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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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BASIC IDEAS IN APPLIED FUNCTIONAL ANALYSIS

(continued)

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this solution. The method of Galerkin, which will be described in the next Chapter, is a first step in this approach.

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III Galerkin's method

III. 1. Introduction.

We started at the first beginning in writing an abstract problem.

(1) $\qquad Au = f.$

which was supposed to represent a large class of functional equations. A operating from a space X into a space Y - f being given in Y - u is the solution that we are looking for in a subset X_0 of X defined by some special conditions.

The Lax-Milgram theorem gave us a method to study the existence and the uniqueness of u and we applied this method to a simple example "The Homogeneous Dirichlet problem"

The idea to approach this solution is to replace the spaces X and Y by some "simpler" ones X_h and Y_h and to associate with the equation (1) a family of approached equations (depending on one parameter h):

(37) $\qquad A_h u_h = f_h$

where A_h is supposed to approach A

$f_h \in Y_h$ " " " " f

If u_h is the solution of (37) we are interested to know if, and in what sense, u_h is a good approximation of u.

On this way we have.

i) to formulate a "convenient" family of problems (37)

ii) to study the existence and uniqueness of their solutions u_h

iii) to study the convergence of the sequence u_h and to see if the eventual limit is or not the solution of (1)

iv) to develop some algorithm to the effective computation of u_h.

V to evaluate the error between u and u_h

On this way the Galerkin's method leads to an approach solution of the problem:

(6) find $u \in V$ s.t $\quad a(u,v) = (f,v) \quad \forall v \in V$

Which is, with the notations and assumptions of the Lax Milgram theorem,

(8) Find $u \in V$ such that $Au = f$ in V'

III . 2 Formulation of the approached problem - existence and uniqueness of its solution

· The notation and assumptions are the same as for Lax-Milgram but we suppose here that V is a separable Hilbert space that is to say that V admits a countable Hilbert basis

What is a Hilbert basis? It is a sequence $(w_i) \subset V$ such that:

i) $\forall m < \infty$, $w_1, \ldots\ldots, w_m$ are linearly independant

ii) the vector space of finite linear combinations of the w_i is dense in V (that is equivalent to say: if v is orthogonal to every w_i then $v = 0$)

A separable Hilbert space admits an infinity of basis and there exists some classic method to obtain one basis - They are not normally orthonormal.

Let then (w_i) be a basis on V and V_m the space generated by $w_1, \ldots w_m$ - We are going to look for each fixed m an approach solution of (6) in V_m that is to say an element of the type:

$$(38) \qquad u_m = \xi_{im} w_i \qquad \xi_{im} \in R$$

such that

$$(39) \qquad a(u_m, v) = (f, v) \qquad \forall v \in V_m.$$

this equation has an unique solution because it is equivalent to the system

$$a(u_m, w_j) = (f, w_j) \qquad \forall j = 1, 2, \ldots m$$

that is to say.

$$a(w_i, w_j)\, \xi_{im} = (f, w_j) \qquad \forall j = 1, 2, \ldots m.$$

which is a linear system of m equations for the m unknown numbers $\xi_{1m}, \xi_{2m}, \ldots, \xi_{mm}$. Now, this system is of Cramer, because the coercivity of a - hence the existence and uniqueness of u_m

III Result of convergence.

Theorem 8: Let V be a real, separable Hilbert space et let $a(\,,\,)$ be a bilinear continuous and coercive form on V. Then the solution $u_m \in V_m$ of (39) converges strongly in V toward the solution u of (6) when $m \to \infty$

Before giving the proof of this result, it is necessary to recall the definition of weak convergence and also the theorem of weak compacity.

Definition 10 Let E be a normed vector space, a sequence $(x_m) \subset E$ is said to be weakly converging if $\exists\ x \in E$, such that

$$\lim_{m \to \infty} \ell(x_m) = \ell(x) \qquad \forall \ \ell \in E'$$

Theorem 9 (of weak compacity) if V is a reflexive Banach space, and therefore if V is a Hilbert space, from all bounded sequence in V, we can extract a subsequence weakly convergent towards an element of V

Proof of the theorem 8: We want to apply the theorem 9. Taking then $v = u_m$ in (39), we obtain

$$(40) \qquad a(u_m, u_m) = (f, u_m)$$

on from this, because the coercivity of a

$$\alpha \|u_m\|^2 \leq a(u_m, u_m) = (f, u_m) \leq \|f\|_* \|u_m\|$$

hence

$$(41) \qquad \|u_m\| \leq \frac{\|f\|_*}{\alpha}.$$

Then, from the preceding theorem, $\exists\ u^* \in V$ and a subsequence $\{u_{m'}\} \subset \{u_m\}$ such that

$$(42) \qquad u_{m'} \longrightarrow u_* \text{ weakly. } \quad \text{when } m' \to \infty .$$

Let v, be a finite linear combination of the elements w_i, then $v \in V_j$ for a fixed j and $v \in V_{m'}$ as soon as $m' \geq j$. We have then

$$a(u_{m'}, v) = (f, v)$$

(43) $\qquad a(u_*, v) = (f, v)$

This equality is true for every v, finite linear combination of w_i and because (w_i) is a Hilbert basis, we know that the set of this combinations is dense in V. The equality (4.3) is therefore still true, by continuity for all $v \in V$. This proves that u_* is solution of (8) and because the uniqueness of this solution

$$u_* = u.$$

($\forall$ sequence extracted from u_m being bounded, we can begin again the same reasoning, we shall obtain the same limite; that proves that the sequence u_m itself converges toward u weakly.

To obtain now the strong convergence, let us consider the sequence

$$X_m = a(u_m - u, u_m - u)$$
$$= a(u_m, u_m) - a(u, u_m) - a(u_m, u) + a(u, u)$$

from (40) we know that

$$a(u_m, u_m) = (f, u_m)$$

and because the week convergence of u_m toward u.

$$\lim a(u_m, u_m) = (f, u) \quad \text{on the one hand}$$
$$\lim X_m = (f, u) - a(u, u) - a(u, u) + a(u, u) = 0.$$

Hence

$$0 \le \alpha \lim \|u_m - u\|^2 \le \lim X_m = 0$$
$$\text{and} \qquad \lim u_m = u.$$

<u>Remark 13</u>: With a view of the approach resolution of the variational problem introduced in I.3:

minimize $\quad J(v) = a(v, v) - 2(f, v) \quad$ on V.

a method due at Ritz, consists in looking for an element $\tilde{u}$ wich realizes the minima on J on a subset $\tilde{V}$ of V. if we choose $\tilde{V} = V_m$, then $\tilde{u} = u_m$. That results from (39) and from the proposition 2