INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23/60

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

IDENTIFICATION OF SYSTEMS

(Part III)

G. Chavent
I R I A
Le Chesnay
France

---

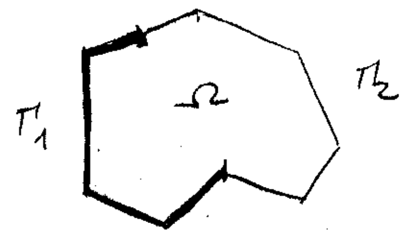
3) Numerical approximation of the PDE of § IV

We come back now to the elliptic PDE of § (IV), and take:

$$(5.39)\left\{\begin{array}{l} f,\ \Omega,\ V=\{v\in H^1(\Omega)\,/\,v=0 \text{ on } \Gamma_1\},\ a(u,v),\ L(v) \\ \text{as defined by } (4.3) \text{ to } (4.7)\end{array}\right.$$

In order to be able to use the results of § (V)-2) we suppose moreover that:

$$(5.40)\left\{\begin{array}{l} \Omega\subset\mathbb{R}^2,\ \Omega \text{ is polygonal}, \\ \Gamma_1 = \text{finite union of intervals}\end{array}\right.$$



and we define a family of triangulation $\mathcal{T}_h$, $h>0$ s.t.:

(5.41) $\{\mathcal{T}_h, h>0\}$ satisfies (5.10) to (5.12)

and

$$(5.42)\left\{\begin{array}{l} \forall h>0,\ \forall K\in\mathcal{T}_h, \text{ the only allowed cases are:} \\ \quad \partial K\cap\Gamma_1=\emptyset \quad \text{or} \quad \partial K\cap\Gamma_1 = \text{a side of } K\end{array}\right.$$

We define then an approximation of the space V in (5.39) by a family of space V_h, $h>0$ s.t. (compare with (5.13))

$$(5.43)\quad V_h = \{v_h\in\mathcal{C}^0(\bar\Omega)\,/\,v_h|_K\in P^1\ \forall K\in\mathcal{T}_h,\ v_h|_{\Gamma_1}=0\}$$

Define (cf fig 8)

$$(5.44)\left\{\begin{array}{l} \Omega_h=\{\text{all vertex } M \text{ of } K \text{ with } K\in\mathcal{T}_h, \text{ such that } M\notin\Gamma_1\} \\ I(h) = \text{card } \Omega_h\end{array}\right.$$

Then it is easy to check that, thanks hypothesis (5.42) we have

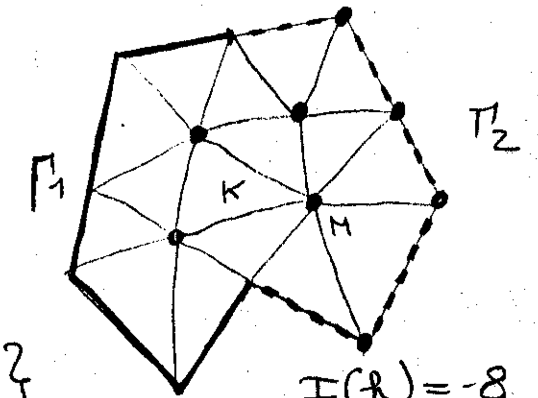
Thm 9 With hyp. and notations (5.39) to (5.44) we have

(5.45) $\dim V_h = I(h)$

and:

(5.46) $\begin{cases} \text{a function } v \in V_h \text{ is uniquely determined by the data of its} \\ \text{values on } \Omega_h \text{, ie the functions } \{w_M, M \in \Omega\} \text{ defined by (5.16)} \\ \text{are a basis of } V_h. \end{cases}$

Figure 8



$\Gamma_1 =$ (thick line)

$\Gamma_2 =$ (dashed line)

$\Omega_h = \{\bullet\}$

In order to prove that this approximation is converging, just define:

(5.47) $\mathcal{V} = \{ v \mid \exists w \in \mathcal{C}^2(\mathbb{R}^2) \text{ st } v = w|_\Omega,\ v|_{\Gamma_1} = 0 \}$

and a family r_h of applications of $\mathcal{V}$ into V_h by (5.22).

