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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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DISCONTINUOUS WAVES
(Part III)

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11. The Growth of Acceleration Waves

Returning now to general properties of acceleration waves in elastic
materials, we take up the important question of the way in which the
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$$\underline{a} = a\underline{m}, \quad \underline{m} \in S, \tag{11·1}$$

where, without loss of generality, $\underline{m}$ is so chosen that $\underline{m}\cdot\underline{n} \geqslant 0$. We refer to a as the <u>strength</u> of the wave Σ. When $\underline{m}\cdot\underline{n} > 0$, Σ is said to be <u>expansive</u> when $a > 0$ and <u>compressive</u> when $a < 0$; when $\underline{m}\cdot\underline{n} = 0$, Σ is (locally) a transverse wave. Motivation for this terminology is provided by the jump condition $(7{\cdot}18)_3$: we see that $\dot{\rho}^- \lesseqgtr \dot{\rho}^+$ according as Σ is expansive, transverse or compressive.

The variation of the strength a as Σ propagates has two causes: mechanical processes intrinsic to the transmitting material and geometrical effects associated with the progressive spreading of Σ_t. In this introductory treatment of the growth of acceleration waves we shall confine attention to mechanical considerations by assuming that

(i) at $t = 0$, Σ_t is plane and is a uniform wave in the sense that

(ii) a region of uniform state (in which $\underset{\sim}{v} = \underset{\sim}{0}$ and $\underset{\sim}{F}$ is uniform and constant) lies ahead of Σ_t.

It then follows from the propagation condition (9.9) and the definition (that V and $\underset{\sim}{m}$ remain constant in time as well as uniform on Σ. In short, Σ is a uniform plane wave at all relevant times in the interval $t \geqslant 0$.

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First we require compatibility conditions for jumps in third partial derivatives of the deformation function $\underset{\sim}{\psi}$ on a singular surf of order two. The derivatives in question are $\ddot{v}_i$ and $\partial L_{ij}/\partial x_k$ and it can be shown that, under the restrictions (i) and (ii) imposed above,

$$[\ddot{v}_i] = 2\frac{da_i}{dt} + V^{-1}a_i a_p n_p + V^2 A_i, \tag{11.2}$$

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where $A_i = [\partial^2 v_i/\partial n^2]$ and d/dt is the rate of change as perceived by an observer moving with Σ_t in the direction of the normal $\underset{\sim}{n}$. may be remarked that when the restriction to uniform plane waves is lifted the derivative on the right-hand side of equation (11.2) is taken along the ray, or bicharacteristic, through the representative

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Secondly it is necessary to introduce the second-order elasticity tensor $\underset{\sim}{\mathcal{B}}$ associated with the choice of stress and deformation measures underlying the linear elasticity tensor $\underset{\sim}{B}$. $\underset{\sim}{\mathcal{B}}$ is the sixth-order tensor with components

$$\mathcal{B}_{ijklmn} = J^{-1} F_{ip} F_{kq} F_{mr} \frac{\partial^2 \hat{S}_{pj}}{\partial F_{lq}\, \partial F_{nr}} = J^{-1} F_{ip} F_{kq} F_{mr} \frac{\partial^3 \hat{W}}{\partial F_{jp}\, \partial F_{lq}\, \partial F_{nr}} \qquad (11.4)$$

cf. equations (9.5)), and we take note of the symmetry properties

$$\mathcal{B}_{ijmnkl} = \mathcal{B}_{klijmn} = \mathcal{B}_{ijklmn}. \qquad (11.5)$$

t follows from (11.5) that the number of independent components of $\underset{\sim}{\mathcal{B}}$ is 65.

b) The growth equation. Stage 1. The central objective of Section 11 is the derivation of an ordinary differential equation for the strength a. It is achieved by forming jumps on Σ in a differentiated form of the equation of motion.

We start by returning to the referential equation of motion (8.8) and taking the material time derivative of each term:

$$\rho_r \ddot{v}_i = \frac{\partial \dot{s}_{qi}}{\partial X_q} + \rho_r \dot{b}_i = \frac{\partial \dot{s}_{qi}}{\partial x_p} F_{pq} + \rho_r \dot{b}_i . \tag{11.6}$$

On multiplying through by J^{-1} and using the integrated continuity equation (8.5) together with the kinematical formula

$$\frac{\partial}{\partial x_p}\left(J^{-1} F_{pi} \right) = 0, \tag{11.7}$$

we can recast (11.6) as

$$\rho \ddot{v}_i = \frac{\partial}{\partial x_p}\left(J^{-1} F_{pq} \dot{s}_{qi} \right) + \rho \dot{b}_i . \tag{11.8}$$

Next we appeal to the constitutive equation $(8.9)_2$, the kinematical result $\dot{\underset{\sim}{F}} = \underset{\sim}{L}\,\underset{\sim}{F}$ (included in the Exercise on p.17) and the definition (9.5) in writing

$$J^{-1} F_{pq} \dot{s}_{qi} = J^{-1} F_{pq} \frac{\partial \hat{s}_{qi}}{\partial F_{st}} \dot{F}_{st} = J^{-1} F_{pq} F_{rt} \frac{\partial \hat{s}_{qi}}{\partial F_{st}} L_{sr} = B_{pirs} L_{sr} . \tag{11.9}$$

