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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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IDENTIFICATION OF SYSTEMS

(Part II)

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(I) Identification in an elliptic PDE: the continuous case.

We come back now to the identification problem of the § I:
Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ with boundary Γ, and let Γ_1, Γ_2 be a partition of Γ, and $\vec{n}$ be the normal to Γ pointing outwards of Ω.
For given functions a on Ω and g on Γ_2, we consider the following elliptic PDE:

$$
(P)\begin{cases}
-\sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(a(x)\dfrac{\partial u}{\partial x_i}\right)=0 & \text{on } \Omega\\
u=0 & \text{on } \Gamma_1\\
a(s)\dfrac{\partial u}{\partial n}(s)=g(s) & \text{on } \Gamma_2
\end{cases}
$$

The function g being supposed known, to every function $a(x)$ we associate the error function $J(a)$ defined by

$$
(4.1)\qquad J(a)=\int_{\Omega}\left(u(x;a)-z(x)\right)^2 dx
$$

where $z(x)$ is a known function (representing some measure of the observed temperature distribution).

We shall first give a unicity and existence theorem for the solution u of (P), so that the function $J(a)$ defined in (4.1) makes sense, and study the derivability of the application $a \rightarrow u$.

Then we shall calculate the derivative of J with respect to a and introduce the notion of adjoint state.

1) Study of the state equation

Multiply the first equation of (P) by a function $v(x)$ and integrate over Ω. Using the Green formula one gets:

$$\int_\Omega a(x) \sum_{i=1}^{n} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - \int_\Gamma a(s) \frac{\partial u}{\partial n}(s) v(s) = 0 \qquad \forall v$$

Taking $v = 0$ on Γ_1 we get, as $a(s)\frac{\partial u}{\partial n}(s) = g(s)$ on Γ_2:

(4.2) $$\int_\Omega a(x) \sum_{i=1}^{n} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = \int_{\Gamma_2} g(s) v(s) \qquad \forall v \text{ s.t. } v = 0 \text{ on } \Gamma_1$$

Suppose that

(4.3) $g \in L^2(\Gamma_2)$

(4.4) Ω is bounded, Γ is "regular", and Γ_1 is of strictly positive $d\Gamma$ measure

Define:

(4.5) $V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \}$

(4.6) $a(u,v) = \int_\Omega a(x) \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \qquad \forall u, v \in V$

(4.7) $L(v) = \int_{\Gamma_2} g(s) v(s)\, ds \qquad \forall v \in V$

One deduces from (4.2) that the <u>variational formulation of problem (P) is</u>

(Q) $\boxed{\text{find } u \in V \text{ such that } a(u,v) = L(v) \ \forall v \in V}$

Define now:

(4.8) $\mathcal{A}_c = \{ a \in L^\infty(\Omega) \mid \exists \alpha > 0, \ a(x) \geq \alpha \text{ a.e. on } \Omega \}$

which is obviously an <u>open subset</u> of $L^\infty(\Omega)$.

<u>Thm 3</u> With the hypothesis and notation (4.2) to (4.8), the problem (Q) has a unique solution $u \in V$ for every $a \in \mathcal{A}_c$, and the mapping $a \to u$ is infinitely differentiable.

<u>Proof</u>: As the trace application is a continuous mapping from $H^1(\Omega)$ in $L^2(\Gamma)$, <u>V is</u> a closed subspace of $H^1(\Omega)$, and hence <u>a Hilbert space</u> when equipped with the norm and scalar product of $H^1(\Omega)$.

The bilinear form $a(u,v)$ and the linear form $L(v)$ are obviously continuous on V.

Due to hypothesis (4.4) and Poincaré inequality, the bilinear form $a(u,v)$ is V elliptic for every $a \in \mathcal{a}_c$, and the first part of the thm follows from Lax-Milgram theorem.

