2_{H^1(\Omega)}.$$

Let us now consider the « mixed problem ». Suppose

$$\partial\Omega = (\partial_1 \Omega) \cup (\partial_2 \Omega) \quad \text{with} \quad (\partial_1 \Omega) \cap (\partial_2 \Omega) = \varnothing.$$

We wish to solve

$$\left.\begin{array}{ll} -\Delta u + \lambda u = f & \text{in } \Omega \\ u = 0, & \text{on } \partial_1 \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial_2 \Omega \end{array}\right\},$$

The operator $-\Delta + \lambda$ associated with this problem is coer
$\lambda > 0$. If $\lambda = 0$ it is coercive if $\partial_1\Omega$ is large enough; that i
enough to guarantee the existence of a $\beta > 0$ such that

$$\int_\Omega v^2 dx \leq \beta \int_\Omega |v_x|^2 dx$$

for all $v \in H^1(\Omega)$ with $v = 0$ on $\partial_1\Omega$.

Let us state formally some problems of variational ineq
Let E be a closed subset of the domain Ω. Let $\psi(x)$
ned on E. We then define

$$\mathbb{K} = \{ v \in H_0^1(\Omega) \mid v(x) \geq \psi(x) \quad \text{for} \quad x \in E \}.$$

$\mathbb{K}$ is a convex subset of $H_0^1(\Omega)$. We then consider the bilinear [illegible] form

$$a(u,v) = \int_\Omega u_{x_i}\, v_{x_i}\, dx.$$

The corresponding variational inequality becomes

$$u \in \mathbb{K}, \quad \int_\Omega u_{x_i}(v-u)_{x_i}\, dx \geq 0 \quad \text{for all} \quad v \in \mathbb{K}$$

Let α be any bounded function on Ω with $(\operatorname{Supp}\alpha) \cap E = \emptyset$. Then we can set $v = u \pm \alpha \in \mathbb{K}$ to obtain

$$\int_\Omega u_{x_i}\alpha_{x_i}\, dx \geq 0$$

or, since both $\pm\alpha$ satisfy this inequality,

$$\int_\Omega u_{x_i}\alpha_{x_i}\, dx = 0.$$

Formally, this means that in the sense of distributions u satisfies $-\Delta u = 0$ in $\Omega - E$.

If $(\operatorname{Supp}\alpha) \cap E \neq \emptyset$ and $\alpha \geq 0$, we get the variational inequality

$$\int_\Omega u_{x_i}\alpha_{x_i}\, dx \geq 0.$$

Therefore, in the sense of distributions, u satisfies

$$-\Delta u \geq 0;$$

that is, u is a super-solution in E.

In other words, the solution u of the previous problem is a function $u \in \mathbb{K}$ *such that* $-\Delta u \geq 0$ *with* $-\Delta u = 0$ *whenever* $u > \psi$.

NOTE: In the case where $\psi(x) \equiv 1$ on E, we call the corresponding solution the capacity potential or equilibrium potential.

Now consider $a(u,v)$ to be any bilinear form on $H_0^1(\Omega)$ which is coercive; that is, for some $\alpha > 0$

$$a(v,v) \geq \alpha \| v \|^2_{H_0^1} = \alpha \| v_x \|_2^2.$$

DEFINITION 5.1: Let $\psi(x) \in C^0(\overline{\Omega})$ and let E be a closed su of Ω. We say $v \geq \psi$ *on* E *in the sense of* $H^{1,p}$ $(1 \leq p < +\infty$ there is a sequence, $\{v_n\}$ with $v_n \in C^1(\overline{\Omega})$ such that $v_n \to$ $H^{1,p}(\Omega)$ and $v_n \geq \psi$ on E.

NOTE: If $\psi(x) \in H_0^{1,p}(\Omega)$, then we say $v \geq \psi$ on E in sense of $H_0^{1,p}(\Omega)$ if there is a sequence, $\{v_n\}$ with $v_n \in C_0^1(\Omega)$ s that $v_n \to v - \psi$ in $H_0^{1,p}(\Omega)$ and $v_n \geq 0$ on E.

Consider the case $p = 2$. Any function of $H^1(\Omega)$ can be defi at all points of Ω except possibly on a set of capacity zero.

$$\operatorname{Cap}(E) = \inf\left\{ \int_{\mathbb{R}^N} |\alpha_x|^2\, dx \;\middle|\; \alpha \in C_0^1(\mathbb{R}^N),\ \alpha \geq 1 \text{ on } E \right\}.$$

Clearly, capacity has the following properties:

i) $\operatorname{Cap}(E) \geq 0$ for all sets E.

ii) $E_1 \subset E_2$ implies $\operatorname{Cap} E_1 \leq \operatorname{Cap} E_2$.

iii) $\operatorname{Cap}(E_1 \cup E_2) \leq \operatorname{Cap}(E_1) + \operatorname{Cap}(E_2)$.

Then $v \geq \psi$ in the sense of $H^1(\Omega)$ means that $v \geq \psi$ except or set of capacity zero.

2. Now let $\mathbb{K} = \{ v \in H_0^1(\Omega) \mid v \geq \psi$ on $E\}$ where, if $\psi \in H^1($ we take $v - \psi \geq 0$ on E. Then clearly $\mathbb{K}$ is a closed, convex in $H_0^1(\Omega)$.

Let us define the bilinear form $a(u,v)$ by

$$a(u,v) = \int_\Omega a_{ij}\, u_{x_i} v_{x_j}\, dx$$

where

$$a_{ij}(x)\xi_i\xi_j \geq v|\xi|^2,\ v > 0 \quad \text{for all } \xi \in \mathbb{R}^N - \{0\} \quad \text{for almost all } x \in$$

$$|a_{ij}(x)| \leq M \quad \text{almost everywhere in}$$

By the coerciveness of $a(u,v)$ in $H_0^1(\Omega)$, there is a $u \in \mathbb{K}$ such th

$$a(u, v-u) \geq 0 \quad \text{for all } v \in \mathbb{K}$$

This implies that

Let us formally define

$$I(u) \equiv \{x \in E \mid u(x) = \psi(x)\}.$$

This is meant formally since we do not know as yet what $u(x) = \psi(x$ means for two functions in $H_0^1(\Omega)$. Once $I(u)$ is defined we must ask what $I(u)$ is and what $I(u)$ looks like.

DEFINITION 5.2: $I(u) \equiv \{x \in E \mid u(x) = \psi(x)\}$ is the complement in E of the set where $u > \psi$; that is, $u > \psi$ at x_0 if there is a neighborhood $\{x \mid |x - x_0| \leq 2\varrho\} = U_{2\varrho}(x_0)$, $\varrho > 0$, and a function $\alpha(x) \in C^1(\mathbb{R}^N)$, $\alpha \geq 0$ with

$$\alpha(x) \equiv 0 \qquad |x - x_0| > 2\varrho$$

$$\alpha(x) > 0 \qquad |x - x_0| < \varrho$$

such that $u - \alpha \geq \psi$ on E.

Note that $\Omega - I(u)$ is open. Therefore, $I(u)$ is closed in Ω. Recall from § 1: $\mathbb{K}_u \equiv \{w \in H_0^1(\Omega) \mid w = \varepsilon(v - u)\}$ for some $\varepsilon > 0$ and some $v \in \mathbb{K}\}$.

$\mathbb{K}_u$ is a convex cone; that is, if $\xi, \eta \in \mathbb{K}_u$ then $\xi + \eta \in \mathbb{K}_u$.

Remember that $\mathbb{K} = \{v \in H_0^1(\Omega) \mid v \geq \psi$ on $E\}$. Clearly $\overline{\mathbb{K}}_u$ is closed and convex. Thus, we need only show $\mathbb{K}_u$ is a cone. Let $\xi, \eta \in \mathbb{K}_u$. Then $\xi = \varepsilon_1(v_1 - u)$ and $\eta = \varepsilon_2(v_2 - u)$ for some $\varepsilon_1, \varepsilon_2 > 0$ and $v_1, v_2 \in \mathbb{K}$. Therefore

$$\frac{1}{\varepsilon_1 + \varepsilon_2}\xi + \frac{1}{\varepsilon_1 + \varepsilon_2}\eta = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}v_1 + \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}v_2 - u,$$

or

$$\left(\frac{1}{\varepsilon_1 + \varepsilon_2}\right)(\xi + \eta) = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}v_1 + \left(1 - \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right)v_2 - u.$$

Therefore, $\xi + \eta = (\varepsilon_1 + \varepsilon_2)\left\{\left[\left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right)v_1 + \left(1 - \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right)v_2\right] - u\right\}$.

Since $\varepsilon_1, \varepsilon_2 > 0$, we have $0 < \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2} < 1$ and $\varepsilon_1 + \varepsilon_2 > 0$. Since $\mathbb{K}$ is convex, $v \equiv \left(\frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right)v_1 + \left(1 - \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right)v_2 \in \mathbb{K}$. Moreover, $\varepsilon \equiv \varepsilon_1 + \varepsilon_2 > 0$. Thus $\xi + \eta = \varepsilon\{v - u\}$ and, hence, $\xi + \eta \in \mathbb{K}_u$.

Suppose $u > \psi$ at x_0; that is, $x_0 \notin I(u)$. Then there is a neighborhood $U_\varrho(x_0)$ such that

$$H_0^1(U_\varrho(x_0)) \subset \mathbb{K}_u. \tag{5.1}$$

Since $x_0 \notin I(u)$, there is an $\alpha(x) \in C_0^1(\mathbb{R}^N)$ with the properties given in the previous definition 5.2 for some $\varrho > 0$. Now let $\zeta \in C^1(\Omega)$ with support in $\{x \mid |x - x_0| < \varrho\}$ with $0 \leq \zeta \leq \alpha$. Then $\psi \leq u - \alpha \leq u - \zeta$ on E. Hence $u - \zeta \in \mathbb{K}$ and, hence, $-\zeta = (u - \zeta) - u \in \overline{\mathbb{K}}_u$.

Now let $\eta \in C_0^1(U_\varrho(x_0))$, then there is an $\varepsilon > 0$ such that $-\varepsilon\eta < \alpha$ and, hence, $\varepsilon\eta \in \mathbb{K}_u$. Therefore, $\eta \in \mathbb{K}_u$. By closure we get $H_0^1(U_\varrho(x_0)) \subset \overline{\mathbb{K}}_u$.

Moreover, we have:

$$\{w \in H_0^1(\Omega) \mid w \geq 0 \text{ on } \Omega\} \subset \overline{\mathbb{K}}_u. \tag{5.2}$$

For, define $v = u + \frac{1}{\varepsilon}w$ for any $\varepsilon > 0$ and $w \in H_0^1(\Omega)$ with $w \geq 0$ on Ω. Then, since $u \geq \psi$ on E, we have $v = u + \frac{1}{\varepsilon}w \geq u \geq \psi$ on E. Hence, $v \in \mathbb{K}$. Therefore, solving for w, we get for any $\varepsilon > 0$, $w = \varepsilon(v - u)$ where $v \in \mathbb{K}$. Hence $w \in \mathbb{K}_u$.

It follows from (5.1) and (5.2) that any function $w \in H_0^1(\Omega)$ such that $w \geq 0$ on $I(u)$ is in $\overline{\mathbb{K}}_u$. For we can write

$$w = \max(w, 0) + \min(w, 0),$$

where $0 \leq \max(w, 0) \in H_0^1(\Omega)$ and $\min(w, 0) \in H_0^1(\Omega)$ with support of $\min(w, 0)$ in $\Omega - I(u)$, an open set. Therefore, from (5.1) and (5.2), respectively, since $\mathbb{K}_u$ is a cone, we have $w \in \overline{\mathbb{K}}_u$.

Let $C_{I(u)} \equiv \{w \in H_0^1(\Omega) \mid w \geq 0$ on $I(u)\}$. We now return to the bilinear form

$$a(u, v) \equiv \int_\Omega a_{ij}(x)\, u_{x_i} v_{x_j}\, dx$$

as described above.

From (5.1) we conclude $H_0^1(\Omega - I(u)) \subset \mathbb{K}_u$. Therefore, we have

$$a(u, w) \geq 0 \qquad \text{for all } w \in H_0^1(\Omega - I(u))$$

and, hence,

$$a(u, w) = 0 \qquad \text{for all } w \in H_0^1(\Omega - I(u)).$$

If $(\operatorname{supp} w) \cap I(u) \neq \emptyset$, then it follows from (5.2) that we can only say

$$a(u, w) \geq 0 \qquad w \in C_{I(u)}.$$

It follows from this discussion that, for fixed $u \in H^1(\Omega)$, $a(u, w)$ represents a non-negative linear functional on $H_0^1(\Omega)$. It follows from a well known theorem of F. Riesz that there is a non-negative measure μ such that

$$a(u, w) = \int w \, d\mu \qquad \text{for all } w \in H_0^1(\Omega).$$

It follows that

$$\operatorname{supp} \mu \subset I(u).$$

Suppose $w \in H_0^1(\Omega)$ is arbitrary with support in $\Omega - I(u)$. We have shown for such w that

$$0 = a(u, w) = \int w \, d\mu.$$

Therefore, $\operatorname{supp} \mu \subset I(u)$.

The solution of our problem satisfies the following problem: let $u \in \mathbb{K}$ be a solution to $a(u, w) \geq 0$ for all $w \in \mathbb{K}_u$. Then there is a non-negative measure μ such that

$$\int_\Omega a_{ij}(x) u_{x_i} w_{x_j} \, dx = \int_\Omega w \, d\mu \qquad \text{for all } w \in \mathbb{K}_u$$

where $\operatorname{supp} \mu \in I(u)$.

If we define the operator L by

$$Lu = -\{a_{ij} u_{x_i}\}_{x_j},$$

then this fact can be stated in the equivalent form:

Let $u \in \mathbb{K}$ be a solution to the variational inequality $a(u, w) \geq 0$ for all $w \in \mathbb{K}_u$. Then there is a non-negative measure μ with $\operatorname{supp} \mu \subset I(u)$ such that

$$Lu = \mu \qquad \text{(in the sense of distributions).}$$

DEFINITION 5.3: Suppose $u \in H^1(\Omega)$ satisfies

$$\int_\Omega a_{ij}(x) u_{x_i} w_{x_j} \, dx \geq 0 \text{ for all } 0 \leq w \in H_0^1(\Omega).$$

Then u is called a *super-solution with respect to the operator* L.