Equation (11.8) then becomes

$$\rho \ddot{v}_i = \frac{\partial B_{pirs}}{\partial x_p} L_{sr} + B_{pirs} \frac{\partial L_{sr}}{\partial x_p} + \rho \dot{b}_i . \tag{11.10}$$

We are now ready to form jumps on Σ and it will be assumed that $\dot{\underline{b}}$, as well as the extrinsic force $\underline{b}$ itself, is continuous. In the first term on the right-hand side of (11.10) both $\partial B_{pirs}/\partial x_p$ and L_{sr} are discontinuous on Σ. However, each of these fields vanishes ahead of Σ by virtue of assumption (i) so the jump of a product in this ase equals minus the product of the individual jumps. The discontin-ity relation derived from (11·10) is accordingly

$$\rho[\ddot{v}_i] = -\left[\frac{\partial B_{pirs}}{\partial x_p}\right][L_{sr}] + B_{pirs}\left[\frac{\partial L_{sr}}{\partial x_p}\right]. \qquad (11·11)$$

:) <u>Calculation of jumps.</u> Of the four jumps appearing in (11·11), $[\ddot{v}_i]$ is iven by equation (11·2), $[L_{sr}]$ by $(7·18)_2$ and $[\partial L_{sr}/\partial x_p]$ by (11·3). To alculate the remaining jump we proceed as follows, using in turn $(9·5)_1$, ·17), $(11·4)_1$, $(9·5)_1$ again and (9·7).

$$\left[\frac{\partial B_{pirs}}{\partial x_p}\right] = \frac{\partial B_{pirs}}{\partial F_{tu}}\left[\frac{\partial F_{tu}}{\partial x_p}\right] = \frac{\partial}{\partial F_{tu}}\left(J^{-1}F_{pv}F_{rw}\frac{\partial \hat{s}_{vi}}{\partial F_{sw}}\right)V^{-2}a_t F_{xu} n_x n_p$$

$$= V^{-2}n_p n_x\left(J^{-1}F_{pv}F_{rw}F_{xu}\frac{\partial^2 \hat{s}_{vi}}{\partial F_{sw}\partial F_{tu}}a_t + J^{-1}F_{xu}F_{rw}\frac{\partial \hat{s}_{ui}}{\partial F_{sw}}a_p\right.$$

$$\left. + J^{-1}F_{pv}F_{xu}\frac{\partial \hat{s}_{vi}}{\partial F_{su}}a_r - J^{-1}F_{pv}F_{rw}\frac{\partial \hat{s}_{vi}}{\partial F_{sw}}a_x\right)$$

$$= V^{-2}\left(\mathcal{B}_{pirsxt}a_t + B_{xirs}a_p + B_{pixs}a_r - B_{pirs}a_x\right)n_p n_x$$

$$= V^{-2}\left\{\mathcal{B}_{pirstu}n_p n_t a_u + Q_{is}(\underline{n})a_r\right\}. \qquad (11·12)$$

d) <u>The growth equation. Stage 2.</u> On substituting from equations (11·2), 1·12), $(7·18)_2$ and (11·3) into the discontinuity relation (11·11) we now nd that

$$\left(2\frac{da_i}{dt} + V^{-1}a_i n_r a_r + V^2 A_i\right) = V^{-3}\mathcal{B}_{pirstu}n_p n_r a_s n_t a_u + V^{-3}Q_{iq}(\underline{n})a_q n_r a_r$$

use being made also of the definition (9·7) and the symmetry property $(9\cdot 8)_2$. To obtain a differential equation in a we mu... through by a_i and sum on the repeated index i. The result is

$$\rho \frac{d}{dt}(a_p a_p) + \rho V^{-1} a_p a_p n_r a_r + \rho V^2 a_p A_p = V^{-3} \mathcal{B}_{pqrstu} n_p a_q n_r a_s n_t a_u$$

$$+ V^{-3} Q_{pq}(\underline{n}) a_q a_p n_r a_r + Q_{pq}(\underline{n}) a_q A_p, \qquad (11\cdot 14)$$

and, on account of the propagation condition (9.9) the last two terms on the left are seen to cancel the second and third terms on the right. Hence, using (11·1) we arrive at the required <u>growth equation</u> in the simple form

$$\frac{da}{dt} = \alpha a^2, \qquad (11\cdot 15)$$

where

$$\alpha = \frac{1}{2\rho V^3} \mathcal{B}_{pqrstu} n_p m_q n_r m_s n_t m_u . \qquad (11\cdot 16)$$

We recall that the propagation speed V and the unit vector $\underline{m}$ defining the direction of the amplitude are determined by the propagation condition (9.9). Under the simplified conditions assumed here, α is a constant: its physical dimensions are $L^{-1}T$.