Let then $A(a)$ be the operator of $\mathcal{L}(VV')$ associated to the bilinear form $a(u,v)$ (or to the function $a(x)$) by:

$$(4.9) \qquad (A(a)\,u, v) = a(u,v) \qquad \forall u, v \in V$$

So the equation (Q) may be rewritten

$$(4.10) \qquad A(a)\,u = L$$

or equivalently, as $A(a)$ is invertible (Lax Milgram):

$$(4.11) \qquad u = [A(a)]^{-1} L$$

Hence the application $a \longrightarrow u$ from $\mathcal{a}_c$ in V is the composition of two applications:

- the application $a \longrightarrow A(a)$ which is obviously linear continuous from $\mathcal{a}_c$ in $\mathrm{Isom}(VV')$ [1]
- the application $A \longrightarrow A^{-1}$ from $\mathrm{Isom}(VV')$ in $\mathrm{Isom}(V'V)$ which is known to be $\mathcal{C}^\infty$

So the application $a \longrightarrow u$ is $\mathcal{C}^\infty$ from $\mathcal{a}_c$ in V, which ends the proof of thm 3. ∎

2) Adjoint state and derivability of J.

Let us denote by $(,)$ and $| \ |$ the scalar product and norm in $L^2(\Omega)$. Then the error function J defined in (4.1) may be rewritten:

$$(4.12) \qquad J(a) = | u(a) - z |^2$$

where $u(a)$ is the solution of (Q) for the value a of the unknown parameter.

It follows from Thm 3 that J is a C^∞ function of a, and we want now to calculate its first derivative.

The most convenient way to do this is to consider J as a function of two variables u and a related by the equation (Q). So we are lead to introduce a Lagrangian function depending on u, a and on a Lagrange multiplier p for the equation (Q):

$$(4.13) \begin{cases} \forall (u,a) \in V \times \mathcal{A}_a, \ \forall p \in V & \text{define:} \\ \mathcal{L}(u, a ; p) = | u - z |^2 + a(u,p) - L(p) \end{cases}$$

Then the error criterion is given (for any $p \in V$!) by:

$$(4.14) \qquad J(a) = \mathcal{L}(u(a), a ; p)$$

As we know from thm 3 that the mapping $a \to u(a)$ is derivable, we may differentiate (4.14) with respect to a:

$$(4.15) \qquad \delta J = \frac{\partial \mathcal{L}}{\partial u}(u(a), a ; p)\, \delta u + \frac{\partial \mathcal{L}}{\partial a}(u(a), a ; p)\, \delta a$$

The equation (4.15) is true for any $p \in V$. So if we choose p such that:

$$(4.16) \qquad \frac{\partial \mathcal{L}}{\partial u}(u(a), a ; p) . v = 0 \qquad \forall v \in V$$

we get from (4.15):

$$(4.17) \qquad \delta J = \frac{\partial \mathcal{L}}{\partial a}\left(u(a), a; p\right)\delta a$$

which is the desired first derivative of J with respect to a!

Let us now interpret (4.16) (4.17).

Using (4.13), (4.16) becomes

$$(\tilde{Q}) \quad \boxed{\text{find } p \in V \text{ s.t. } \quad a(v, p) = -2(u - z, v) \quad \forall v \in V}$$

which has effectively a unique solution $p \in V$ when $a \in \mathcal{O}_c$ (the bilinear form $a(u,v)$ is still V-elliptic), and (4.17) gives then:

$$(4.18) \qquad \delta J = \delta a\,(u, p) = \int_\Omega \delta a(x) \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial p}{\partial x_i}\,dx$$

The function p is called the adjoint state, and the equation $(\tilde{Q})$ is the adjoint state equation. It's easy to check that the adjoint state p is a weak solution of:

$$(\tilde{P}) \begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial p}{\partial x_i}\right) = -2\,(u - z) & \text{in } \Omega \\ p = 0 & \text{on } \Gamma_1' \\ a(s)\frac{\partial p}{\partial n}(s) = 0 & \text{on } \Gamma_2' \end{cases}$$

which is to be compared with the (direct) state equation (P).