As thm 8 applies obviously so that (5.26) holds for every $v \in \mathcal{V}$.

Thm 10 With hypothesis and notation (5.39) to (5.44) and (5.37) or (5.38), the approximation of V by the family of spaces V_h, $h > 0$ is converging ie:

(5.48) $\forall v \in \mathcal{V} \qquad v_h = r_h v \to v$ in V as $h \to 0$.

Following now the §1) we define the "approximate solution u_h of the state equation (P) by the equation (Q_h) of §1), and, as as $\mathcal{V}$ is dense in V, it results from Thm 10 and 6 the

Thm 11 Under the hyp of Thm 10, we have

(5.49) $\qquad u_h \to u$ in V as $h \to 0$

when u and u_h are the solution of (Q) and (Q_h).

As we are now sure that our approximation is converging, let us turn now to its practical determination

The determination of the function $u_h \in V_h$ resumes, using (5.46), in the determination of its $I(h)$ values v_M at each point M of Ω_h (M is called a node of the approximation).

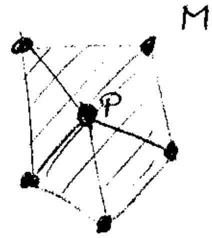
(5.50) $$u_h = \sum_{M \in \Omega_h} u_M \, w_M$$

Due to the bilinearity of $a(u,v)$, the definition (Φ_h) of u_h is equivalent, as $(w_P, P \in \Omega)$ is a basis of V_h, to:

(5.51) $$\boxed{\sum_{M \in \Omega_h} u_M \, a(w_M, w_P) = L(w_P) \qquad \forall P \in \Omega_h}$$

Remark 10

As the bilinear form a is an integral of product of derivatives of w_M and of w_P, the summation in the left-hand side of (5.51) may be reduced to the only nodes M of Ω_h belonging to the support of w_P, so that in the linear system (5.51), a lot of coefficients $a(w_M, w_P)$ are equal to zero! ∎



Supp w_P

Once a numerotation of the nodes of Ω_h has been choosen (and this is possible in many different ways!), we may define:

(5.52)
$$\begin{cases} \underset{\sim}{u}_h = \begin{bmatrix} u_1 \\ \vdots \\ u_{I(h)} \end{bmatrix} \quad , \quad \underset{\sim}{L}_h = \begin{bmatrix} L(w_1) \\ \vdots \\ L(w_{I(h)}) \end{bmatrix} \\ A_h = [a_{ij}] \quad \text{with} \quad a_{h\,ij} = a(w_j, w_i) \end{cases}$$

so that eq. (5.51) may be rewritten in matrix form:

(5.53) $$A_h \underset{\sim}{u}_h = \underset{\sim}{L}_h$$

It results from (5.52) and from the V-ellipticity of $a(u,v)$ that the matrix A_h is symmetric, positive-definite, and from remark 10 that it contains many zero terms. So the numerotation of the nodes of Ω_h has to be choosen, when possible, to minimize the band-width of the matrix A_h, in order to minimize the number of memory required for the numerical resolution of (5.53) and to have a better speed of resolution.

Exercise 5 : Choose a numerotation of Ω_h on fig 8, and give the structure of the matrix A_h (ie find the location in A_h of zero and a-priori non zero elements) ■

Exercise 6 For Ω, Γ_1, Γ_2 and $\mathcal{C}_h$ given by fig 9, find the numerotation of Ω_h that minimizes the band-width of A_h (Ω is a crown). ■