(e) <u>Solution of the growth equation.</u> Subject to the initial condition the growth equation (II.15) has the unique solution

$$a = \frac{a_0}{1-\alpha a_0 t}. \tag{II.18}$$

The following possibilities (illustrated in Figure 3) arise.

(i) If $\alpha \neq 0$ and $\alpha a_0 > 0$, a/a_0 increases monotonically and without bound over the finite time interval $0 \leq t < (\alpha a_0)^{-1}$.

(ii) If $\alpha \neq 0$ and $\alpha a_0 < 0$, a/a_0 decreases monotonically, approaching zero as $t \to \infty$.

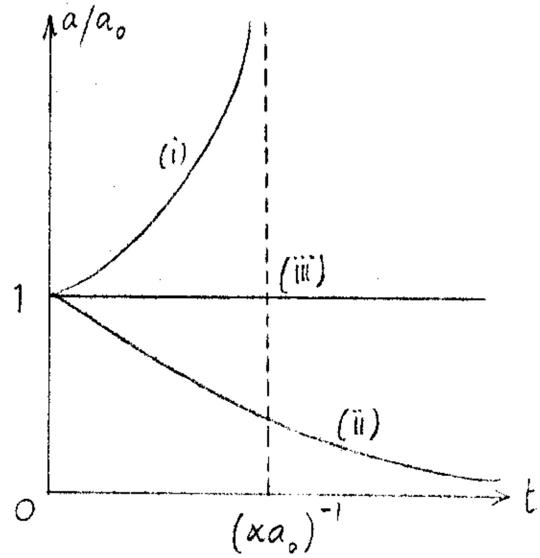

Figure 3

(iii) If $\alpha = 0$, then $a = a_0$ for all $t \geq 0$ and Σ propagates with constant strength.

We conclude from (i) and (ii) that an expansive wave (for which $a_0 > 0$) grows if $\alpha > 0$ and decays if $\alpha < 0$, while a compressive wave ($a_0 < 0$) grows if $\alpha < 0$ and decays if $\alpha > 0$.

As regards the interpretation of the results just obtained, we first observe, from (II.16), that when geometrical effects are excluded the growth of an acceleration wave in an elastic material is entirely controlled by the second-order elasticity tensor $\mathcal{B}$. We are therefore concerned here with an inherently non-linear phenomenon. Next we note that the breakdown of solutions after a finite time is a well-known feature of problems governed by non-linear partial differential equations of hyperbolic type. Familiar examples are the breaking of a water wave and the formatio[n] of a normal shock wave in gasdynamics. In order to continue the solution beyond the appearance of a singularity it is necessary to consider distinct régimes in the space spanned by the space and time variables a[nd] to carry through an appropriate matching process. The quoted examples suggest that when possibility (i) applies the steadily "steepening" accelerat[ion] wave degenerates into a shock wave at $t = (\alpha a_0)^{-1}$. A rigorous continua[t]ion of an elastic acceleration wave solution into a shock régime has ye[t] to be constructed, however. Finally we recall that shock formation occur[s] in a gas when wavelets in the tail of a disturbance travel faster tha[n] wavelets near the head. The criterion for the growth of an acceleration wave in an elastic material can given a similar interpretation, but whereas in a compressible inviscid fluid the local speed of sound is normally an increasing function of pressure, it appears that the speed of body waves in an elastic solid is usually raised by tension, not

Exercise. Investigate the growth of plane uniform acceleration waves in the elastic material defined in the last Exercise (p.29). Show, in particular, that transverse waves propagate with constant strength and that, in he case of a longitudinal wave, $\alpha = (2\rho V_L^3)^{-1} J^2 f'''(J)$.

2. Concluding Remarks

The discussion of acceleration waves in elastic materials given in he last three sections is quite general. No assumptions have been rade about the symmetry of the material considered and it has ot been necessary to place objectivity restrictions on the constitutive quation. Even the notion of a strain-energy function could have been ispensed with had we so wished. The results derived in these ectures therefore require only the most rudimentary characterization f elastic behaviour.

From these beginnings there are many directions in which the urther study of discontinuous waves may be pursued. Within the mbit of the mechanical theory of elasticity the most important levelopments concern isotropic materials and materials with internal onstraints, notably incompressibility and inextensibility. A logical framework of the thermo-mechanical theory of elastic materials. Looking further afield, there is a fairly long list of particular branches of continuum mechanics in which acceleration wave studies have been carried out. It includes

Hypoelasticity	Elastic dielectrics
Plasticity	Magnetoelasticity
Materials with memory	Chemically reacting continua
Fluids with internal state variables	

Further details and references can be found in two recent review articles.

P. J. Chen. Growth and decay of waves in solids. Handbuch der Physik (ed. S. Flügge), Vol. VIa/3. Springer, Berlin etc., 1973.

M. F. McCarthy. Singular surfaces and waves. Continuum Physics (ed. A. C. Eringen), Vol. II. Academic Press, New York etc., 1975.