So we have proved the

Thm 4: Under the hypothesis of thm 3, the error function J is C^∞ with respect to a on $\mathcal{O}_c$. Its first derivative is given by (4.18), where $u \in V$ is the (direct) state solution of (Q), and where $p \in V$ is the adjoint state solution of $(\tilde{Q})$.

Remark 6 : As $a \in L^\infty(\Omega)$, $J'(a)$ is in $[L^\infty(\Omega)]' \supset L^1(\Omega)$, and (2.18) shows that $J'(a)$ is in fact in $L^1(\Omega)$: define

$$(4.19) \qquad \frac{\partial J}{\partial a}(x) = \sum_{i=1}^{n} \frac{\partial u}{\partial x}(x) \frac{\partial p}{\partial x}(x) \qquad \forall x \in \Omega$$

so that (4.18) rewrites:

$$(4.20) \qquad \delta J = \int_\Omega \frac{\partial J}{\partial a}(x)\, \delta a(x)\, dx$$

The function $\frac{\partial J}{\partial a}$, which belongs to $L^1(\Omega)$, is termed the partial functional derivative of J with respect to a. Its value $\frac{\partial J}{\partial a}(x)$ at a point x of Ω may be seen as the partial derivative of J with respect to the value $a(x)$ of a at x.

If the functions a and $\frac{\partial J}{\partial a}$ would belong to $L^2(\Omega)$, then $\frac{\partial J}{\partial a}$ would be exactly the gradient of J with respect to a, and we could design a gradient method by defining a sequence a^0, a^1 ... etc .. by

$$(4.21) \qquad a^{n+1}(x) = a^n(x) - \rho \frac{\partial J}{\partial a}(a^n)(x) \qquad \forall x \in \Omega$$

As the function $\frac{\partial J}{\partial a}(x)$ is in fact only in $L^1(\Omega)$ when $a(x)$ is in $L^\infty(\Omega)$, a method such as (4.21) does not make sense (a^{n+1} does not belong to L^∞!), and the gradient method sketched in § II and III has to be applied on the discretized problem (see § V) ■

Remark 7 : The calculation of the partial functional derivative $\frac{\partial J}{\partial a}(x)$ requires only the knowledge of u and p, i.e. the resolution of two partial differential equations, though the parameter space is infinite dimensional! ■

Exercise 1 Prove thm 4 for the state equation (P) – $a(\varphi)$ – but with a boundary observation on Γ_2, ie replace (4.2) by

(4.22) $$J(a) = \int_{\Gamma_2} \left(u(s;a) - z(s)\right)^2 ds$$

where z is a given function of $L^2(\Gamma_2)$

Exercise 2 Prove thm 4 for the state equation:

$$(P') \begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a(x) \frac{\partial u}{\partial x_i} \right) = 0 & \text{on } \Omega \\ u = 0 & \text{on } \Gamma_1 \\ a(s) \frac{\partial u}{\partial n}(s) = \tau(s)\left(u_e(s) - u(s) \right) & \text{on } \Gamma_2 . \end{cases}$$

where $a \in L^\infty(\Omega)$ and $u_e \in L^2(\Gamma_2)$ are known, with $a(x) \geq \alpha > 0$, and where $\tau(s)$ is an unknown positive function; the observation beeing on the boundary Γ_2 as in Ex 1.

Exercise 3 Prove thm 4 for the state equation:

$$(P'') \begin{cases} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i} \right) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

where $f \in L^2(\Omega)$ is known, and where $a_{ij} \in L^\infty(\Omega)$, $i,j = 1 \ldots n$ are unknown functions, such that for almost every $x \in \Omega$, the $n \times n$ matrix of coefficients a_{ij} is positive definite; The observation being taken at your convenience.