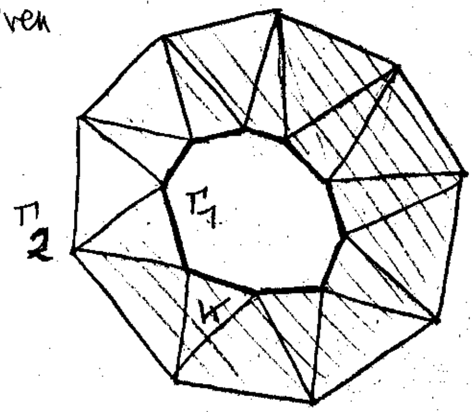


Figure 9 $\Omega =$

Exercise 7 For Ω, Γ_1, Γ_2, g and $\mathcal{C}_h$ given by fig 9, calculate completely the matrix A_h and right hand side L_h in the following cases:

i) $a(x) = a$ is constant on Ω

ii) as i) but using for all integrations the quadrature formula:

$$\begin{cases} \forall f \text{ continuous } \displaystyle\int_K f\,dx \simeq \frac{1}{3\,\text{meas}K}\left(f(n_1)+f(n_2)+f(n_3)\right) \\ \text{(triangle } K \text{ with vertices } n_1, n_2, n_3) \end{cases}$$

iii) what happens if a is not constant? ■

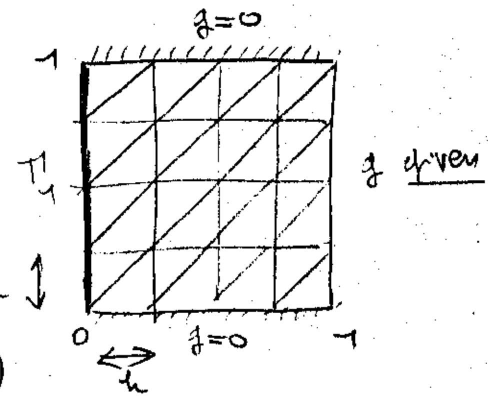


Fig 10

4) The approximate error function and its gradient.

Let us first approximate the cost function J as defined in (4.1) by:

(5.54) $$J_h(a) = \int_\Omega (u_h(x;a) - z(x))^2 \, dx \quad \forall a \in \mathcal{O}_c.$$

As u_h is given by the variational formulation (Q_h) on the Hilbert space V_h, all the "continuous" theory of the paragraph IV applies here by simply replacing V by V_h:

(5.55) the functional J_h is C^∞ on the set $\mathcal{O}_c$ and we know how to calculate its derivative with the help of a Lagrangian. Define:

(5.56) $$\begin{cases} \forall (u_h, a) \in V_h \times \mathcal{O}_c, \ \forall p_h \in V_h \text{ let} \\ \mathcal{L}(u_h, a; p_h) = |u_h - z|^2 + a(u_h, p_h) - L(p_h) \end{cases}$$

Then the adjoint state p_h is then defined by:

(5.57) $$\frac{\partial \mathcal{L}}{\partial u_h}(u_h(a), a; p_h)\, v_h = 0 \qquad \forall v_h \in V_h$$

that is (compare to (Q_h)):

$(\tilde{Q}_h)$ $$\boxed{\text{find } p_h \in V_h \text{ s.t. } a_h(v_h, p_h) = -2(v_h, u_h - z) \quad \forall v_h \in V_h}$$

Let us denote by $\underset{\sim}{P_h}$ the vector of the coordinate of p_h on the basis $(w_i, \ i = 1, 2 \ldots I(h))$ of V_h. As the bilinear

$$(5.58) \qquad A_h \, p_h = \underset{\sim}{S}_h$$

where

$$(5.59) \qquad \underset{\sim}{S}_h = -2 \begin{bmatrix} (w_1, u_h - z) \\ \vdots \\ (w_{I(h)}, u_h - z) \end{bmatrix}$$

Remark 11 The resolution of eq. (5.58) may be performed by the same program than that of (5.53), as only the right-hand side change! Moreover, the direct state equation (5.53) has to be solved first, as the knowledge of its solution u_h is required for the computation of the right-hand side $\underset{\sim}{S}_h$ of the adjoint equation (5.59)