THEOREM 5.1. *Let g be any super-solution in $H^1(\Omega)$ with respect to L such that $g \geq \psi$ on E and $g \geq 0$ on $\partial\Omega$.*

Then $u \leq g$ a.e. in Ω where u is a solution of the problem

$$\int_\Omega a_{ij}(x) u_{x_i} (v - u)_{x_j} \, dx \geq 0 \text{ for all } v \in \mathbb{K}.$$

PROOF: Let $\zeta = \min(u, g)$. Since $u \in H_0^1(\Omega)$ and $g \in H^1(\Omega)$, we have $\zeta \in H_0^1(\Omega)$. Moreover, since $u \geq \psi$ and $g \geq \psi$ on E, $\zeta \in \mathbb{K}$. Let $v = \zeta$. Then

$$\int_\Omega a_{ij} u_{x_i} (\zeta - u)_{x_j} \, dx \geq 0.$$

But g is a super-solution implying that

$$\int_\Omega a_{ij} g_{x_i} w_{x_j} \, dx \geq 0 \qquad \text{for all } w \geq 0, \; w \in H_0^1(\Omega).$$

Set $w = \zeta - u \leq 0$. Then

$$\int_\Omega a_{ij} g_{x_i} (\zeta - u)_{x_j} \, dx \leq 0.$$

Using the coerciveness of $\mathcal{a}(u, v)$, we get

$$0 \leq \alpha \| \zeta - u \|^2_{H_0^1} \leq \int_\Omega a_{ij} (\zeta - u)_{x_i} (\zeta - u)_{x_j} \, dx =$$

$$= \int_\Omega a_{ij} \zeta_{x_i} (\zeta - u)_{x_j} \, dx - \int_\Omega a_{ij} u_{x_i} (\zeta - u)_{x_j} \, dx$$

But $\zeta = g$ if $\zeta - u \neq 0$.

Thus

$$\alpha \| \zeta - u \|^2_{H_0^1} \leq \int_\Omega a_{ij} g_{x_i} (\zeta - u)_{x_j} dx - \int_\Omega a_{ij} u_{x_i} (\zeta - u)_{x_j} dx \leq 0.$$

Therefore $\| \zeta - u \|_{H_0^1} = 0$ and, hence, $\zeta = u$ a. e. . It follows that $u \leq g$ a. e. in Ω.

COROLLARY 5.1: *If $\psi \leq N$, then $u \leq N$.*

PROOF: Since $g \equiv N$ is a solution, this follows immediately.

COROLLARY 5.1′: *The solution u of the variational inequality $a(u, w) \geq 0$ for all $w \in \mathbb{K}_u$ is $u(x) = \mathrm{Min}\, g(x)$ {g satisfies the hypotheses of the theorem 5.1}.*

NOTE: The unique solution u of the coercive bilinear form

$$a(u, v) \equiv \int_\Omega a_{ij}(x)\, u_{x_i} v_{x_j}\, dx$$

gives rise to a non-negative measure μ with support in $I(u)$ such that

$$a(u, w) = \int_\Omega w\, d\mu \qquad \text{for all } w \in H_0^1(\Omega).$$

The measure μ is unique from the uniqueness of the solution u. From our statements we have: supp $\mu \subset I(u) \subset E$.

Thus, if A is any set in $\Omega - E$, then $\mu(A) = 0$.

Suppose Cap $E = 0$. Then the solution of our problem is $u \equiv 0$. For, if $w \in C_0^1(\Omega)$ and $w = 1$ on E, then

$$\int_\Omega d\mu \leq \text{const } \| u_x \|_{L^2(\Omega)} \cdot \| w \|_{L^2(\Omega)}$$

and therefore, since the infimum of the right side is zero, it follows that $\int d\mu = 0$, and thus u is solution of the Dirichlet problem:

$$Lu = 0, \text{ with } u \in H_0^1(\Omega).$$

3. In order to prove that the solution of the problem consid red in 2. is smooth, we set down the following assumptions:

i) *a.* Assume Ω is smooth; say $\partial\Omega \in C^2$.

b. Assume the a_{ij} are smooth; say, $a_{ij} \in C^1(\overline{\Omega})$ since fo mally we could differentiate to obtain

$$\{a_{ij}(x)\, u_{x_i}\}_{x_j} = a_{ij}(x)\, u_{x_i x_j} + \frac{\partial a_{ij}(x)}{\partial x_j} u_{x_i}$$

with all coefficients at least continuous on $\overline{\Omega}$; $a_{ij} \xi_i \xi_j \geq \nu |\xi|^2$.

c. Assume that Ω and the a_{ij} are so smooth that we ha

$$\| u_{xx} \|_{L^p} + \| u_x \|_{L^p} + \| u \|_{L^p} \leq C(\Omega, p) \| f \|_{L^p}$$

if we consider $Lu = -\{a_{ij} u_{x_i}\}_{x_j} = f$ in Ω, $u = 0$ on $\partial\Omega$.

ii) Let $\psi \in H^{2,p}(\Omega)$ for some $p > N$, where $\Omega \subset \mathbb{R}^N$. Th $\psi \in C^{1,\alpha}(\Omega)$ where $\alpha = 1 - \dfrac{N}{p}$.

Also assume $\psi < 0$ on $\partial\Omega$.

Recall that our variational inequality is the following: let

$$\mathbb{K} = \{v \in H_0^1(\Omega) \mid v \geq \psi\}.$$

To find $u \in \mathbb{K}$ such that $a(u, v - u) \geq 0$ for all $v \in \mathbb{K}$ where

$$a(u, v) \equiv \int_\Omega a_{ij}(x)\, u_{x_i} v_{x_j}\, dx.$$

We are going to prove the following.

The assumptions that we actually have, imply that the soluti u satisfies $u \in H^{2,p}$ and, hence, $u \in C^{1,\alpha}(\Omega)$ with $\alpha = 1 - \dfrac{N}{p}$.

We will construct a solution u by considering a certain no linear problem and then show that this solution is actually a s lution of the variational inequality which we are considering.

By previous considerations we have shown that the solution actually a super-solution in Ω with u satisfying

$$Lu = 0 \quad \text{if } u > \psi$$

where we note that $u \in H_0^1(\Omega)$.

Let us define the function $\theta = \theta(s)$ by

$$\theta(s) = \begin{Bmatrix} 1 & \text{if } s \leq 0 \\ 0 & \text{if } s > 0 \end{Bmatrix}.$$

We will smooth θ by considering a sequence of function $\{\theta_n\}$, each of which is continuous, which converges pointwise to θ.

Let us define

$$\theta_n(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ -ns + 1 & \text{if } 0 \leq s \leq \frac{1}{n} \\ 0 & \text{if } s > \frac{1}{n} \end{cases}$$

Therefore we consider the problem

$$Lu = \operatorname{Max}[L\psi, 0]\, \theta(u - \psi) \qquad u \in H_0^1(\Omega)$$

and its approximations,

$$Lu_0^{(n)} = \operatorname{Max}[L\psi, 0]\, \theta_n(u^{(n)} - \psi) \qquad u^{(n)} \in H_0^1(\Omega).$$

We first consider the case where θ is a Lipschitz continuous function.

THEOREM 5.2: *Let θ be a Lipschitz continuous function with $0 \leq \theta \leq 1$. Let $u = 0$ on $\partial\Omega$. Then there is a $u \in H^{2,p}(\Omega)$ such that*

(5.3) $$Lu = \operatorname{Max}(L\varphi, 0)\, \theta(u - \psi).$$

PROOF: We look for $u \in H_0^1(\Omega)$ such that

(5.3′) $$\int_\Omega a_{ij} u_{x_i} v_{x_j}\, dx = \int_\Omega \operatorname{Max}(L\psi, 0)\, \theta(u - \psi)\, v\, dx$$

for all $v \in H_0^1(\Omega)$. For each $w \in H_0^1(\Omega)$ we consider the problem of finding $u \in H_0^1(\Omega)$ such that

$$\int_\Omega a_{ij} u_{x_i} v_{x_j}\, dx = \int_\Omega \operatorname{Max}[L\psi, 0]\, \theta(w - \psi)\, v\, dx.$$

But L is a coercive operator on $H_0^1(\Omega)$ and, therefore, there is a solution $u \in H_0^1(\Omega)$ of this equation. This defines a map

$$T: H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$$

given by

$$u = Tw.$$

Setting $v = u$ and noting that $\theta(w - \psi) \leq 1$, we have

$$\nu \| u_x \|_2^2 \leq \int_\Omega a_{ij} u_{x_i} u_{x_j}\, dx \leq \int_\Omega \operatorname{Max}[L\psi, 0]\, | u |\, dx$$

$$\leq \left\{ \int_\Omega [\operatorname{Max}(L\psi, 0)]^2\, dx \right\}^{1/2} \left\{ \int_\Omega | u |^2\, dx \right\}^{1/2}$$

$$\leq \mathcal{C} \left\{ \int_\Omega [\operatorname{Max}(L\psi, 0)]^2\, dx \right\}^{1/2} \| u_x \|_2 .$$

Since $|\Omega| < \infty$, we have used Poincaré's inequality. Hence

$$\| u \|_{H_0^1(\Omega)} \leq \mathcal{C}(\nu, \Omega, p)\, \| L\psi \|_p .$$

Let

$$\Sigma = \{ v \in H_0^1(\Omega) \mid \| v \|_{H_0^1(\Omega)} \leq \mathcal{C}(\nu, \Omega, \varrho)\, \| L\psi \|_p \}.$$

Then $T: H_0^1(\Omega) \to \Sigma$ and, therefore, $T: \Sigma \to \Sigma$. T maps a convex set Σ of $H_0^1(\Omega)$ in itself.

We show that T is continuous. Let $u_n = Tw_n$, $u = Tw$, and suppose $w_n \to w$ in $H_0^1(\Omega)$. We wish to show $u_n \to u$ in $H_0^1(\Omega)$. We have

$$\int_\Omega a_{ij} (u_n)_{x_i} v_{x_j}\, dx = \int_\Omega \operatorname{Max}(L\psi, 0)\, \theta(w_n - \psi)\, v\, dx$$

and

$$\int_\Omega a_{ij} u_{x_i} v_{x_j}\, dx = \int_\Omega \operatorname{Max}(L\psi, 0)\, \theta(w - \psi)\, v\, dx.$$

Hence

$$\int_\Omega a_{ij} (u_n - u)_{x_i} v_{x_j}\, dx = \int_\Omega \operatorname{Max}(L\psi, 0)\, [\theta(w_n - \psi) - \theta(w - \psi)]\, v\, dx$$

$$\leq K \int_\Omega \operatorname{Max}(L\psi, 0)\, | (w_n - \psi) - (w - \psi) |\, | v |\, dx$$

$$= K \int_\Omega \operatorname{Max}(L\psi, 0)\, | w_n - w |\, | v |\, dx.$$

where K is the Lipschitz constant of θ.

Now let $v = u_n - u$. Also $\text{Max}\,(L\psi, 0) \in L^p$ for $p > N$. Thus

$$\int_\Omega a_{ij}(u_n - u)_{x_j}(u_n - u)_{x_j}\,dx \le K \int_\Omega \text{Max}\,(L\psi, 0)\,|w_n - w|\,|u_n - u|\,dx$$

$$\le K' \left\{\int_\Omega [\text{Max}\,(L\psi, 0)]^p\,dx\right\}^{\frac{1}{p}} \left\{\int_\Omega |w_n - w|^2\,dx\right\}^{\frac{1}{2}} \left\{\int_\Omega |u_n - u|^{2^*}\,dx\right\}^{\frac{1}{2^*}},$$

where
$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N} < \frac{1}{2} - \frac{1}{p}.$$

Using Sobolev inequality, we have

$$\nu\,\|(u_n - u)_x\|_2^2 \le K'\,\|\text{Max}\,(L\psi, 0)\|_p\,\|w_n - w\|_2 \cdot \mathcal{S}\,\|(u_n - u)_x\|_2 .$$

Thus,

$$\|(u_n - u)_x\|_2 \le \frac{K\mathcal{S}}{\nu}\,\|\text{Max}\,(L\psi, 0)\|_p\,\|w_n - w\|_2 .$$

Using Poincaré again, we have

$$\|u_n - u\|_{H_0^1(\Omega)} \le \mathcal{C}(\nu, \Omega, \mathcal{S}, K)\,\|\text{Max}\,(L\psi, 0)\|_p\,\|w_n - w\|_{H_0^1(\Omega)} .$$

Hence, since $w \to w$ in $H_0^1(\Omega)$, we have $u_n \to u$ in $H_0^1(\Omega)$.

From the assumption i), ii) it follows that $Tw \in H^{2,p}$ and thus T is a compact map. Therefore, from Schauder fixed point, it follows that T has a fixed point u; i.e. $u = Tu$ and hence (5.3′) holds. This proves theorem 5.2.

We now return to the general problem posed before theorem 5.2:

$$u \in H_0^1(\Omega) : Lu = \text{Max}\,[L\psi, 0]\,\theta\,(u - \psi)$$

where
$$\theta(s) = \begin{cases} 1 & s \le 0 \\ 0 & s > 0 \end{cases}.$$

Letting θ_n be as defined above, we know by the theorem 5.2 that there is a solution $u^{(n)}$ of

$$u^{(n)} \in H_0^1(\Omega) : Lu^{(n)} = \text{Max}\,[L\psi, 0]\,\theta_n\,(u^{(n)} - \psi).$$

We claim that: $u^{(n)} \ge \psi$. In fact we have

$$Lu^{(n)} - L\psi = \text{Max}\,[L\psi, 0]\,\theta_n\,(u^{(n)} - \psi) - L\psi.$$

For contradiction, we assume there is an $x_0 \in \Omega$ such that $u^{(n)}($
$< \psi(x_0)$. Then there is an open set A such that $u^{(n)}(x) < \psi($
A with $u^{(n)} = \psi$ on ∂A. Considering $L(u^{(n)} - \psi) = Lu^{(n)} - L$
A, we have $u^{(n)} - \psi = 0$ on ∂A.

Moreover, since $u^{(n)} \le \psi$ on A, $\theta_n(u^{(n)} - \psi) = 1$ and,
if $L\psi \ge 0$ we have $L(u^{(n)} - \psi) = 0$ and if $L\psi \le 0$ we
$L(u^{(n)} - \psi) = -L\psi \ge 0$. Therefore, by the maximum pri
$(u^{(n)} - \psi)(x) \ge \text{Min}\,(\{(u^{(n)} - \psi)(y) \mid y \in \partial A\} = 0$, for all $x \in A$.
refore, $u^{(n)}(x) \ge \psi(x)$ on A contradicting the assumption $u^{(n)}$
$< \psi(x_0)$. Therefore $u^{(n)}(x) \ge \psi(x)$ on Ω.