Exercise 4 Prove thm 4 for the state equations (P) and for point-measurements. A point-measurement of u at point x_i may be seen as the measure of the mean value of u on a small domain Ω_i surrounding x_i:

(4.23) $$J(a) = \sum_{i=1}^{k} \left(\frac{1}{\operatorname{meas} \Omega_i} \int_{\Omega_i} u(x;a)\,dx - z_i \right)^2$$

where k is the number of point-measurements and $z \in \mathbb{R}^k$ is a

(V) Identification in an elliptic PDE: the discretized case.

As it is impossible to "compute" the function $u(x)$ solution of (Q), we have to replace it by a function $u_h(x)$ depending on a finite number of degree of liberty, so that we may determine it on a computer. We will show in this § how the use of a variational formulation as (Q) makes it possible to design easily numerical approximation schemes.

1) A flavour of approximation theory for variational problems

As suggested by the example of § (IV), let

(5.1) $V =$ Hilbert space, with norm $\| \ \|$ and scalar product $((\ , \ ))$

(5.2) $a : V \times V \longrightarrow \mathbb{R}$ a bilinear continuous form

(5.3) $L : V \longrightarrow \mathbb{R}$ a linear continuous form.

We know that, under the hypothesis

(5.4) $\exists \alpha > 0, \quad a(v,v) \geq \alpha \|v\|^2$ (ie a is V-elliptic)

the problem

(Q) find $u \in V$ s.t. $a(u,v) = L(v) \quad \forall v \in V$

has a unique solution.

Let h be an index (which shall tend to zero), which, in practical applications, shall be related to the "mesh size" of the approximation.

To every such h, we associate a finite dimensional subspace V_h of V, ie:

(5.5) $\forall h, \quad \dim V_h = I(h) < +\infty$

and we define the approximated problem (Q_h) by:

(Q_h) find $u_h \in V_h$ s.t. $a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h$

We have first the:

Thm 5 With the hypothesis and notation (5.1) to (5.5), the problem (Q_h) has a unique solution u_h, related to the solution u of (Q) by:

(5.6) $$\|u_h - u\| \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|v_h - u\|$$

where M and α are respectively the continuity and V-ellipticity constants of the bilinear form a:

Proof. The uniqueness and existence of u_h results obviously from the application of the Lax Milgram theorem to the problem (Q_h).

Taking $v = v_h$ in (Q) and substracting from (Q_h) gives:

$$a(u_h - u, v_h) = 0 \qquad \forall\, v_h \in V_h$$

So that

$$a(u_h - u, u_h - u) = a(u_h - u, v_h - u) \quad \forall v_h \in V$$

Using the continuity and V-ellipticity of a gives

$$\alpha \|u_h - u\|^2 \leq M \|u_h - u\| \, \|v_h - u\| \qquad \forall v_h \in V$$

which gives (5.6). ∎

As we want that the approximate solution u_h tends toward u when $h \to 0$, we have to make some hypothesis on the choice of the approximation spaces V_h. A reasonable hypothesis seems to ask that there exists a family of linear applications I_h from V in V_h such that, for every $v \in V$ (as we don't know u!), $I_h v \to v$ in V as $h \to 0$.

In fact its sufficient that this property holds on a dense subset $\mathcal{V}$ of V, as stated in the

Thm 6 Under the hypothesis of thm 5 and if there exists a dense subspace $\mathcal{V}$ of V and a family of linear continuous applications r_h of $\mathcal{V}$ in V_h such that

(5.7) $\forall v \in \mathcal{V} \qquad r_h v \longrightarrow v$ in V as $h \to 0$

then

(5.8) $$u_h \longrightarrow u \quad \text{in } V \text{ as } h \to 0$$

i.e. the approximated solution u_h tends toward the exact solution u as the "mesh size" h tends to zero.