Let us now give the expression of the derivative of J_h with respect to a:

$$\delta J_h = \frac{\partial \mathcal{L}}{\partial a}\left(u_h(a), a; p_h\right) \delta a = \delta a\left(u_h, v_h\right)$$

that is

$$(5.60) \qquad \delta J_h = \int_\Omega \delta a(x) \sum_{i=1}^{2} \frac{\partial u_h}{\partial x_i} \frac{\partial p_h}{\partial x_i}$$

as u_h and p_h are linear on each triangle K of $\mathcal{T}_h$, (5.60) may be rewritten:

$$(5.61) \qquad \delta J_h = \sum_{K \in \mathcal{T}_h} \left\{ \frac{1}{\operatorname{meas} K} \int_K \delta a(x)\, dx \right\} \left\{ \operatorname{meas} K \sum_{i=1}^{2} \frac{\partial u_h}{\partial x_i} \frac{\partial p_h}{\partial x_i} \right\}.$$

We see from (5.61) that only the mean values of a on each triangle K of $\mathcal{T}_h$ influence the first order variation of the

In fact, the coefficients $a(w_M, w_P)$ of the linear system giving the approximate state u_h are:

$$(5.62) \quad a(w_M, w_P) = \sum_{K \in \mathcal{T}_h} \int_K a(x) \sum_{i=1}^{2} \frac{\partial w_M}{\partial x_i} \frac{\partial w_P}{\partial x_i} \, dx$$

As the basis functions w_M and w_P are linear functions on K, their derivatives are constants, so that $a(w_M \, w_P)$ depends only on the mean values a_K of the function a on the triangles K; hence u_h and J_h depends only on those mean values $(a_K, \, K \in \mathcal{T}_h)$.

So there is no hope to recover, by minimization of the error functional $J_h(a)$ defined in (5.54), more than the mean values a_K of a on every triangle K of $\mathcal{T}_h$.

So we shall:

(5.63) $\begin{cases} \text{choose a numerotation for the triangle } K \text{ of } \mathcal{T}_h, \\ \text{say from } 1 \text{ to } N(h) \end{cases}$

and take, as numerical unknown for the optimization problem the vector $\underset{\sim}{a}_h = (a_1, \dots, a_{N(h)})$ of the mean values of a on each triangle K of $\mathcal{T}_h$.

The functional $J_h(a)$ of (5.54) is now replaced (and equal to!) the function of $N(h)$ variables:

$$(5.64) \quad J_h(\underset{\sim}{a}_h) = \int_\Omega \left(u_h(x, \underset{\sim}{a}_h) - z(x) \right)^2 dx$$

defined $\forall \, \underset{\sim}{a}_h \in \mathcal{A}_{ch}$ with and from (5.61) we see that

$$(5.66)\quad \frac{\partial J_h}{\partial a_K}(\underset{\sim}{a}_h) = \text{meas}\,K \sum_{i=1}^{2} \frac{\partial u_h}{\partial x_i}\frac{\partial p_h}{\partial x_i} \quad \forall K \in \mathcal{C}_h.$$

Remark 12 : The formula (5.66) gives the exact gradient of the approximated error function J_h of (5.64), so that optimizations algorithms as stated in § II and III may work satisfactorily.

This may not have been the case if, starting from continuous calculation made in § IV, we had designed
i) an approximation of the state eq. (Q),
ii) ——————————— (Q^*),
iii) an approximation of the formula (4.19) which gives the continuous gradient of J.

As at each step different choices are possible, the general result would be that the computed gradient would only be an approximation of the gradient of the approximated error function J_h, which may cause difficulty in the use of gradient methods.

Remark 13 : All the integrals appearing in the calculation of $J_h(\underset{\sim}{a}_h)$, $a(w_H, w_P)$, $\underset{\sim}{L}_h$, $\underset{\sim}{S}_h$ are sums of integral on K of polynomials (in two variables) of maximum degree two.

Such integrals can be computed exactly, so that the formulas given above are effective for computation purposes.