Now since $\|u^{(n)}\|_{H^{2,p}(\Omega)} \le \mathcal{C}$, there is a subsequence $\{u^{(n)}\}$ that

$$u^{(n')} \to u \text{ strongly in } C^{1,\alpha} \qquad (\alpha = 1 -$$

$$u^{(n')} \to u \text{ strongly in } H^{2,p}$$

Clearly $u \ge \psi$ since $u^{(n)} \ge \psi$ for all n. Therefore, we nee
show that

$$Lu = \text{Max}\,(L\psi, 0)\,\theta\,(u - \psi)$$

in the variational sense; that is

$$0 = \int_\Omega a_{ij} u_{x_j} v_{x_j} - \text{Max}\,(L\psi, 0)\,\theta\,(u - \psi)\,v\}\,dx$$

for all $v \in H_0^1(\Omega)$.

We first show that the above operator L is monotone
Since L is monotone we have $(Lu - Lv, u - v) \ge 0$.

Since θ is increasing

$$(\theta(t') - \theta(t''), t' - t'') \le 0$$

for all t', t''. Therefore,

$$(Lu - \text{Max}\,(L\psi, 0)\,\theta\,(u - \psi) - Lv + \text{Max}\,(L\psi, 0)\,\theta\,(v - \psi), u -$$

$$= (Lu - Lv, u - v) - (\text{Max}\,(L\psi, 0)\,[\theta\,(u - \psi) - \theta\,(v - \psi)], u -$$

$$\ge -\int_\Omega \text{Max}\,(L\psi, 0)\,[\theta\,(u - \psi) - \theta\,(v - \psi)]\,(u - v)\,dx \ge 0$$

since $\text{Max}\,(L\psi, 0) \geq 0$. It follows that all of the approximating operators are monotone. But, since, for each n, θ_n is Lipschitz continuous, there is a solution $u^{(n)}$ of

$$\int_\Omega \{a_{ij}(u^{(n)})_{x_i}(v - u^{(n)})_{x_j} - \text{Max}\,(L\psi, 0)\,\theta_n\,(u^{(n)} - \psi)\,(v - u^{(n)})\}\,dx = 0,$$

for all $v \in H_0^1(\Omega)$. Since the operators involved are monotone, from lemma 2.2 we have

$$\int_\Omega \{a_{ij}\,v_{x_i}(v - u^{(n)})_{x_j} - \text{Max}\,(L\psi, 0)\,\theta_n\,(v - \psi)\,(v - u^{(n)})\}\,dx = 0$$

for all $v \in H_0^1(\Omega)$.

Suppose $v > \psi$, then clearly $\theta_n(v - \psi) \to 0$ as $n \to \infty$ and, from the dominated convergence theorem since

$$\text{Max}\,(L\psi, 0) \in L^p(\Omega) \text{ and } v, u^{(n)} \in H_0^1(\Omega)$$

we have

$$\lim_{n\to\infty} \int_\Omega \text{Max}\,(L\psi, 0)\,\theta_n(v - \psi)\,(v - u^{(n)})\,dx = 0.$$

Since we no longer permit all $v \in H_0^1(\Omega)$ we thus have

$$\lim_{n\to\infty} \int_\Omega a_{ij}\,v_{x_i}(v - u^{(n)})_{x_j}\,dx = \int_\Omega a_{ij}\,v_{x_i}(v - u)_{x_j}\,dx \geq 0.$$

Thus, applying lemma 2.2 again, we have

$$\int_\Omega a_{ij}\,u_{x_i}(v - u)_{x_j}\,dx \geq 0.$$

For any $v \geq \psi$ we let $\{v_n\}$ be a sequence of $H_0^1(\Omega)$ functions, such that $v_n > \psi$ and $v_n \to v$ in $H_0^1(\Omega)$. Then we have

$$\int a_{ij}\,u_{x_i}(v_n - u)_{x_j}\,dx \geq 0$$

for all n. Thus, again we get

$$\int_\Omega a_{ij}\,u_{x_i}(v - u)_{x_j}\,dx \geq 0.$$

Therefore,

$$\int_\Omega a_{ij}\,u_{x_i}(v - u)_{x_j}\,dx \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

§ 6. An abstract theorem of regularity.

1. In this section let us define the following:

V a real reflexive Banach space.

V' the dual of V.

A a monotone, hemicontinuous operator with $A : V \to V'$.

$\mathbb{K}$ a closed convex subset of V.

$(\cdot,\cdot)$ the pairing between V' and V.

We have already proved the following theorem:

Let $f \in V'$ and assume $\mathbb{K}$ is bounded. Then there is a $u \in \mathbb{K}$ such that

$$(Au - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

In this section we shall be concerned with the following question: If W is a dense subset of V' with $\|\;\|_{V'} \leq C\,\|\;\|_W$, when does $f \in W$ imply that $Au \in W$ where u is a solution of the variational inequality?

It is clear that if $u \in \mathring{\mathbb{K}}$ the variational inequality implies $Au = f$ and the question is answered. Thus, the question actually reduces to the case when $u \in \partial\mathbb{K}$.

We formally note:

W a dense subset of V' with $\|\;\|_{V'} \leq C\,\|\;\|_W$ for some constant C.

W' the set of all linear functionals on W.

W, W' are reflexive Banach spaces which are strictly convex.

$\varphi = \varphi(r)$ a strictly increasing function of R^1_+ into R^1_+ with

$$\varphi(0) = 0 \text{ and } \varphi(r) \to +\infty \qquad \text{as } r \to +\infty.$$

The following theorem will be used.

THEOREM 6.1 *Letting W, W' and $(\,,)$ be as above. Then, there is a duality map $J: W \to W'$ such that*

i) $(Jw, w) = \varphi(\| w \|)\cdot\| w \|$ *for all* $w \in W$

ii) $\| Jw \|_{W'} = \varphi(\| w \|)$ *for all* $w \in W$.

It is clear that ii) implies that J is one to-one and, therefore, there is an inverse mapping $J^{-1}: W' \to W$.

What might this mapping J look like?

EXAMPLE *a*). Suppose $W = H$, a Hilbert space. Then we can take $\varphi(r) = r$ and define J by

$$\langle Jw, v \rangle = (w, v)_H$$

for $w \in H$ and $v \in H$. Here $\langle\,,\rangle$ is the pairing of H' and H and $(\,;)$ is the inner product in H.

Then, clearly,

$$\langle Jw, w \rangle = (w, w) = \| w \|\cdot\| w \|$$

and

$$\| Jw \|_H = \sup_{\|v\| \leq 1} \langle Jw, v \rangle = \sup_{\|v\| \leq 1} (w, v) = \| w \|_H .$$

EXAMPLE *b*). Suppose $W = L^p$, $p \geq 2$. Letting $\varphi(r) = r^{p-1}$, we define

$$J : L^p \to L^{p'}$$

by

$$Jw = | w |^{p-2} w.$$

For then

$$(Jw, w) = \int (| w |^{p-2} w)\, w \, dx = \int | w |^p dx = \| w \|^{p-1}_{L^p}\cdot\| w \|.$$

Also,

$$\| Jw \|_{L^{p'}} = \left[\int \| w |^{p-2} w |^{p'} dx\right]^{\frac{1}{p'}} = \left[\int | w |^{p'(p-1)} dx\right]^{\frac{1}{p'}} =$$

$$= \left(\left[\int | w |^p dx\right]^{\frac{1}{p}}\right)^{\frac{p}{p'}} = \| w \|^{p-1}_{L^p}$$

where $\dfrac{1}{p'} + \dfrac{1}{p} = 1$.

DEFINITION 6.1. The Triplet $\{u, A, \mathbb{K}\}$ where u is a solution of the variational inequality and A and $\mathbb{K}$ are as given above, is *J-compatible* if for each $\varepsilon > 0$ we have

i) There exist maps

$$B_\varepsilon : V \to W, \qquad C_\varepsilon : V \to W'$$

such that $\| B_\varepsilon v \|_W , \| C_\varepsilon v \|_{W'} \leq C$ for all $v \in V$.

ii) The solution u_ε of

$$u_\varepsilon + \varepsilon J(Au_\varepsilon + B_\varepsilon u_\varepsilon) = u + \varepsilon C u_\varepsilon$$

satisfies $u_\varepsilon \in \mathbb{K}$ and $Au_\varepsilon \in W$.

The following is one of the desired result of this section:

THEOREM 6.2. *Assume $\mathbb{K}$ is bounded and $f \in W$. Let $u \in \mathbb{K}$ be a solution of the variational inequality*

$$(Au - f, v - u) \geq 0 \qquad \textit{for all } v \in K.$$

If $\{u, A, \mathbb{K}\}$ is J-compatible, then $Au \in W$.

Let us recall the following lemma:

LEMMA 6.1 *Let $u_i \in V$ and assume $u_i \to u$ (weakly) in V. Assume $\lim_{i\to\infty} (Au_i, u - u_i) = 0$ where A is a monotone operator. Then*

$$Au_i \to Au \ (\textit{weakly}) \ \textit{in } V'.$$

PROOF. Let $v \in V$ and let $w = (1 - t) u + tv$ for $0 \leq t \leq 1$. From the monotonicity we have

$$0 \leq (Au_i - Aw, u_i - w) =$$

$$= (Au_i, u_i - u) - t(Au_i, v - u) - (A(u + t(v - u)), u_i - u + t(u - v)).$$

Letting $i \to \infty$ we see that the weak convergence implies

$$(A(u + t(v - u)), u_i - u + t(u - v)) \to t(A(u + t(v - u)), u - v).$$

Therefore, assuming $t > 0$,

$$t \limsup_{i\to\infty} (Au_i, v - u) \leq t(A(u + t(v - u)), v - u).$$

Dividing by t and then letting $t \to 0$, we get

$$\limsup_{i\to\infty} (Au_i, v-u) \leq (Au, v-u).$$

Since this is true for all v, it is also true for $\bar{v} \equiv 2u - v$. Then, since $u - \bar{v} = v - u$, we have

$$\limsup_{i\to\infty} -(Au_i \bar{v} - u) = \limsup_{i\to\infty} (Au_i, v-u) \leq$$

$$\leq (Au, v-u) = -(Au, \bar{v} - u).$$

and, hence,

$$(Au, \bar{v} - u) \leq \liminf_{i\to\infty} (Au_i, \bar{v} - u).$$

Therefore,

$$\lim_{i\to\infty} (Au_i, v-u) = (Au, v-u)$$

and, thus

$$\lim_{i\to\infty} (Au_i, z) = (Au, z) \qquad \text{for all } z \in V.$$

Proof of Theorem 6.2. By lemma 2.2 u is a solution of the variational inequality

$$u \in \mathbb{K}, \ (Au - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

if and only if

$$u \in \mathbb{K}, \ (Av - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

Therefore, assuming the J-compatibility, let us take $v = u$. Then, since $u_\varepsilon - u = \varepsilon C_\varepsilon u_\varepsilon - \varepsilon J(Au_\varepsilon + B_\varepsilon u_\varepsilon)$, we see that

$$0 \leq (Au_\varepsilon - f, C_\varepsilon u_\varepsilon - J(Au_\varepsilon + B_\varepsilon u_\varepsilon)) =$$

$$= (Au_\varepsilon + B_\varepsilon u_\varepsilon, C_\varepsilon u_\varepsilon - J(Au_\varepsilon + B_\varepsilon u_\varepsilon)) -$$

$$- (B_\varepsilon u_\varepsilon + f, C_\varepsilon u_\varepsilon - J(Au_\varepsilon + B_\varepsilon u_\varepsilon)).$$

Let $\xi_\varepsilon \equiv Au_\varepsilon + B_\varepsilon u_\varepsilon$. Then

$$0 \leq -(\xi_\varepsilon, J\xi_\varepsilon) + (\xi_\varepsilon, C_\varepsilon u_\varepsilon) - (B_\varepsilon u_\varepsilon + f, C_\varepsilon u_\varepsilon) + (B_\varepsilon u_\varepsilon + f, J\xi_\varepsilon).$$

ging terms, we get

$$(J\xi_\varepsilon, \xi_\varepsilon) \leq (J\xi_\varepsilon, B_\varepsilon u_\varepsilon + f) + (C_\varepsilon u_\varepsilon, \xi_\varepsilon - B_\varepsilon u_\varepsilon - f).$$

Therefore, since $\| B_\varepsilon u_\varepsilon \|, \| C_\varepsilon u_\varepsilon \| \leq C$, we have

$$\varphi(\|\xi_\varepsilon\|) \|\xi_\varepsilon\| \leq \varphi(\|\xi_\varepsilon\|)(\|f\| + C) + C(\|\xi_\varepsilon\| + C + \|f\|)$$

or

$$\varphi(\|\xi_\varepsilon\|)(\|\xi_\varepsilon\| - C_1) \leq C_2(\|\xi_\varepsilon\| + C_3).$$

We wish to show that $\|\xi_\varepsilon\|$ is uniformly bounded. If $\|\xi_\varepsilon\| \leq$ then we have done. If not, then

$$\varphi(\|\xi_\varepsilon\|) \leq C_2 \cdot \frac{1 + \dfrac{C_3}{\|\xi_\varepsilon\|}}{1 - \dfrac{C_1}{\|\xi_\varepsilon\|}}.$$

Clearly, as $\|\xi_\varepsilon\| \to \infty$, $\varphi(\|\xi_\varepsilon\|) \to C_2 < \infty$. This contradict property of φ and, therefore, there is a constant C_0 such that

$$\|\xi_\varepsilon\| \leq C_0, \qquad C_0 \text{ independent}$$

But then, for all $\varepsilon > 0$,

$$\|Au_\varepsilon\| \leq \|\xi_\varepsilon\| + \|B_\varepsilon u_\varepsilon\| \leq C_0 + C.$$