Proof: Let $\varepsilon > 0$ be given. As $\mathcal{V}$ is dense in V, there exists a $v \in \mathcal{V}$ such that

$$\| u - v \| \leq \frac{\alpha \varepsilon}{2M}$$

As $r_h v \to v$, $\exists\, h_0$ such that

$$h \leq h_0 \quad \Rightarrow \quad \| r_h v - v \| \leq \frac{\alpha \varepsilon}{2M}$$

so that

$$\| u - r_h v \| \leq \frac{\alpha \varepsilon}{M}$$

taking then $v_h = r_h v$ in (5.6) gives:

$$\| u_h - u \| \leq \varepsilon \quad \text{when } h \leq h_0$$

■

Remark 8 The definition of the approximated problem (Q_h) requires only the choice of the family of spaces V_h. The choice of $\mathcal{V}$ and r_h is "necessary" only for the proof of the convergence of the approximation ■

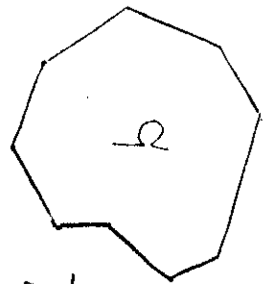
Remark 9: The thm 6 proves only that $u_h \to u$, but do not give any estimation for the error made when replacing u by u_h. As follows from formula (5.6) such estimation of error always requires some a-priori knowledge on the exact solution u, as u appears in the right-hand side of (5.6)!

■

2) Finite-elements approximation of the space $H^1(\Omega)$

We shall construct, in that paragraph, a family of subspaces V_h of $H^1(\Omega)$ for the case:

(5.9) $\quad \Omega \subset \mathbb{R}^2$, Ω is polygonal

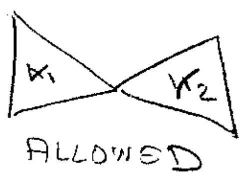


using the technique of finite elements.

To every $h>0$ we make correspond a finite family $\mathcal{C}_h$ of closed triangles K such that:

(5.10) $$\bar{\Omega} = \bigcup_{K\in\mathcal{C}_h} K$$

(5.11) $\begin{cases} \text{given two triangles } K_1 \text{ and } K_2 \text{ of } \mathcal{C}_h \text{, the only possible situations are:} \\ K_1 \cap K_2 = \emptyset \\ K_1 \cap K_2 = 1 \text{ common side} \\ K_1 \cap K_2 = 1 \text{ common vertex} \end{cases}$



(5.12) $$\operatorname{diam} K \leq h \quad \forall K\in\mathcal{C}_h$$

The subspaces V_h of $H^1(\Omega)$ are then defined by:(1)

(5.13) $$V_h = \{ v_h \in \mathcal{C}^0(\bar{\Omega}) \mid v_h|_K \in \mathbb{P}^1 \ \forall K\in\mathcal{C}_h \}$$

where $\mathbb{P}^1$ denotes the space of polynomials of degree ≤ 1 in two variables.

Define (cf fig 7)

(5.14) $\quad \Omega_h = \{\text{all vertex of } K \text{ with } K\in\mathcal{C}_h\}$, $\quad I(h) = \operatorname{card} \Omega_h$

Then obviously we have the

Thm 7: The subspaces V_h of $H^1(\Omega)$ defined by (5.13) are finite-dimensional, and precisely:

(5.15) $$\dim V_h = I(h) < +\infty \quad \forall h > 0$$

A function $v_h \in V_h$ is uniquely determined by the data of its values on the vertex in Ω_h (those values are called "the degree of freedom of the space V_h")

The corresponding basis of V_h is the collection of functions $\{w_M,\ M \in \Omega_h\}$ defined by:

(5.16) $$w_M(P) = \delta_{MP} = \begin{cases} 0 & \text{if } M \neq P \\ 1 & \text{if } M = P \end{cases} \qquad \forall P \in \Omega_h$$