However, it can be proved that, if all those integrals are computed using the quadrature formula given in Ex 7 p. 35 (and which is exact for polynomials up to degree 1), then

- the state approximation remains converging.
- the formula (5.66) still gives the exact gradient of $J_h(\underset{\sim}{a}_h)$
- The computation of the cost function J_h, of the matrix A_h and of the adjoint state second member $\underset{\sim}{S}_h$ are much simplified, and hence less time consuming ∎

Remark 14 The steepest descent algorithm with projection given in the fig 6 of § (III) applies directly to the minimization of the approximated error function $J_h(\underset{\sim}{a}_h)$ given in (5.64) on the closed bounded set (for $0<\alpha<M$ given)

$$(5.67)\quad \mathcal{O}_{ad\,h} = \left\{ a_h \in \mathbb{R}^{N(h)} \;\middle|\; \alpha \le a_i \le M \quad \forall i=1..N(h) \right\}$$

The block "Computation of J^k" and "Computation of ∇J^k" of fig 6 are then detailed on the fig 11. ∎

Computation of J^k

knowing $\underset{\sim}{a}_h^k$, compute the matrix A_h^k

Solve the linear system

$$A_h^k u_h^k = \underset{\sim}{L}_h$$

to get the state u_h^k

Compute

$$J_h^k = \int_e \left(u_h^k(x) - z(x)\right)^2 dx$$

computation of ∇J^k

knowing u_h^k, compute the right-hand side $\underset{\sim}{S}_h^k$

Solve the linear system:

$$A_h^k p_h^k = \underset{\sim}{S}_h^k$$

Compute, for any $K \in \mathcal{E}_h$

$$\frac{\partial J_h^k}{\partial a_K} = \text{meas}\, K \sum_{i=1}^{2} \frac{\partial u_h^k}{\partial x_i} \frac{\partial p_h^k}{\partial x_i}$$

Fig 11 : Details of two blocks of the fig 6.

given in Ex 7 p. 35 (and which is exact for polynomials up to degree 1), then

- the state approximation remains converging.
- the formula (5.66) still gives the exact gradient of $J_h(\underset{\sim}{a}_h)$
- The computation of the cost function J_h, of the matrix A_h and of the adjoint state second member $\underset{\sim}{S}_h$ are much simplified, and hence less time consuming ∎

<u>Remark 14</u> <u>The steepest descent algorithm</u> with projection given in the fig 6 of § III <u>applies directly</u> to the <u>minimization</u> of the approximated <u>error function</u> $J_h(\underset{\sim}{a}_h)$ given in (5.64) on the closed bounded set (for $0<\alpha<M$ given)

$$(5.67)\quad \mathcal{C}_{ad\,h} = \left\{ \underset{\sim}{a}_h \in \mathbb{R}^{N(h)} \;\middle|\; \alpha \le a_i \le M \quad \forall i = 1..N(h) \right\}$$

The block "Computation of J^k" and "Computation of ∇J^k" of fig 6 are then detailed on the fig 11. ∎

Computation of J^k

knowing $\underset{\sim}{a}_h^k$, compute the matrix A_h^k

$\downarrow$

Solve the linear system

$$A_h^k \, u_h^k = \underset{\sim}{L}_h$$

to get the state u_h^k

$\downarrow$

Compute

$$J_h^k = \int_\Omega \left(u_h^k(x) - z(x)\right)^2 dx$$

$\downarrow$

Computation of ∇J^k

knowing u_h^k, compute the right-hand side $\underset{\sim}{S}_h^k$

Solve the linear system:

$$A_h^k \, p_h^k = \underset{\sim}{S}_h^k$$

Compute, for any $K \in \mathcal{B}_h$

$$\frac{\partial J_h^k}{\partial a_K} = \text{meas}\, K \sum_{i=1}^{2} \frac{\partial u_h^k}{\partial x_i} \frac{\partial p_h^k}{\partial x_i}$$

$\downarrow$

Fig 11 : Details of two blocks of the fig 6.