Therefore, there is a constant L, independent of ε, such that

$$\|Au_\varepsilon\|_W \leq L.$$

Now since J maps bounded sets into bounded sets, we see

$$u_\varepsilon - u = \varepsilon [C_\varepsilon u_\varepsilon - J(Au_\varepsilon + B_\varepsilon u_\varepsilon)] \equiv \varepsilon \theta_\varepsilon u_\varepsilon$$

where θ_ε maps bounded sets into bounded sets. Therefore,

$$u_\varepsilon \to u \text{ (strongly) in } W'.$$

Now, we wish to show $Au \in W$. Since $u_\varepsilon \in \mathbb{K}$, a bounded subse

It follows that there is a subsequence which converges weakly to some $\zeta \in V$; $u_{\varepsilon'} \longrightarrow \zeta$ (weakly) in V. Therefore

$$(u_{\varepsilon'}, z) \longrightarrow (\zeta, z) \qquad \text{for all } z \in V'.$$

But

$$(u_{\varepsilon}, z) \longrightarrow (u, z) \qquad \text{for all } z \in W.$$

Therefore, since $W \subset V'$

$$(\zeta, z) = (u, z) \qquad \text{for all } z \in W.$$

Since W is a dense subspace of V', we have $\zeta = u$. Since $u_{\varepsilon'} \longrightarrow u$ (weakly) in V' for ~~some~~ any subsequence, we have

$$u_{\varepsilon} \longrightarrow u \text{ (weakly) in } V.$$

Now we have shown that

$$\| Au_{\varepsilon} \|_W \leq L.$$

Thus, there is a subsequence $Au_{\varepsilon''}$ and an $\eta \in W$ such that

$$Au_{\varepsilon''} \longrightarrow \eta \qquad \text{(weakly) in } W.$$

But

$$|(Au_{\varepsilon}, u_{\varepsilon} - u)| \leq \| Au_{\varepsilon} \|_W \| u_{\varepsilon} - u \|_{W'} \leq L \| u_{\varepsilon} - u \|_{W'} \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

Therefore,

$$\lim_{\varepsilon \to 0} (Au_{\varepsilon}, u_{\varepsilon} - u) = 0.$$

Since the $u_{\varepsilon} \in V$ and $u_{\varepsilon} \longrightarrow u$ weakly in V, we have from lemma 6.1 that

$$Au_{\varepsilon} \longrightarrow Au \qquad \text{(weakly) in } V'.$$

Therefore, ~~$Au = \eta$ in W and~~ since W is dense in V', $Au = \eta$. Thus,

$$Au_{\varepsilon} \longrightarrow Au \qquad \text{(weakly) in } W.$$

Since W is reflexive and since $\{w \in W \mid \| w \|_W \leq L\}$ is bounded, $\{w \in W \mid \| w \|_W \leq L\}$ is weakly sequentially compact and, therefore, since $Au_{\varepsilon} \longrightarrow Au$ in W we have

$$Au \in W.$$

Let us now prove a result analogous to theorem 6.2 in the case where $\mathbb{K}$ is not bounded.

THEOREM 6.3. *Assume there is a $v_0 \in \mathbb{K}$ such that*

$$\frac{(Av, v - v_0)}{\| v \|} \longrightarrow +\infty \qquad \text{as } \| v \| \longrightarrow +\infty,\ v \in \mathbb{K}.$$

If $f \in W$ and if $u \in \mathbb{K}$ is a solution of the variational inequality

$$(Au - f, v - u) \geq 0 \text{ for all } v \in \mathbb{K},$$

then $\{u, A, \mathbb{K}\}$ is J-compatible implies $Au \in W$.

PROOF. The argument is the same as in theorem 6.2 up through the fact that $u_{\varepsilon} \longrightarrow u$ (strongly) in W'.

To show $\| u_{\varepsilon} \|_V \leq L_0$, we consider the equality

$$(Au_{\varepsilon}, u_{\varepsilon} - v_0) = (Au_{\varepsilon}, u_{\varepsilon} - u) + (Au_{\varepsilon}, u - v_0).$$

But $(Au_{\varepsilon}, u_{\varepsilon} - u) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Moreover, $(Au_{\varepsilon}, u - v_0)$ is bounded as $\varepsilon \longrightarrow 0$. Therefore, $(Au_{\varepsilon}, u_{\varepsilon} - v_0)$ is bounded as $\varepsilon \longrightarrow 0$. Thus, we must have $\| u_{\varepsilon} \|$ is bounded as $\varepsilon \longrightarrow 0$; that is, there is an L_0 such that $\| u_{\varepsilon} \| \leq L_0$. Therefore, there is a ball Σ_{L_0} such that $u_{\varepsilon} \in \mathbb{K} \cap \Sigma_{L_0}$.

Since $u \in \mathbb{K} \cap \Sigma_R$ for some R, we have that $u_{\varepsilon} \longrightarrow u$ (weakly) in V where $u_{\varepsilon}, u \in \overline{\mathbb{K}}$, $\overline{\mathbb{K}} \equiv \mathbb{K} \cap \Sigma_{\widetilde{R}}$ ($\Sigma_{\widetilde{R}} \supset \Sigma_R \cap \Sigma_{L_0}$ is bounded ball).

Also it is clear from the definition that the triplet $\{u, A, \overline{\mathbb{K}}\}$ is J-compatible. Therefore, from theorem 6.2 we have

$$Au \in W.$$

DEFINITION 6.2. The operator $A : H^{1,p}(\Omega) \longrightarrow H^{-1,p'}$ will be called *T-monotone* if for all $v_1, v_2 \in H^{1,p}(\Omega)$ with $v_1 \leq v_2$ on $\partial\Omega$, we have

$$(Av_1 - Av_2, w) \geq 0$$

where $w = \text{Max}(v_1 - v_2, 0)$ and $(\,,\,)$ denotes the pairing between $H^{-1,p'}$ and $H^{1,p}$.

NOTE: The above pairing will also be written as

$$\int_{[v_1 \geq v_2]} (Av_1 - Av_2) \cdot (v_1 - v_2)\, dx = (Av_1 - Av_2, w).$$

This notation has no meaning in general other than a notation for the pairing on the right.

We now adopt the following assumptions:

$$\mathbb{K} = \{v \in H_0^{1,p}(\Omega) \mid v \geq \psi\}$$

$A : H^{1,p}(\Omega) \to H^{1,p'}$ generally non-linear, $p > 1$

$T \in H^{-1,p'}$ if $T = f_0 + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(f_i)$ where $f_0, f_i \in L^{p'}$.

LEMMA 6.2. *If $A : H^{1,p}(\Omega) \to H^{-1,p'}$ is T-monotone, then $A : H_0^{1,p}(\Omega) \to H_0^{-1,p'}$ is monotone.*

PROOF. Suppose $v_1, v_2 \in H_0^{1,p}(\Omega)$. Then $v_1 \leq v_2$ and $v_2 \leq v_1$ on $\partial\Omega$. Thus, we can apply the definition of T-monotone in the orders $v_1 - v_2$ and $v_2 - v_1$; that is

$$(Av_1 - Av_2, \operatorname{Max}(v_1 - v_2, 0)) \geq 0 \text{ and } (Av_2 - Av_1, \operatorname{Max}(v_2 - v_1, 0)) \geq 0.$$

But

$$v_1 - v_2 = \operatorname{Max}(v_1 - v_2, 0) - \operatorname{Max}(v_2 - v_1, 0).$$

Therefore

$$(Av_1 - Av_2, v_1 - v_2) =$$

$$= (Av_1 - Av_2, \operatorname{Max}(v_1 - v_2, 0)) - (Av_1 - Av_2, \operatorname{Max}(v_2 - v_1, 0))$$

$$= (Av_1 - Av_2, \operatorname{Max}(v_1 - v_2, 0)) + (Av_2 - Av_1, \operatorname{Max}(v_2 - v_1, 0)) \geq 0.$$

THEOREM 6.4. *Let $A : H^{1,p}(\Omega) \to H^{-1,p'}$ be T-monotone with A hemicontinuous on $H_0^{1,p}(\Omega)$. Suppose there is a $v_0 \in \mathbb{K}$ such that*

$$\frac{(Av, v - v_0)}{\|v\|_{1,p}} \to +\infty \quad \text{as } \|v\|_{1,p} \to +\infty \text{ for } v \in \mathbb{K}.$$

For q so large that $L^q \subset H^{-1,p'}$ assume $f \in L^q(\Omega)$ and $A\psi$ is a measure such that $\operatorname{Max}(A\psi, 0) \in L^q(\Omega)$. Then $Au \in L^q(\Omega)$.

PROOF. According to a previous theorem, we need only define J in such a way that $\{u, A, \mathbb{K}\}$ is J-compatible where $u \in \mathbb{K}$ is a solution of the variational inequality $(Au - f, v - u) \geq 0$ for all $v \in \mathbb{K}$.

Let $W = L^q(\Omega)$ and define $J_q : L^q(\Omega) \to L^{q'}(\Omega)$ by

$$J_q w = |w|^{q-2} w.$$

Now since A is coercive and hemicontinuous on $H_0^{1,p}(\Omega)$, there is a solution $u \in \mathbb{K}$ of the variational inequality

$$(Au - f, v - u) \geq 0 \qquad \text{for all } v \in K.$$

By defining B_ε and C_ε for each $\varepsilon > 0$, we will show that the solution u_ε of

$$u_\varepsilon + \varepsilon J_q (Au_\varepsilon + B_\varepsilon u_\varepsilon) = u + \varepsilon C_\varepsilon u_\varepsilon$$

satisfies $u_\varepsilon \in \mathbb{K}$ and $Au_\varepsilon \in W = L^q(\Omega)$. Let us define

$$-B_\varepsilon u_\varepsilon \equiv \operatorname{Max}(A\psi, 0) = k \in L^q(\Omega), \qquad C_\varepsilon u_\varepsilon \equiv 0.$$

Then clearly B_ε and C_ε satisfy the hypotheses of « J-compatibility ». Now we consider

$$u_\varepsilon + \varepsilon J_q (Au_\varepsilon - k) = u.$$

Note that $J_q^{-1} = J_{q'}$. Then we can write this equation as

$$Au_\varepsilon - k - J_{q'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right) = 0.$$

For each fixed u, since $J_{p'}$ is monotone and since A is coercive, this operator is coercive. Therefore, there is a solution u_ε of the variational inequality

$$\left(Au_\varepsilon - k - J_{q'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right), v - u_\varepsilon\right) \geq 0 \quad \text{for all } v \in H_0^{1,p};$$

that is, a solution u_ε of

$$Au_\varepsilon - k - J_{q'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right) = 0,$$

where $u_\varepsilon \in H_0^{1,p} \subset L^{q'}$. But $u \in \mathbb{K} \subset H_0^{1,p}(\Omega) \subset L^{q'}$ implies $\frac{u - u_\varepsilon}{\varepsilon} \in L^{q'}$. Since $J_{q'} : L^{q'} \to L^q$ we have

$$Au_\varepsilon - k = J_{q'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right) \in L^q(\Omega).$$

Since $k \in L^q(\Omega)$, we have $Au_\varepsilon \in L^q(\Omega)$.

Now we must show $u_\varepsilon \in \mathbb{K}$; that is, $u_\varepsilon \geq \psi$. Define

$$E = \{x \in \Omega \mid u_\varepsilon(x) < \psi(x)\}.$$

For contradiction, assume $m(E) > 0$. Since $u \in \mathbb{K}$, we then have $u_\varepsilon \leq \psi \leq u$ on E in the sense of $H^{1,p}$. Since the operatos $Au_\varepsilon - k - J_{q'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right)$ is coercive, it is T-monotone. Letting $z_\varepsilon = \operatorname{Max}(\psi - u_\varepsilon, 0)$, we have

$$\int_{\{\psi \geq u_\varepsilon\}} \left[Au_\varepsilon - k - J_{q'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right)\right]\cdot(\psi - u_\varepsilon)\,dx = 0$$

and, hence,

$$\int_{\{\psi \geq u_\varepsilon\}} \left[Au_\varepsilon - k - J_{q'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right)\right]\cdot(u_\varepsilon - \psi)\,dx = 0.$$

Therefore,

$$0 = \int_{\{\psi \geq u_\varepsilon\}} \left[Au_\varepsilon - A\psi + A\psi - k - J_{q'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right)\right]\cdot(u_\varepsilon - \psi)\,dx$$

$$= \int_{\{\psi \geq u_\varepsilon\}} [Au_\varepsilon - A\psi]\cdot(u_\varepsilon - \psi)\,dx$$

$$+ \int_{\{\psi \geq u_\varepsilon\}} [A\psi - k]\cdot(u_\varepsilon - \psi)\,dx$$

$$+ \int_{\{\psi \geq u_\varepsilon\}} \left[J_{q'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right)\right]\cdot(u_\varepsilon - \psi)\,dx.$$

Since A is T-monotone,

$$\int_{\{\psi \geq u_\varepsilon\}} [Au_\varepsilon - A\psi]\cdot(u_\varepsilon - \psi)\,dx \geq 0.$$

Since $A\psi \leq k$ and $u_\varepsilon \leq \psi$,

$$\int_{\{\psi \geq u_\varepsilon\}} [A\psi - k]\cdot(u_\varepsilon - \psi)\,dx \geq 0.$$

Therefore

$$\int_{\{\psi \geq u_\varepsilon\}} \left[J_{q'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right)\right]\cdot(u_\varepsilon - \psi)\,dx \leq 0.$$

From our definition of J_q we have

$$\int_{\{\psi \geq u_\varepsilon\}} \left|\frac{u_\varepsilon - u}{\varepsilon}\right|^{q'-2}\left(\frac{u_\varepsilon - u}{\varepsilon}\right)\left(\frac{u_\varepsilon - \psi}{1}\right)dx \leq 0.$$

But $u_\varepsilon \leq \psi \leq u$ implies either $u_\varepsilon = u$ or $u_\varepsilon = \psi$ in the sense of $H^{1,p}$. But this implies $u_\varepsilon = \psi$ in the sense of $H^{1,p}$. Hence, $m(E) = 0$ and we have the desired contradiction. Therefore, $\{u, A, \mathbb{K}\}$ is J-compatible and, hence, $Au \in W = L^q(\Omega)$ from the theorem 6.3.

2. This theorem can be applied in the following ways:

Let us consider A defined for $u \in H_0^1(\Omega)$ by

$$Au \equiv -\{a_{ij}u_{x_i}\}_{x_j} + b_i u_{x_i} + cu$$

where $a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$, $\alpha > 0$ and $c(x) \geq \lambda$ for λ sufficiently large. Then A is clearly T-monotone.