It is easy to check that

(5.17) $$\operatorname{Supp} w_M = \bigcup_{M \text{ is a vertex of } K} K$$

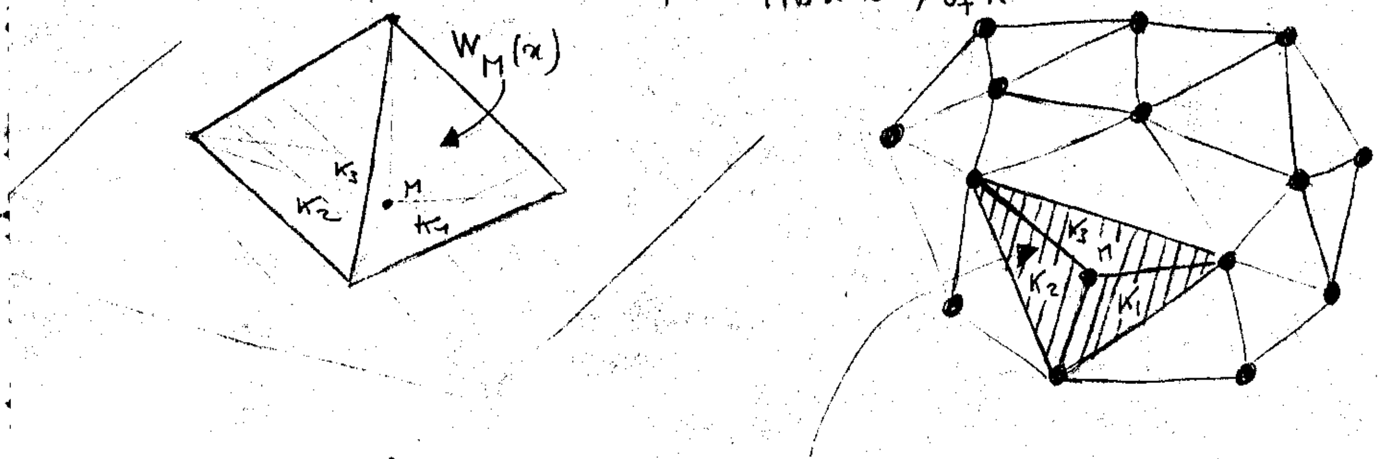


Fig 7 the basis function w_M, its support, $\mathcal{C} = \bigcup$ triangles, $\Omega_h = \{\bullet\}$

With that choice of basis functions we have:

(5.18) $$\forall v_h \in V_h, \quad v_h(x) = \sum_{M \in \Omega_h} v_h(M)\, w_M(x)$$

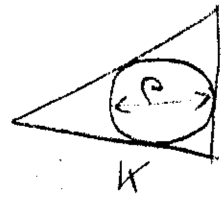
Let us now turn to the convergence of this approximation of $H^1(\Omega)$: we have to construct a dense subspace $\mathcal{V}$ of V and a family r_h of restriction of $\mathcal{V}$ to V_h.

Define the inferior diameter ρ of any triangle K by:

(5.19) $$\rho(K) = \sup_{\rho>0 \text{ s.t. } \exists \text{ disk of diameter } \rho \text{ included in } K} \rho$$

and:

(5.20) $$\rho_h = \min_{K\in\mathcal{C}_h} \rho(K)$$



Then if we choice

(5.21) $$\mathcal{V} = \{v \mid \exists w \in \mathcal{C}^2(\mathbb{R}^2), \text{ s.t. } v = w|_{\bar\Omega}\}$$

and:

(5.22) $$\pi_h : \forall v \in \mathcal{C}^2(\bar\Omega) \longrightarrow v_h = \pi_h v = \sum_{M\in\Omega_h} v(M)\, w_M$$

we have the

Theorem 8: With the hypothesis and notation (5.9) to (5.14) and (5.19) to (5.22), for every $v\in\mathcal{V}$, the function $v_h = \pi_h v$ is related to v by the following majorations:

(5.23) $\forall x \in \Omega$, $$|v_h(x) - v(x)| \leq \frac{M}{2} h^2$$

(5.24) $\forall K\in\mathcal{C}_h, \forall x\in \mathring{K}$, $$\|v_h'(x) - v'(x)\| \leq \frac{M}{2} h^2 \frac{3C}{\rho}$$

where C is a constant independant of h, and where:

(5.25) $$M = \max_{x\in\overline{\Omega}} \|v''(x)\|$$

Hence:

(5.26) $$\|v_h - v\|_{H^1(\Omega)} \leq \frac{1}{2} M \operatorname{meas}\Omega\; h \sqrt{h + 3C\frac{h}{\rho}}$$

Let us first give some lemmas:

Lemma 1: $\forall v \in \mathcal{V}$ and $v_h = \pi_h v$ defined by (5.22) we have:

(5.27) $$v_h(x) = v(x) + \sum_{M\in\Omega_h} \frac{1}{2} v''(\tilde{x}_M)(M-x)^2\, w_M(x) \qquad \forall x \in \Omega$$

(5.28) $$v_h'(x) = v'(x) + \sum_{M\in\Omega_h} \frac{1}{2} v''(\tilde{x}_M)(M-x)^2\, w_M'(x) \qquad \begin{matrix}\forall K\in\mathcal{C}_h\\ \forall x\in\mathring{K}\end{matrix}$$

where $\tilde{x}_M \in [x, M]$ $\forall M \in \Omega_h$.

Proof (Exercise!) For every $M\in\Omega_h$, substitute the second order development

(5.29) $$v(M) = v(x) + v'(x)(M-x) + \frac{1}{2} v''(\tilde{x}_M)(M-x)^2$$

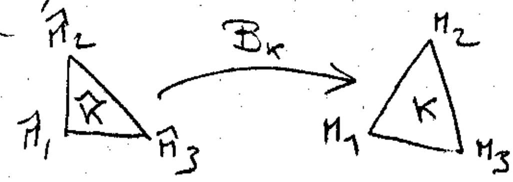
in the definition (5.22) of $v_h(x)$. Taking successively ...

$v \in P^1$ gives (5.27). Similarly, substituting (5.29) in (5.30)

$$v'_h(x) = \sum_{M \in \Omega_h} v(M)\, w'_M(x)$$

gives (5.28). ■

<u>Lemma 2</u>: Let $\hat{K}$ be a reference triangle in a reference plane with coordinates $\hat{x} = (\hat{x}_1, \hat{x}_2)$. For every $K \in \mathcal{T}_h$, we denote by B_K the (uniquely defined) linear application from $\mathbb{R}^2$ in $\mathbb{R}^2$ such that



(5.31) $\begin{cases} B_K \hat{M}_i = M_i & i = 1,2,3 \\ \text{where } \hat{M}_i \text{ and } M_i \text{ are the vertex of } \hat{K} \text{ and } K \end{cases}$

Then

(5.32) $\|B_K\| \leq \dfrac{h}{\hat{\rho}}, \quad \|B_K^{-1}\| \leq \dfrac{\hat{h}}{\rho}$

where $\hat{h}$ and $\hat{\rho}$ are associated to $\hat{K}$ as h and ρ to K.

<u>Proof</u>: $\|B_K\| = \sup_{\hat{\xi} \neq 0} \dfrac{\|B_K \hat{\xi}\|}{\|\hat{\xi}\|} = \sup_{\|\hat{\xi}\| = \hat{\rho}} \dfrac{\|B_K \hat{\xi}\|}{\hat{\rho}}$