A second example would be to define A on $H^{1,t}(\Omega)$ to $H^{-1,t'}$ by

$$Av = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left\{\left[1 + \sum_{j=1}^{N}\left|\frac{\partial v}{\partial x_j}\right|^2\right]^{\frac{t-2}{2}}\frac{\partial v}{\partial x_i}\right\}.$$

If $t = 2$, this reduces to Laplace's equation. If $t = 1$, this is the equation of minimal surfaces. We would solve the variational inequality relative to A for $1 < t \leq 2$; that is, for $1 < t \leq 2$, there is $u \in \mathbb{K} \equiv \{v \in H_0^{1,t} \mid v \geq \psi\}$ such that

$$(Au, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}.$$

It is clear, however, that A is T-monotone from $H^{1,t}(\Omega)$ and hemicontinuous from $H_0^{1,t}(\Omega)$. Since it is also coercive, we know there is such a solution.

Let us now consider a problem which places more restrictions on the solutions. We denote the following: $\psi_1(x)$, $\psi_2(x)$ defined on $\bar{\Omega}$ with $\psi_i \in H^{1,p}(\Omega)$ such that $\psi_1 \leq \psi_2$ on Ω in the sense of $H^1(\Omega)$, $\psi_1 \leq 0 \leq \psi_2$ on $\partial\Omega$ in the sense of $H^{1,p}(\Omega)$.

$$\mathbb{K}_1 = \{v \in H_0^{1,p}(\Omega) \mid \psi_1(x) \leq v(x) \leq \psi_2(x)\}$$

$A : H^{1,p}(\Omega) \to H^{-1,p'}$ is T-monotone with

$A(v+k) = A(v)$ for all constants k,

A hemicontinuous on $H_0^{1,p}(\Omega)$, and

A coercive; that is, there is $v_0 \in K_1$ such that

$$\frac{(Av, v - v_0)}{\| v \|_{1,p}} \to +\infty \text{ as } \| v \|_{1,p} \to +\infty \text{ with } v \in \mathbb{K}_1 .$$

THEOREM 6.5. *Let p be so large that $L^q(\Omega) \subset H^{-1,p'}$. If $f \in L^q(\Omega)$, then $Au \in L^q(\Omega)$ where $u \in \mathbb{K}_1$ is the solution of the variational inequality*

$$(Au - f, v - u) \geq 0 \quad \textit{for all} \quad v \in \mathbb{K}_1 .$$

REMARK: Before proving this theorem we must put the following restrictions on the functions ψ_i:

$$\operatorname{Max}(A\psi_1, 0) \in L^q(\Omega), \quad \operatorname{Min}(A\psi_2, 0) \in L^q(\Omega),$$

and $A\psi_i$ is a measure in the sense that

$$(A\psi, g) = \int (A\psi)\, g\, dx, \quad g \in L^{q'}(\Omega)$$

where the integral exists in the usual sense.

PROOF. Let us define $J_q : L^q(\Omega) \to L^{q'}(\Omega)$ by setting $\varphi(v) = v^{q-1}$. Then $J_q v = |v|^{q-2} v$. Note that J_q^{-1} exists and $J_q^{-1} = J_{q'}$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

It is clear from the previous theorem that if $\{u, A, \mathbb{K}_1\}$ is J_q-compatible we will have $Au \in L^p(\Omega)$. Here we will take $\{u, A, \mathbb{K}_1\}$ to be J_q-compatible if, for each $\varepsilon > 0$, there is an operator $B_\varepsilon : H_0^{1,p}(\Omega) \to L^q(\Omega)$ such that

$$\| B_\varepsilon v \|_{L^q} \leq \mathcal{C} \quad \text{for all} \quad v \in H_0^{1,p}(\Omega),$$

and there is a solution $u_\varepsilon \in \mathbb{K}_1$ with $Au_\varepsilon \in L^q$ of the equation

Thus, to show that $\{u, A, \mathbb{K}_1\}$ is J_q-compatible, we must exibit B_ε. Assume $\varepsilon > 0$ is fixed. Let us define for each n the functions $\theta_1^{(n)}(x)$, $\theta_2^{(n)}(x)$:

$$\theta_1^{(n)}(x) = \begin{cases} 1 & \text{for} \quad x \leq -\frac{1}{n} \\ -nx & \text{for} \quad -\frac{1}{n} < x < 0, \\ 0 & \text{for} \quad x \geq 0 \end{cases} \qquad \theta_2^{(n)}(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ nx & \text{for } 0 \leq x \leq \frac{1}{n} . \\ 1 & \text{for } x \geq \frac{1}{n} \end{cases}$$

Then for eauch n we consider

$$v_n + \varepsilon J_q (Av_n - B_1^{(n)} v_n - B_2^{(n)} v_n) = u$$

where

$$B_1^{(n)} v \equiv \operatorname{Max}(A\psi_1, 0) \cdot \theta_1^{(n)}(v - \psi_1),$$

$$B_2^{(n)} v \equiv \operatorname{Min}(A\psi_2, 0) \cdot \theta_2^{(n)}(v - \psi_2).$$

We can rewrite this equations as

$$Av_n - B_1^{(n)} v_n - B_2^{(n)} v_n + J_{q'}\left(\frac{v_n - u}{\varepsilon}\right) = 0.$$

Note that this operator is coercive on $H_0^{1,p}(\Omega)$ since

A is coercive on $H_0^{1,p}(\Omega)$,

$-B_1^{(n)}$ is monotone since θ_1 is decreasing and $\operatorname{Max}(A\psi_1, 0) \geq 0$,

$-B_2^{(n)}$ is monotone since θ_2 is increasing and $\operatorname{Min}(A\psi_2, 0) \leq 0$,

$J_{q'}$ is monotone.

Since this operator is hemicontinuous, there is a solution $v_n \in H_0^{1,p}(\Omega)$.

Moreover,

$$\| v_n \|_{H_0^{1,p}, \Omega} \leq \mathcal{C}(\varepsilon)$$

since $0 < \theta_i^{(n)} < 1$ and $\operatorname{Max}(A\psi_1, 0)$, $\operatorname{Min}(A\psi_2, 0) \in L^q(\Omega)$. Therefore there is a subsequence $\{v_{n'}\}$ for which

$$v_{n'} \longrightarrow u_\varepsilon \text{ (weakly) in } H_0^{1,p}(\Omega)$$

$$v_{n'} \longrightarrow u_\varepsilon \text{ (strongly) in } L^s(\Omega), \quad s < p^* = \frac{n-p}{np}.$$

It follows that

$$v_n \longrightarrow u_\varepsilon \text{ (weakly) in } H_0^{1,p}(\Omega),$$

$$v_n \longrightarrow u_\varepsilon \text{ (strongly) in } L^s(\Omega),\ s < p^*.$$

We wish to show that $u_\varepsilon \in \mathbb{K}_1$ and $Au_\varepsilon \in L^q(\Omega)$ where B_ε is appropriately defined. We first show the latter.

We are going to show that there is an operator $B_\varepsilon : H_0^{1,p}(\Omega) \longrightarrow L^q(\Omega)$ such that

$$Au_\varepsilon - B_\varepsilon u_\varepsilon + J_{q'}\left(\frac{u_\varepsilon - u}{\varepsilon}\right) = 0. \tag{6.1}$$

Now for each $x \in \Omega$, we have

$$0 \leq \theta_i^{(n)}(v_n(x) - \psi_i(x)) \leq 1, \qquad i = 1, 2,$$

and, hence, there is a subsequence $\{n'\}$ such that

$$\theta_i^{(n')}(v_{n'}(x) - \psi_i(x)) \longrightarrow \theta_i(x), \qquad i = 1, 2.$$

In view of this, let us define

$$B_\varepsilon u_\varepsilon \equiv \text{Max}(A\psi_1, 0)\,\theta_1(x) + \text{Min}(A\psi_2, 0)\,\theta_2(x).$$

We must first show that B_ε satisfies the equation (6.1) above. Now we have

$$Av_n - B_1^{(n)} v_n - B_2^{(n)} v_n + J_{q'}\left(\frac{v_n - u}{\varepsilon}\right) = 0.$$

Hence, we have

$$(Av_n, w) - (B_1^{(n)} v_n + B_2^{(n)} v_n, w) + \left(J_{q'}\left(\frac{v_n - u}{\varepsilon}\right), w\right) = 0$$

for all $w \in H_0^{1,p}(\Omega)$. By lemma 2.2 this becomes

$$(Av, v - v_n) - (B_1^{(n)} v_n + B_2^{(n)} v_n, v - v_n) + \left(J_{q'}\left(\frac{v - u}{\varepsilon}\right), v - v_n\right) \geq 0$$

for all $v \in H_0^{1,p}(\Omega)$. We now let $n \to \infty$. In the first and last terms we clearly have

$$(Av, v - v_n) \longrightarrow (Av, v - v_\varepsilon)$$

and

$$\left(J_{p'}\left(\frac{v - u}{\varepsilon}\right), v - v_n\right) \longrightarrow \left(J_{p'}\left(\frac{v - u}{\varepsilon}\right), v - u_\varepsilon\right).$$

Moreover, since $B_i^{(n)} v_n$ is uniformly in $L^q(\Omega)$ and since $v - v_n \longrightarrow v - u_\varepsilon$ in $L^{q'}$ if q in large enough, we have

$$(B_1^{(n)} v_n - B_2^{(n)} v_n, v - v_n) \longrightarrow (B_\varepsilon u_\varepsilon, vu_\varepsilon).$$

Hence,

$$(Av, v - u_\varepsilon) - (B_\varepsilon u_\varepsilon, v - u_\varepsilon) + \left(J_{q'}\left(\frac{v - u}{\varepsilon}\right), v - u_\varepsilon\right) \geq 0$$

for all $v \in H_0^{1,p}(\Omega)$ (hence $= 0$) and, by Minty,

$$(Au_\varepsilon, v - u_\varepsilon) - (B_\varepsilon u_\varepsilon, v - u_\varepsilon) + \left(J_{q'}\left(\frac{u_\varepsilon - u}{\varepsilon}\right), v - u_\varepsilon\right) = 0$$

for all $v \in H_0^{1,p}(\Omega)$.

Now since (6.1) holds, we see that

$$J_{q'}\left(\frac{u_\varepsilon - u}{\varepsilon}\right) \in L^q(\Omega) \quad \text{and} \quad B_\varepsilon u_\varepsilon \in L^q(\Omega) \quad \text{implies} \quad Au_\varepsilon \in L^q(\Omega).$$

Now we prove that for each n we have: $\psi_1 - \frac{1}{n} \leq v_n \leq \psi_2 + \frac{1}{n}$. Since $v_n \longrightarrow u_\varepsilon$ in $H_0^{1,p}(\Omega)$, it will follow that $\psi_1 \leq u_\varepsilon \leq \psi_2$ and hence, $u_\varepsilon \in \mathbb{K}_1$. Note that v_n is a solution of the equation

$$v_n + \varepsilon J_q(Av_n - B_1^{(n)} v_n - B_2^{(n)} v_n) = u.$$

We first wish to show $v_n \geq \psi_1 - \frac{1}{n}$; the other inequality will fol low similarly. Suppose there is a set E of positive measure such that $v_n < \psi_1 - \frac{1}{n}$ on E. Since $v_n \geq \psi_1$ on $\partial\Omega$ we have $E \subset \bar{E} \subset \Omega$. Now let $\mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)$ be a test function for the above equation. Since $v_n \leq \psi_2 - \frac{1}{n}$ on E, we have $v_n - \psi_2 \leq -\frac{1}{n}$ and, hence, $\theta_2^{(n)}(v_n - \psi_2) = 0$ on E. Hence,

$$0 = \left(Av_n, \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right) - \left(B_1^{(n)} v_n, \mathrm{Min}\left(v_n \psi_1 + \frac{1}{n}, 0\right)\right) +$$

$$+ \left(J_{q'}\left(\frac{v_n - u}{\varepsilon}, \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right)\right).$$

Noting that $A\left(\psi_1 - \frac{1}{n}\right) = A(\psi_1)$, we subtract and add this term from the first and second terms, respectively, to get

$$\left(Av_n - A\left(\psi_1 - \frac{1}{n}\right), \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right) -$$

$$- \left(B_1^{(n)} v_n - A\psi_1, \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right) +$$

$$+ \left(J_{q'}\left(\frac{v_n - u}{\varepsilon}\right), \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right) = 0.$$

Note that $v_n \leq \psi_1 - \frac{1}{n}$ implies $\theta_1^{(n)}(v_n - \psi_1) = 1$ and, hence, $B_1^{(n)} v_n = \mathrm{Max}(A\psi_1, 0)$. Thus, $A\psi_1 - \mathrm{Max}(A\psi_1, 0) \leq 0$. Therefore,

$$\left(A\psi_1 - B_1^{(n)} v_n, \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right) \geq 0.$$

Since $u \geq \psi_1$, $J_{q'}\left(\frac{v_n - u}{\varepsilon}\right) \leq J_{q'}\left(\frac{v_n - \psi_1 + \frac{1}{n}}{\varepsilon}\right)$ and hence

$$J_{q'}\left(\frac{v_n - u}{\varepsilon}\right) \cdot \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right) \geq$$

$$\geq J_{q'}\left(\frac{v_n - \psi_1 + \frac{1}{n}}{\varepsilon}\right) \cdot \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right) \geq 0$$

Therefore, it follows that

$$\left(Av_n - A\left(\psi_1 - \frac{1}{n}\right), \mathrm{Min}\left(v_n - \psi_1 + \frac{1}{n}, 0\right)\right) \leq 0.$$

Thus, $v_n = \psi_1 - \frac{1}{n}$. Therefore, E has zero measure.

3. In $H_0^1(\Omega)$ let v_x denote $(v_{x_1}, \dots, v_{x_N})$ and define the operator

$$Av = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}[a_i(v_x)]$$

where the $a_i(\xi)$, $i = 1, \dots, N$, define a map

$$\vec{a}(\xi): \mathbb{R}^N \to \mathbb{R}^N$$

which is monotone and continuous and locally Lipschitzian; here monotone means

$$\sum_{i=1}^{N} [a_i(\xi') - a_i(\xi'')](\xi_i' - \xi_i'') \geq 0 \quad \text{for all} \quad \xi \in \mathbb{R}^N.$$

Define the closed, convex set $\mathbb{K}_2$ by

$$\mathbb{K}_2 \equiv \{v \in H_0^1(\Omega) \mid |\mathrm{grad}\, v| \leq 1 \quad \text{a. e. in } \Omega\}.$$

Then $A: \mathbb{K}_2 \to H^{-1,2}$; in fact, $A: \mathbb{K}_2 \to H^{-1,\infty}$, however we will use $H^{-1,2}$. Since $\vec{a}$ is monotone, A is monotone. Moreover, A is continuous on finite dimentional subspaces of $\mathbb{K}_2$. Since $\mathbb{K}_2$ is a closed, convex, bounded set in $H_0^1(\Omega)$, we know from our theory that for any $f \in H^{-1,2}$ we have a solution $u \in \mathbb{K}_2$ such that

$$(Au - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}_2.$$

We wish to show the following property of u:

THEOREM 6.6. *Assume Ω is convex. If $f \in L^p(\Omega)$, then $Au \in L^p(\Omega)$ for $p > 1$.*

PROOF. Let $\varepsilon > 0$ be arbitrarily chosen. As in the previous cases let $J_p: L^p(\Omega) \to L^{p'}(\Omega)$ be the canonical map defined by

We look for a solution $u_\varepsilon \in \mathbb{K}_2$ of the equation

$$u_\varepsilon + \varepsilon J_p A u_\varepsilon = u$$

where $Au_\varepsilon \in L^p(\Omega)$. Clearly, if $u_\varepsilon \in \mathbb{K}_2$, then u_ε is bounded in Ω and, hence, $Au_\varepsilon \in L^\infty(\Omega)$ since A is continuous. But then we have $Au_\varepsilon \in L^p(\Omega)$. Hence, we need only show that $u_\varepsilon \in \mathbb{K}_2$. However instead of looking at the above equation, we consider

$$Au_\varepsilon = J_{p'}\left(\frac{u-u_\varepsilon}{\varepsilon}\right) = \left|\frac{u-u_\varepsilon}{\varepsilon}\right|^{p'-2}\left(\frac{u-u_\varepsilon}{\varepsilon}\right).$$

We will show this by actually considering a more general problem. Let $\theta : \mathbb{R}^1 \to \mathbb{R}^1$ be strictly increasing with $\theta(0) = 0$. We wish to look for a solution $v \in H_0^1(\Omega)$ of

$$Av = \theta(F - v); \tag{6.2}$$

that is, we wish to show:

1. There is a solution of (6.2)
2. If $F \in \mathbb{K}_2$, then $v \in \mathbb{K}_2$.

We first prove the following lemma:

LEMMA 6.3. *Let $F_1, F_2 \in L^\infty(\Omega)$ and $u_1, u_2 \in H^{1,\infty}(\Omega)$ be such that*

$$Au_1 = \theta(F_1 - u_1), \quad Au_2 = \theta(F_2 - u_2).$$

Then

$$\operatorname{Min}\{\inf_{\partial\Omega}(u_2 - u_1), \inf_\Omega(F_2 - F_1)\} \le u_2 - u_1 \le$$

$$\le \operatorname{Max}\{\sup_{\partial\Omega}(u_2 - u_1), \sup_\Omega(F_2 - F_1)\}.$$

PROOF. We will only show

$$u_2 - u_1 \le \operatorname{Max}\{\sup_{\partial\Omega}(u_2 - u_1), \sup_\Omega(F_2 - F_1)\} \equiv T,$$

the other inequality would be proved similarly. Let

$$E = \{x \in \Omega \mid u_2(x) - u_1(x) > T\}.$$

We wish to show that the measure of E is zero. Now

$$\int_\Omega a_i(u_{1x})\eta_{x_i}\,dx = \int_\Omega \theta(F_1 - u_1)\eta\,dx \quad \text{for all} \quad \eta \in H_0^1(\Omega),$$

$$\int_\Omega a_i(u_{2x})\eta_{x_i}\,dx = \int_\Omega \theta(F_2 - u_2)\eta\,dx \quad \text{for all} \quad \eta \in H_0^1(\Omega).$$

Subtracting we get,

$$\int_\Omega [a_i(u_{2x}) - a_i(u_{1x})]\,\eta_{x_i}\,dx = \int_\Omega [\theta(F_2 - u_2) - \theta(F_1 - u_1)]\,\eta\,dx,$$

for all $\eta \in H_0^1(\Omega)$. Hence, in particular we have by setting

$$\eta = \operatorname{Max}(u_2 - u_1, T) - T = \begin{cases} u_2 - u_1 - T & \text{on } E \\ 0 & \text{on } \Omega - E \end{cases},$$

$$0 \le \int_E [a_i(u_{2x}) - a_i(u_{1x})](u_{2x_i} - u_{1x_i})\,dx =$$

$$= \int_E [\theta(F_2 - u_2) - \theta(F_1 - u_1)](u_2 - u_1 - T)\,dx.$$

Since $\sup_\Omega(F_2 - F_1) \le T$, we have $F_2 - F_1 \le T$. But on E we have $u_2 - u_1 > T$. Thus, on E, $F_2 - F_1 \le u_2 - u_1$, or, $F_2 - u_2 \le F_1 - u_1$. Since θ is increasing we have

$$\theta(F_2 - u_2) \le \theta(F_1 - u_1).$$

Thus, $\theta(F_2 - u_2) - \theta(F_1 - u_1) \le 0$. Since $u_2 - u_1 - T \ge 0$ on E,

$$[\theta(F_2 - u_2) - \theta(F_1 - u_1)](u_2 - u_1 - T) = 0 \quad \text{a. e. on } E.$$

Thus, either $u_2 - u_1 = T$, or $\theta(F_2 - u_2) = \theta(F_1 - u_1)$ in which case we have $F_2 - u_2 = F_1 - u_1$, a. e. on E. Hence $F_2 - F_1 = u_2 - u_1 = T$. Thus, in either case, we have $u_2 - u_1 = T$ a. e. on E and, hence, E has measure zero.

We now return to the proof of theorem 6.6. We now wish to show 2, assuming that the solution v exists. Assume $F \in \mathbb{K}_2$. Then $|\operatorname{grad} F| \leq 1$ a.e. on Ω. Note that $F \in H_0^1(\Omega)$. Now since Ω is convex, for $x_0 \in \partial\Omega$, there is a «linear» function $\pi(x)$ such that $\pi(x_0) = 0$,

$$-\pi(x) \leq F(x) \leq \pi(x), \qquad x \in \Omega,$$

and

$$|\operatorname{grad} \pi(x)| = 1.$$

We wish to show that we then have $-\pi(x) \leq v(x) \leq \pi(x)$. Since π is linear, we have $A\pi = \theta(\pi - \pi)$. Now let $F_2 = F$, $u_2 = v$, $F_1 = \pi$, and $u_1 = \pi$ in the lemma. Then $F_2 - F_1 = F - \pi \leq 0$ on Ω. Moreover, since $v = 0$ on $\partial\Omega$ and, on $\partial\Omega$, $0 = F \leq \pi$, we have $u_2 - u_1 = v - \pi \leq 0$ on Ω, it follows that

$$v - \pi \leq \operatorname{Max}\{\sup_{\partial\Omega}(v - \pi), \sup_{\Omega}(F - \pi)\} \leq 0.$$

Similarly, by considering $A(-\pi) = \theta((-\pi) - (-\pi))$, we obtain

$$0 \leq v + \pi.$$

Hence

$$-\pi(x) \leq v(x) \leq \pi(x), \qquad x \in \Omega.$$

Now suppose $x \in \Omega$ and $x_0 \in \partial\Omega$. Since $v(x_0) = \pi(x_0) = 0$ and since $|\operatorname{grad} \pi| = 1$, we have

$$|v(x) - v(x_0)| \leq |\pi(x)| \leq |\pi(x) - \pi(x_0)| \leq |x - x_0|.$$

Now let $x_2, x_1 \in \Omega$. We wish to show

$$|v(x_2) - v(x_1)| \leq |x_2 - x_1|.$$

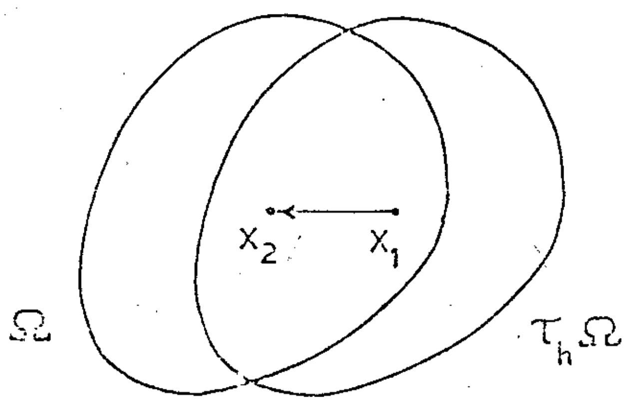

Fig. 3

Let $h = x_2 - x_1$ and define

$$\tau_h \Omega = \{y = x - h \mid x \in \Omega\},$$

$$\tau_{-h} v(x) = v(x + h),$$

$$\tau_{-h} F(x) = F(x + h).$$

We wish to consider the region $\Omega \cap \tau_h \Omega$.

Now if $x \in \Omega \cap \tau_h \Omega$, then $x + h \in \Omega$ and, hence

$$Av(x + h) = \theta(F(x + h) - v(x + h));$$

that is,

$$A\tau_{-h} v = \theta(\tau_{-h} F - \tau_{-h} v), \qquad \text{on } \Omega \cap \tau_h \Omega.$$

Applying the lemma 6.3 we get

$$\tau_{-h} v - v \leq \operatorname{Max}\{\sup_{\partial(\Omega \cap \tau_h \Omega)} (\tau_{-h} v - v), \sup_{\Omega \cap \tau_{-h}\Omega} (\tau_{-h} F - F)\}.$$

Suppose $x \in \partial(\Omega \cap \tau_h \Omega)$ Then either $x \in \partial\Omega$ and $x + h \in \Omega$ or $x \in$ and $x + h \in \partial\Omega$. Since either $x \in \partial\Omega$ or $x + h \in \partial\Omega$,

$$|v(x + h) - v(x)| \leq |h|.$$

Moreover, since $F \in \mathbb{K}_2$, $|\operatorname{grad} F| \leq 1$. Thus

$$|F(x + h) - F(x)| \leq |h|$$

for all $x \in \Omega \cap \tau_h \Omega$. Therefore,

that is

$$\tau_{-h} v(x) - v(x) \leq |h|, \qquad x \in \Omega \cap \tau_h \Omega;$$

$$v(x + h) - v(x) \leq |h|, \qquad x, x + h \in \Omega.$$

Since $x_1 \in \Omega$ and $x_1 + h = x_2 \in \Omega$,

$$v(x_2) - v(x_1) \leq |h| = |x_2 - x_1|.$$

Similarly we get the lower bound. This completes the proof of 2

We now wish to prove 1. Since $v \in H_0^1(\Omega)$, v vanishes in a neighborhood of $\partial\Omega$. From what we have shown we have

$$|v| \leq \operatorname{diam} \Omega.$$

Let

$$\tilde\theta(t) = \begin{cases} \theta(t) & |t| \leq 2 \operatorname{diam} \Omega \\ \text{strictly monotone and bounded.} \end{cases}$$

Let $\vec{a} : \mathbb{R}^N \to \mathbb{R}^N$ be monotone. Let ψ and g be defined on $\mathbb{R}^1$ by

$$\psi(t) = \begin{cases} 1 & 0 \leq t \leq 2 \\ \text{smooth} & 0 \leq t < \infty \\ 0 & 3 \leq t \end{cases} \qquad g(t) = \begin{cases} 0 & 0 \leq t < 1 \\ \text{smooth} & 0 \leq t < \infty \text{ convex.} \\ \text{linear (slope} = a > 0) & 3 \leq t \end{cases}$$

Define

$$\vec{\tilde{a}}(\xi) = \psi(|\xi|^2)\,\vec{a}(\xi) + kg'(|\xi|^2)\,\vec{\xi}.$$

We wish to show $\vec{\tilde{a}}$ is monotone. Let $\xi, \eta \in \mathbb{R}^N$, and, without loss of generality, assume $|\xi| \leq |\eta|$.

a). If $|\eta| \leq 1$, then $\vec{\tilde{a}}(\xi) = \vec{a}(\xi)$ and $\vec{\tilde{a}}(\eta) = \vec{a}(\eta)$ and it is obvious $\vec{\tilde{a}}$ is monotone.

b). If $|\xi| \geq 3$, then $\vec{\tilde{a}}(\xi) = ka\,\vec{\xi}$ and $a(\eta) = ka\,\vec{\eta}$ and it is obvious $\vec{\tilde{a}}$ is monotone.

c). If $1 \leq |\xi| \leq |\eta| \leq 2$ then $\vec{\tilde{a}}(\xi) = \vec{a}(\xi) + kg'(|\xi|^2)\vec{\xi}$, $\vec{\tilde{a}}(\eta) = \vec{a}(\eta) + kg'(|\eta|^2)\vec{\eta}$.