As $\|\hat{\xi}\| = \hat{\rho}$, $\exists\, \hat{x}_1$ and $\hat{x}_2 \in \hat{K}$ s.t. $\hat{\xi} = \hat{x}_1 - \hat{x}_2$. Hence:

$$\|B_K \hat{\xi}\| = \|B_K \hat{x}_1 - B_K \hat{x}_2\|$$

$\leq h$ as both $B_K \hat{x}_1$ and $B_K \hat{x}_2$ belong to K. ■

<u>Proof of thm 8</u> Using lemma 1 and (5.25) gives

$$|v_h(x) - v(x)| \leq \frac{M}{2} \sum_{M \in \Omega_h} \|M - x\|^2 \, |w_M(x)| \qquad \forall x \in \Omega$$

$$\|v'_h(x) - v'(x)\| \leq \frac{M}{2} \sum_{M \in \Omega_h} \|M - x\|^2 \, \|w'_M(x)\| \quad \forall K \in \mathcal{T}_h, \forall x \in \mathring{K}$$

For a given $M \in \Omega_h$, either $x \in \operatorname{supp} w_M$, and then

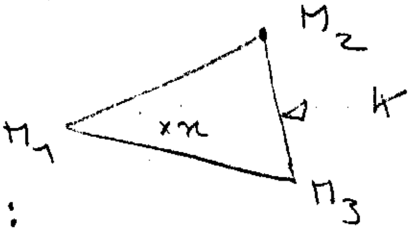
$$|v_h(x)-v(x)| \leq \frac{M}{2} h^2 \sum_{M\in\Omega_h} |w_M(x)| \qquad \forall x \in \Omega$$

(5.33) $$\|v_h'(x)-v'(x)\| \leq \frac{M}{2} h^2 \sum_{M\in\Omega_h} \|w_M'(x)\| \qquad \forall K\in\mathcal{C}_h,\ \forall x \in \mathring{K}$$

It's easy to check that $\sum_{M\in\Omega_h} |w_M(x)| = 1$ which gives (5.25).

We have now to get a majoration for $\sum_{M\in\Omega_h} \|w_M'(x)\|$.

As $x\in\mathring{K}$, this summation contains only three non-zero terms, corresponding to the three vertex $M_1\ M_2\ M_3$ of K.



Then we have, with the notations of Lemma 2:

$$\begin{cases} w_{M_i}(x) = \hat{w}_i(B_K^{-1}x) & i=1,2,3 \\ \text{where } \hat{w}_i \text{ is the linear function of } (\hat{x}_1,\hat{x}_2) \text{ s.t. } \hat{w}_i(\hat{M}_j)=\delta_{ij} \end{cases}$$

so that:

(5.34) $$w'_{M_i}(x) = \hat{w}_i'(\hat{x})\, B_K^{-1} \qquad \text{with } \hat{x} = B_K^{-1}x \in \hat{K}$$

and, using (5.32)

(5.35) $$\|w'_{M_i}(x)\| \leq C/\rho$$

with

(5.36) $$C = \hat{h} \max_{i=1,2,3} \|\hat{w}_i'\|$$ <u>is independent of h</u>

independent of $\hat{x}$ as $\hat{w}_i$ is linear!

Then using (5.35) we get:

$$\sum_{M\in\Omega_h} \|w'_M(x)\| \leq 3C/\rho$$ with <u>C = cst indep of h.</u>

which gives (5.24)

The majoration (5.26) is a simple corollary of (5.23)(5.24). This ends the proof of thm 8. ∎

The thm 8 shows that, if we want that the approximation V_h is converging, it is not sufficient that the size h of all triangles K tends to zero : we see from (5.26) that the ration h/ρ of exterior to interior diameter of all triangles has to be bounded.

Corollary of thm 8: With the hypothesis of thm 8 and

(5.37) $\quad \frac{h}{\rho}$ bounded as $h \to 0$

or equivalently

(5.38) the smallest angle of the triangles K of $\mathcal{T}_h$ is bounded below as $h \to 0$

then the approximation of $H^1(\Omega)$ by the family of spaces V_h defined in (5.13) is convergent when $h \to 0$.

Proof: It results from (5.26) and thm 6, and from the fact that the space $\mathcal{V}$ defined by (5.21) is dense in $H^1(\Omega)$ (see) ■