Thus,

$$M \equiv [\tilde{a}_i(\xi) - \tilde{a}_i(\eta)](\xi_i - \eta_i) = [a_i(\xi) - a_i(\eta)](\xi - \eta_i) +$$

$$+ k[g'(|\xi|^2)\xi_i - g'(|\eta|^2)\eta_i](\xi_i - \eta_i) \geq k[g'(|\xi|^2)\xi_i - g'(|\eta|^2)\eta_i](\xi_i - \eta_i)$$

$$= kg'(|\xi|^2)(\xi_i - \eta_i)(\xi_i - \eta_i) + k[g'(|\xi|^2) - g'(|\eta|^2)]\,\eta_i(\xi_i - \eta_i).$$

But

$$\eta_i \xi_i - \eta_i^2 = \frac{\xi_i^2}{4} - \frac{\xi_i^2}{4} + \xi_i\eta_i - \eta_i^2 = \left|\frac{\xi}{2}\right|^2 - \left|\eta - \frac{\xi}{2}\right|^2.$$

Since $|\xi \leq |\eta|$, $\left|\eta - \frac{\xi}{2}\right| \geq \left|\frac{\xi}{2}\right|$. Hence. $\eta_i\xi_i - \eta_i^2 \leq 0$. But $|\xi| \leq |\eta|$ implies $g'(|\xi|^2) \leq g'(|\eta|^2)$. Thus,

$$M \geq kg'(|\xi|^2)|\xi - \eta|^2 + k(g'(|\eta|^2) - g'(|\xi|^2))\left[\left|\eta - \frac{\xi}{2}\right|^2 - \left|\frac{\xi}{2}\right|^2\right] \geq$$

d). If $2 \leq |\xi|^2 \leq |\eta|^2 \leq 3$, then

$$\vec{\tilde{a}}_i(\xi) = \psi(|\xi|^2)\,\vec{a}(\xi) + kg'(|\xi|^2)\,\vec{\xi}$$

$$\text{and } \vec{\tilde{a}}_i(\eta) = \psi(|\eta|^2)\,\vec{a}(\eta) + kg'(|\eta|^2)\,\vec{\eta}.$$

Hence

$$M \equiv [\tilde{a}_i(\xi) - \tilde{a}_i(\eta)](\xi_i - \eta_i) = [\psi(|\xi|^2)\,a_i(\xi) - \psi(|\eta|^2)\,a_i(\eta)]\,\xi_i - \eta_i) -$$

$$+ k[g'(|\xi|^2)\xi_i - g'(|\eta^2|)\eta_i](\xi_i - \eta_i) =$$

$$= \psi(|\xi|^2)[a_i(\xi) - a_i(\eta)](\xi_i - \eta_i) + [\psi(|\xi|^2) - \psi(|\eta|^2)]\,a_i(\eta)(\xi_i - \eta_i)\cdot$$

$$+ kg'(|\xi|^2)|\xi - \eta|^2 + k[g'(|\xi|^2) - g'(|\eta|^2)]\left[\left|\frac{\xi}{2}\right|^2 - \left|\frac{\xi}{2} - \eta\right|^2\right];$$

$$\geq [\psi(|\xi|^2) - \psi(|\eta|^2)]\,a_i(\eta)(\xi_i - \eta_i) + kg'(|\xi|^2)|\xi - \eta$$

from case c). From $g'(|\xi|^2) \geq c_0 > 0$ for $2 \leq |\xi|^2 \leq 3$.
Also,

$$\sup_{2 \leq |\xi|^2 \leq |\eta|^2 \leq 3} \left| \frac{[\psi|\xi|^2 - \psi|\eta|^2]\,a_i(\eta)(\xi_i - \eta_i)}{|\xi - \eta|^2} \right| = K < \infty$$

since ψ is smooth. Therefore, for $k > \frac{K}{C_0}$, we have

$$M \geq -K|\xi - \eta|^2 + kg'(|\xi|^2)|\xi - \eta|^2 \geq 0.$$

e). Let ξ and η be any two points in $\mathbb{R}^N$ with $|\xi| < |$
If $|\xi| = |\eta|$, then the previous arguments apply. There exists
most six point ξ^i on the line joining ξ and η intersecting t
circles $|z| = 1, 2, 3$.
Then

$$M \equiv [\tilde{a}_i(\xi) - \tilde{a}_i(\eta)](\xi_i - \eta_i) =$$

$$= [(\tilde{a}_i(\xi) - \tilde{a}_i(\xi^1)] + [\tilde{a}_i(\xi^1) - \tilde{a}_i(\xi^2)] + [\tilde{a}_i(\xi^2) - \tilde{a}_i(\xi^3)] +$$

$$+ [\tilde{a}_i(\xi^3) - \tilde{a}_i(\xi^4)] + [\tilde{a}_i(\xi^4) - \tilde{a}_i(\xi^5)] + [\tilde{a}_i(\xi^5) - \tilde{a}_i(\xi^6)]$$

$$+ [\tilde{a}_i(\xi^6) - \tilde{a}_i(\xi\eta)]](\xi_i - \eta_i).$$

But there exist constants $t_i^j \in (0,1)$, $i = 1, 2, \dots, N$; $j = 0, 1, 2, \dots, 6$; such that $\xi_i - \xi_i^1 = t_i^0(\xi_i - \eta_i)$, $\xi_i^1 - \xi_i^2 = t_i^1(\xi_i - \eta_i)$, ..., $\xi_i^6 - \eta_i = t_i^6(\xi_i - \eta_i)$ These t_i^j are clearly ≤ 1 and can be taken > 0.

Then

$$M = \frac{1}{t_i^0}[\widetilde{a}_i(\xi) - \widetilde{a}_i(\xi^1)](\xi_i - \xi_i^1) + \frac{1}{t_i^1}[\widetilde{a}_i(\xi^1) - \widetilde{a}_i(\xi^2)](\xi_i^1 - \xi_i^2) + \dots$$

$$\dots + \frac{1}{t_i^6}[\widetilde{a}_i(\xi^6) - \widetilde{a}_i(\eta)](\xi_i^6 - \eta_i) \geq 0.$$

From what we have already shown.

We now define the operator $\widetilde{A} : H_0^1(\Omega) \to H^{-1}$ by

$$\widetilde{A}v = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\{\widetilde{a}_i(v_x)\}.$$

Then $\widetilde{A}$ is monotone and hemicontinuous. For small positive ν consider the equation

$$-\nu\Delta v^\nu + \widetilde{A}v^\nu = \widetilde{\theta}(F - v^\nu)$$

where $F \in \mathbb{K}_2$. Since $-\widetilde{\theta}$ and $\widetilde{A}$ are monotone, the orerator given by

$$-\nu\Delta v^\nu + \widetilde{A}v^\nu - \widetilde{\theta}(F - v)$$

is coercive and hemicontinuous on $H_0^1(\Omega)$. Hence, there is a solution $v^\nu \in H_0^1(\Omega)$. Moreover, since $F \in \mathbb{K}_2$, we have $v^\nu \in \mathbb{K}_2$ and, hence, $|\operatorname{grad} v^\nu| \leq 1$.

Since $|\operatorname{grad} v^\nu| \leq 1$, the set $\{v^\nu\}$ are uniformly bounded. Hence, as $\nu \to 0$, $v^\nu \rightharpoonup v$ (weakly) and, moreover,

$$\widetilde{A}v = \widetilde{\theta}(F - v).$$

Thus, $|\operatorname{grad} v| \leq 1$. But since $F, v \in \mathbb{K}_2$, we have $\widetilde{A} = A$ and $\widetilde{\theta} = \theta$. Therefore,

$$Av = \theta(F - v).$$

This completes the proof of 1. Hence, the proof of the theore is complete and $Au \in L^p(\Omega)$.

4. We now consider a problem related to the previous theore As before let

$$\mathbb{K}_2 = \{v \in H_0^1(\Omega) \mid |\operatorname{grad} v| \leq 1 \text{ a. e. on } \Omega\}.$$

Define the operator A by

$$Au \equiv -\sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

where

$$a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{a. e. in } \Omega \text{ for all } \xi \in \mathbb{R}^N,$$

$$a_{ij}(x) \in C^1(\overline{\Omega}),$$

$$b_i, C \in L^\infty(\Omega).$$

We also assume Ω to be smooth (in a sense to be defined lat The questions we ask are:

Is there $u \in \mathbb{K}_2$ such that $(Au - f, v - u) \geq 0$ for all $v \in \mathbb{K}$

If $f \in L^p(\Omega)$, do we have $Au \in L^p(\Omega)$?

The first question is easily handled and will be taken care immediately. The second, however, will be more difficult.

Note that $Au + \lambda u$ is coercive if λ is chosen large enou For each $w \in H_0^1(\Omega)$ we know there is a solution $u \in \mathbb{K}_2$ of

$$(Au + \lambda u - (f + \lambda w), v - u) \geq 0 \text{ for all } v \in \mathbb{K}_2.$$

Thus, this defines a map T given by

$$u = Tw.$$

It has been shown that this map is continuous and compact v

$$T : H_0^1(\Omega) \to \mathbb{K}_2 \subset \{u \in L^2(\Omega) \mid \|u\|_{L^2} \leq (\operatorname{diam} \Omega)|\Omega|^{1/2}\}.$$

Hence, there is a fixed point $u \in \mathbb{K}_2$; that is

We now wish to answer the second question. Let us first look at the operator $\widehat{A}$ defined by

$$\widehat{A}u \equiv -\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \widehat{b}_i \frac{\partial u}{\partial x_i} + \lambda u$$

where λ is chosen large enough. We will actually show that $f \in L^p(\Omega)$ implies $\widehat{A}u \in L^p(\Omega)$. Since, however,

$$Au - \widehat{A}u = \sum_{i=1}^{N} (b_i - \widehat{b}_i)\frac{\partial u}{\partial x_i} + (c - \lambda)u \in L^\infty(\Omega),$$

this will imply $Au \in L^p(\Omega)$. Therefore, we need only show $\widehat{A}u \in L^p(\Omega)$. For ease of notation we just write

$$Au = -\sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \widehat{b}_i \frac{\partial u}{\partial x_i} + \lambda u.$$

These $\widehat{b}_i$ will be appropriately chosen soon.

We note that to show $Au \in L^p(\Omega)$, it is only necessary to show the J-compatibility condition; that is, for each $\varepsilon > 0$, there is a solution $u_\varepsilon \in \mathbb{K}_2$ of $u_\varepsilon + \varepsilon J_p . Au_\varepsilon = u$.

Let us define

$$\delta(x) = \text{dist}\{x, \partial\Omega\}, \qquad x \in \overline{\Omega}.$$

We assume that $\partial\Omega$ is smooth in such a way that there is a $r > 0$ such that

$$\delta(x) \in H^{2,\infty}(\Omega_r)$$

where $\Omega_r = \{x \mid \delta(x) \leq r\}$. Moreover, since

$$|\text{grad}\, u| \leq 1$$

with $u = 0$ in a neighborhood of $\partial\Omega$, we see that

$$-\delta(x) \leq u(x) \leq \delta(x);$$

that is $\delta(x)$ is a barrier.

Lemma 6.4: *There is a function $\beta(x) \in H^{2,\infty}(\Omega)$ such that $\beta(x) = \delta(x)$ for $x \in \Omega_{r/2}$ and $\beta(x) \geq \delta(x)$ in Ω_r.*

Proof: Let us define $\chi(t)$ by

$$\chi(t) = \begin{cases} t & \text{for } 0 \leq t \leq \frac{r}{2} \\ \geq t & \text{for } 0 \leq t \leq \text{diam}\,\Omega \\ \text{diam}\,\Omega & \text{for } \frac{2r}{3} \leq t \leq \text{diam}\,\Omega. \end{cases}$$

Then let $\beta(x) = \chi(\delta(x))$. Clearly $\beta(x) = \delta(x)$ for $x \in \Omega_{r/2}$ and $\beta(x) \geq \delta(x)$ in Ω_r. Moreover, since χ is smooth with $\chi(\delta(x)) \equiv \text{diam}\,\Omega$ for $x \in \Omega - \Omega_{2r/3} \supset \Omega - \Omega_r$. Thus, $\beta(x) \in H^{2,\infty}(\Omega)$.

We wish to show

$$Au_\varepsilon = J_{p'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right) \in L^p(\Omega).$$

We will do this by looking at

$$Av = \theta(F - v), \qquad F \in \mathbb{K}_2,$$

where θ is an approximation to $J_{p'}$; prove that $Av \in L^p$, show that $v \to u_\varepsilon$ and that $\theta \to J_p$.

Define $\theta = \theta(t)$ so that θ is strictly increasing, $\theta(0) = 0$, $\theta \in L^\infty(-\infty, \infty)$ and $\theta' \in L^\infty(-\infty, \infty)$.

Note now that $Av - \theta(F - v)$ is a coercive operator since θ monotone and A is coercive. Moreover, it is hemicontinuous if the $\widehat{b}_i$ are smooth enough. Since we will take $\widehat{b}_i = M\beta x_i$ where

$$M = \text{Max}\left(\sup \Sigma\, a_{ij}\frac{\partial^2\beta}{\partial x_i \partial x_j}, 0\right),$$

this operator is hemicontinuous. Thus, there is a solution $v \in H_0^1(\Omega)$. We wish to show that $|\text{grad}\, v| \leq 1$ a.e. on Ω.

Lemma 6.5: *Assuming $\lambda > \frac{2M}{r}$ where M and r are given above, then $A\beta \geq 0$.*

Proof. If $x \in \Omega_{r/2}$, then, since $|\text{grad}\,\beta| = |\text{grad}\,\delta| = 1$,

$$A\beta = -\sum_{i,j=1}^{N} a_{ij}\beta_{x_i x_j} + M\beta_{x_i}\beta_{x_i} + \lambda\beta \geq -M + M + 0 = 0.$$

If $x \in \Omega - \Omega_{r/2}$, then $\lambda\beta \geq \lambda\delta(x) \geq \frac{\lambda r}{2} > M$. Thus

$$A\beta = -\sum_{i,j=1}^{N} a_{ij}\beta_{x_i x_j} + M|\operatorname{grad}\beta|^2 + \lambda\beta \geq -M + M|\operatorname{grad}\beta|^2 + M \geq 0.$$

Now since A is a linear operator, we have

$$A(v-\beta) + A\beta = \theta(F-v),$$

or

$$A(v-\beta) \leq \theta(F-v).$$

Multiply by $\operatorname{Max}(v-\beta, 0)$ and integrate. We get

$$\lambda_0 \int_{v\geq\beta} |v-\beta|^2\,dx \leq \int_{v\geq\beta} A(v-\beta)\cdot(v-\beta)\,dx \leq \int_{v\geq\beta} \theta(F-v)\cdot(v-\beta)\,dx.$$

Since $F \in \mathbb{K}_2$ we have $|\operatorname{grad} F| \leq 1$ and, thus, $F \leq \beta$. Hence, on the set where $v \geq \beta$ we have $F \leq \beta \leq v$. Since $F - v \leq 0$, we have $\theta(F-v) \leq 0$. Therefore,

$$\lambda_0 \int_{v\geq\beta} |v-\beta|^2\,dx \leq \int_{v\geq\beta} \theta(F-v)\cdot(v-\beta)\,dx \leq 0$$

and it follows that $v \leq \beta$. Similarly we can show $v \geq -\beta$. Therefore, in a neighborhood of $\partial\Omega$, we have $|\operatorname{grad} v| \leq 1$. We now wish to show $|\operatorname{grad} v| \leq 1$ in Ω.

Let

$$\gamma = |\operatorname{grad} v|^2 = \sum_{k=1}^{N}\left(\frac{\partial v}{\partial x_k}\right)^2 \quad \text{on} \quad \Omega.$$

Now since $\theta \in L^\infty$ we have $v \in H^{2,t}(\Omega)$ for all t. Moreover, since $\theta' \in L^\infty$ we have $v \in H^{3,t}(\Omega)$ for all t. Thus, since $a_{ij} \in C^1(\Omega)$, we have

$$\frac{\partial\gamma}{\partial x_i} = 2\sum_{k=1}^{N} \frac{\partial^2 v}{\partial x_i \partial x_k}\frac{\partial v}{\partial x_k}$$

$$(6.3) \qquad \frac{\partial}{\partial x_j}\left[a_{ij}(x)\frac{\partial\gamma}{\partial x_i}\right] = 2\frac{\partial a_{ij}}{\partial x_j}\sum_k \frac{\partial^2 v}{\partial x_i\partial x_k}\frac{\partial v}{\partial x_k} + 2a_{ij}\sum_k \frac{\partial^3 v}{\partial x_j\partial x_i\partial x_k}\frac{\partial v}{\partial x_k} + 2a_{ij}\sum_k \frac{\partial^2 v}{\partial x_i\partial x_k}\frac{\partial^2 v}{\partial x_j\partial x_k}.$$

Now take $\frac{\partial}{\partial x_k}$ of Av,

$$Av \equiv -\sum_{i,j} a_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j} + \sum_i \widehat{b_i}\frac{\partial v}{\partial x_i} + \lambda v = \theta(F-v),$$

and multiply by $\frac{\partial v}{\partial x_k}$. We get

$$(6.4) \quad -\sum_{i,j} a_{ij}\frac{\partial^3 v}{\partial x_i\partial x_j\partial x_k}\frac{\partial v}{\partial v_k} - \sum_{i,j}\frac{\partial a_{ij}}{\partial x_k}\frac{\partial^2 v}{\partial x_i\partial x_j}\frac{\partial v}{\partial x_k} + \sum_i \frac{\partial\widehat{b}}{\partial x_k}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_k} + \sum_i \widehat{b_i}\frac{\partial^2 v}{\partial x_k\partial x_i}\frac{\partial v}{\partial x_k} + \lambda\left(\frac{\partial v}{\partial x_k}\right)^2 = \theta'(F-v)\left(\frac{\partial F}{\partial x_k} - \frac{\partial v}{\partial x_k}\right)\frac{\partial v}{\partial x_k}.$$

Sum the expressions in (6.4) over k and use (6.3) to obtain

$$-\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial\gamma}{\partial x_i}\right] + \sum_{i,j,k}\left(\frac{\partial a_{ij}}{\partial x_j}\frac{\partial^2 v}{\partial x_i\partial x_k} - \frac{\partial a_{ij}}{\partial x_k}\frac{\partial^2 v}{\partial x_i\partial x_j}\right)\frac{\partial v}{\partial x_k} + \sum_{i,j,k} a_{ij}\frac{\partial^2 v}{\partial x_i\partial x_k}\frac{\partial^2 v}{\partial x_j\partial x_k} + \sum_{i,k}\left(\frac{\partial\widehat{b_i}}{\partial x_k}\frac{\partial v}{\partial x_i} + \widehat{b_i}\frac{\partial^2 v}{\partial x_k\partial x_i}\right)\frac{\partial v}{\partial x_k} + \lambda|\operatorname{grad} v|^2 = \theta'(F-v)[\operatorname{grad} F\cdot\operatorname{grad} v - |\operatorname{grad} v|^2].$$

We rewrite this as

$$-\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x_j}\left[a_{ij}\frac{\partial\gamma}{\partial x_i}\right] + R + (\lambda-\lambda_1)\gamma = \theta'(F-v)[\operatorname{grad} F\cdot\operatorname{grad} v - |\operatorname{grad} v|^2]$$

where

$$R = \sum_{i,j,k} a_{ij}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial^2 v}{\partial x_k}\right)\right]\left[\frac{\partial}{\partial x_j}\left(\frac{\partial v}{\partial x_k}\right)\right] + \sum_{i,j,k}\left(\frac{\partial a_{ij}}{\partial x_j}\frac{\partial^2 v}{\partial x_i\partial x_k} - \frac{\partial a_{ij}}{\partial x_k}\frac{\partial^2 v}{\partial x_i\partial x_j}\right)\frac{\partial v}{\partial x_k} + \sum_{i,k}\frac{\partial\widehat{b_i}}{\partial x_k}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_k} + \sum_{i,j}\widehat{b_i}\frac{\partial^2 v}{\partial x_k\partial x_i}\frac{\partial v}{\partial x_k} + \lambda_1\gamma$$

$$\geq \nu \sum_{i,j} \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - \frac{\nu}{4} \sum_{i,j} \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - C(\nu) \sum_k \left(\frac{\partial v}{\partial x_k}\right)^2$$

$$- L \sum_k \left(\frac{\partial v}{\partial x_k}\right)^2 - \frac{\nu}{4} \sum_{i,j} \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 - \widehat{C}(\nu) \sum_k \left(\frac{\partial v}{\partial x_k}\right)^2 + \lambda_1 \gamma$$

$$= \frac{\nu}{2} \sum_{i,j} \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)^2 + (\lambda_1 - C(\nu) - L - \widehat{C}(\nu))\gamma \geq 0$$

for $\lambda_1 \geq C(\nu) + \widehat{C}(\nu) + L$. Thus,

$$-\frac{1}{2} \sum_{i,j} \frac{\partial}{\partial x_j}\left[a_{ij} \frac{\partial \gamma}{\partial x_i}\right] + (\lambda - \lambda_1)\gamma \leq \theta'(F - v)[\operatorname{grad} F \cdot \operatorname{grad} v - |\operatorname{grad} v|^2].$$

Now clearly $\theta'(F - v) \geq 0$ and $\operatorname{grad} F \cdot \operatorname{grad} v \leq \frac{1}{2} |\operatorname{grad} F|^2 + \frac{1}{2} |\operatorname{grad} v|^2$. Thus

$$-\frac{1}{2} \sum_{i,j} \frac{\partial}{\partial x_j}\left[a_{ij} \frac{\partial \gamma}{\partial x_i}\right] + (\lambda - \lambda_1)\gamma \leq \frac{1}{2} \theta'(F - v)[1 - \gamma].$$

We now take $\lambda > \lambda_1$. Since $\gamma \geq 0$, we multiply by

$$\operatorname{Max}(\gamma, 1) - 1 \ (\in H_0^1(\Omega))$$

and integrate to get

$$\int_{\{\gamma \geq 1\}} a_{ij} \gamma_{x_i} \gamma_{x_j} dx + (\lambda - \lambda_1) \int_{\{\gamma \geq 1\}} \gamma(\gamma - 1)\, dx \leq$$

$$\leq \frac{1}{2} \int_{\{\gamma \geq 1\}} \theta'[F - v][1 - \gamma][\gamma - 1]\, dx.$$

Since the left hand side is non negative and the right hand side is non positive, we must have $\gamma \leq 1$ on Ω: that is

$$|\operatorname{grad} v| \leq 1 \qquad \text{in } \Omega.$$

Hence, $v \in \mathbb{K}_2$.

Our original problem was to consider

$$A u_\varepsilon = J_{p'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right).$$

Let $\{\theta_n\}$ be a sequence of bounded strictly increasing functions such that

$$\theta_n(t) \to \left|\frac{t}{\varepsilon}\right|^{p'-2} \left(\frac{t}{\varepsilon}\right) \qquad \text{as } n \to \infty.$$

on $-2 \operatorname{diam} \Omega \leq t \leq 2 \operatorname{diam} \Omega$. For each n there is a solution $u_n \in \mathbb{K}_2$ of

$$A u_n = \theta_n(u - u_n) \qquad \text{for all } u \in \mathbb{K}_2.$$

Since $\{u_n\}$ is uniformly bounded by $(\operatorname{diam} \Omega)$, there is a weakly convergent subsequence for which $u_n \to u_\varepsilon$ in $\mathbb{K}_2$.

Thus

$$A u_\varepsilon = J_{p'}\left(\frac{u - u_\varepsilon}{\varepsilon}\right)$$

from previous arguments.

5. We have shown in these last sections that, if the closed convex set is either $\mathbb{K}_1$ or $\mathbb{K}_2$, the solution u of the variational inequality

$$(Au - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

satisfies $u \in H^{2,p}$ but, in general, $u \notin H^3$.

We considered the operator $Au \equiv \{a_{ij} u_{x_i}\}_{x_j}$ where $a_{ij} \xi_i \xi_j \geq \nu |\xi|$ and the convex set

$$\mathbb{K}_3 \equiv \{v \in H^1(\Omega) \mid v \geq 0 \text{ on } \partial\Omega\}$$

by looking for a solution $u \in \mathbb{K}_3$ of the variational inequality

$$(Au - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}$$

Note that there is no restriction on u in Ω, only on $\partial\Omega$. Thus $Au = f$ in Ω. We can decompose $\partial\Omega = \partial_1 \Omega \cup \partial_2 \Omega$ so that $\partial_1 \Omega \cap \partial_2 \Omega = \Phi$ where

$$u = 0 \quad \text{on} \quad \partial_1 \Omega,$$

$$u \geq 0 \quad \text{on} \quad \partial_2 \Omega.$$

On $\partial_2 \Omega$ we must then require $\frac{\partial u}{\partial n} = 0$; that is the derivative of taken normal to $\partial\Omega$. Hence our boundary requirement becomes

$$u \frac{du}{dn} = 0 \quad \text{on} \quad \partial\Omega.$$

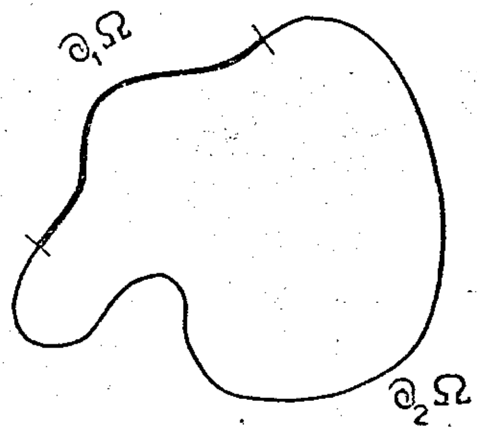

Fig. 4

This, of course, leaves the question: what does $\{x \mid u = 0\}$ look like? We mentioned that if $f \in L^2(\Omega)$, then $u \in H^2(\Omega)$. This was shown by Lions. To generalize this, we ask if $f \in L^p(\Omega)$ implies $u \in H^{2,p}(\Omega)$ for all p. E. Shamir has shown the answer to be **no**. He has given a counter-example in the case where $p = 4$.

6. Now define

$$\mathbb{K}_4 = \{v \in H_0^1(\Omega) \mid \| v \|_{L^2} \leq 1.$$

Since $\mathbb{K}_4$ is a closed, convex set, there is a $u \in \mathbb{K}_4$ such that

$$(Au - f, v - u) \geq 0 \qquad \text{for all } v \in \mathbb{K}_4$$

Noting that, for any convex set $\mathbb{K}$ and $u \in \mathbb{K}$,

$$\mathbb{K}_u = \{w = \varepsilon(v - u) \mid \varepsilon > 0,\ v \in \mathbb{K}\},$$

we have that

$$(Au - f, w) \geq 0 \qquad \text{for all } w \in (\mathbb{K}_4)_u.$$

If $\| u \|_{L^2} < 1$, then $u \in \mathring{\mathbb{K}}_4$ and, hence, $Au = f$.
If $\| u \|_{L^2} = 1$, then $(u, w) \leq 0$ for all $w \in (\mathbb{K}_4)_u$.
For,

$$\begin{aligned}(u, w) &= \varepsilon[(u, v) - (u, u)] \\ &\leq \varepsilon[\| u \| \| v \| - \| u \|^2] \\ &= \varepsilon[\| v \|_{L^2} - 1] \leq 0.\end{aligned}$$

Since $v \in \mathbb{K}_4$.

Thus there exists a number $\lambda \geq 0$ such that

$$Au - f = -\lambda u$$

or

$$Au + \lambda u = f.$$

The solution of the variational inequality related to the con $\mathbb{K}_4$ is thus solution of a differential equation.

We can apply the theory of regularity for solutions of di rential equations and thus the solution is smooth if the data smooth.

Notes for references.

The first theorem of existence and uniqueness of the solut of variational inequalities was proved in [12]. The first proof theorem 2.1 is contained in [12]. The second proof of the sa theorem is contained in [8].

The statement of theorem 2.2 is contained in [6], the proof in [1]. Theorems 2.3 and 2.3′ are in [6] and [3]. Lemma 2.2 proved by Minty. Theorem 2.5 is studied in [6] and [8]. Theor 3.1 is proved in [8], while theorem 3.2 is in [5]. For the relati between variational inequalities and non expansive operators for references, [11]. Theorem 4.1 is part of [8]. Theorem 4.2 proved by Mosco [10]. § 5 is part of results of [7] while § 6 is c tained in [2].

REFERENCES

[1] H. Brezis, *Equations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier, t. 18, *1968*, 115-175.

[2] H. Brezis et G. Stampacchia, *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. France, 96, *1968*, 153-180.

[3] F. E. Browder, *Non linear monotone operators and convex sets in Banach spaces*, Bull. Am. Math. Soc. 71, *1965*, 780-785.

[4] F. E. Browder, *On the unification of the calculus of variations and the theory of monotone nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sc. U. S 56, *1966*, 419-425.

[5] F. E. Browder, *Existence and approximation of solution of nonlinear variational inequalities*, Proc. Nat. Acad. Sc. U. S. 56, *1966*, 1080-1086.

[6] P. Hartmam and G. Stampacchia, *On some nonlinear elliptic differential-functional equations*, Acta Math. 115, *1966*, 271-310.

[7] H. Lewy and G. Stampacchia, *On the regularity of the solution of a variational inequality*, Comm. Pure and Appl. Math. XXII, *1969*, 153-188.

[8] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure and Appl. Math. 20, *1967*, 493-519.

[9] G. J. Minty, *On the generalization of a direct method of the calculus of variations*, Bull Am. Math. Soc. 73, *1967*, 315-321.

[10] U. Mosco, *A remark of a theorem of F. E. Browder*, J. Math. Anal. Appl. 20, *1967*, 90-93.

[11] Z. Opial, *Non expansive and monotone mappings in Banach spaces*, Brown University.

[12] G. Stampacchia, *Formes bilinéairss coercitives sur les ensembles convexes*, Compt. Rend. Acad. Sci. Paris 258, *1964*, 4413-4416.