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BOUNDARY VALUE PROBLEMS FOR PERIODIC HETEROGENEOUS MEDIUM AND APPLICATION FOR COMPOSITE MATERIALS

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Some details about the talk of J.F. BOURGAT and H. LANCHON at the Euromech 71 (Bath, 29th March 1rst April 1976).

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S U M M A R Y

INTRODUCTION :

We are here concerned by a method of *homogenization* :

. The origine of this word is related to the question of a replacement of the heterogeneous medium by an "equivalent" homogeneous one.

. If we have to resolve a boundary value problem, *homogenization* means, replacement of the corresponding differential operator with variable coefficients, by an "equivalent" one with constant coefficients such that the solutions respectively obtained with the two operators are as near as possible. The constants coefficients can then be considered as a new kind of *"effective modulus"* because they are corresponding at the formulation of the same boundary value problem for an homogeneous medium.

It seems that very few papers were yet written in this direction ; the following list of authors, of course non extantive, is that of our own sources.

A. BENSOUSSAN - J.L. LIONS - G. PAPANICOLAOU [2] [3] [6] and L. TARTAR [8], of Paris and New-York, are the people than we are representating here ; they are specialized in functional and numerical analysis.

I. BABUSKA [1] (institute for fluid dynamics and applied mathematics - University of Maryland) seems to work exactly in the same way, but we have not yet obtained the detailed papers about his results.

E. SANCHEZ-PALENCIA [9] (Institut de Mécanique Théorique et Appliquée de l'Université Paris VI) using some asymptotic approach, with the help of physical interpretations, obtained the same "homogenized" coefficients ; he applied them to some boundary value problems in porous media, in electromagnetism and acoustic.

G. DUVAUT [5] [6] (same institut as Sanchez-Palencia) applied recently the method for some problem of composites plates which leads to differential operator of 4th order ; furthermore, he also obtain the homogenized coefficients for general three dimensional elasticity problems.

It is lastly indispeneable to mention E. DE GIORGI and S. SPAGNOLO [4] of Pisa (Italy) whose were probably the 1st mathematiciens to give (in 1967) the theorems of convergence necessary to the following theory ; the small paramator being here the size of the repr-

I - DEFINITION OF A PERIODIC HETEROGENEOUS MEDIUM :

Let consider an heterogeneous medium made of a relatively regular distribution of several components ; then we can idealize this medium, admitting a kind of space periodicity ; in this way, if $C(\underline{x})$ is a variable coefficient characterizing the (mechanical, thermical, electromagnétical..) comportment of the material, and if Ω is the bounded domain of R^N ($N = 1, 2, \ldots 3$) occupied by it, the space periodicity is defined by :

(1) $$C(\underline{x} + \underline{k\varepsilon}) = C(\underline{x})$$

$\forall\, k$, N- integer and $\forall\, \underline{x} \in \Omega$ such that $x + k\varepsilon \in \Omega$;

$\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N)$; ε_i being the period in the x_i direction

$\underline{k\varepsilon} = (k_1\varepsilon_1, \ldots k_N\varepsilon_N)$

Then we can, for instance, introduce;

(2) $$\varepsilon = max\ \varepsilon_i \quad , \quad y_i = \frac{x_i}{\varepsilon} \quad \text{and} \quad p_i = \frac{\varepsilon_i}{\varepsilon}$$

$1 \leqslant i \leqslant N$

in such a way that :

$$C(\underline{x}) = C(\varepsilon\underline{y}) = a(\underline{y})$$

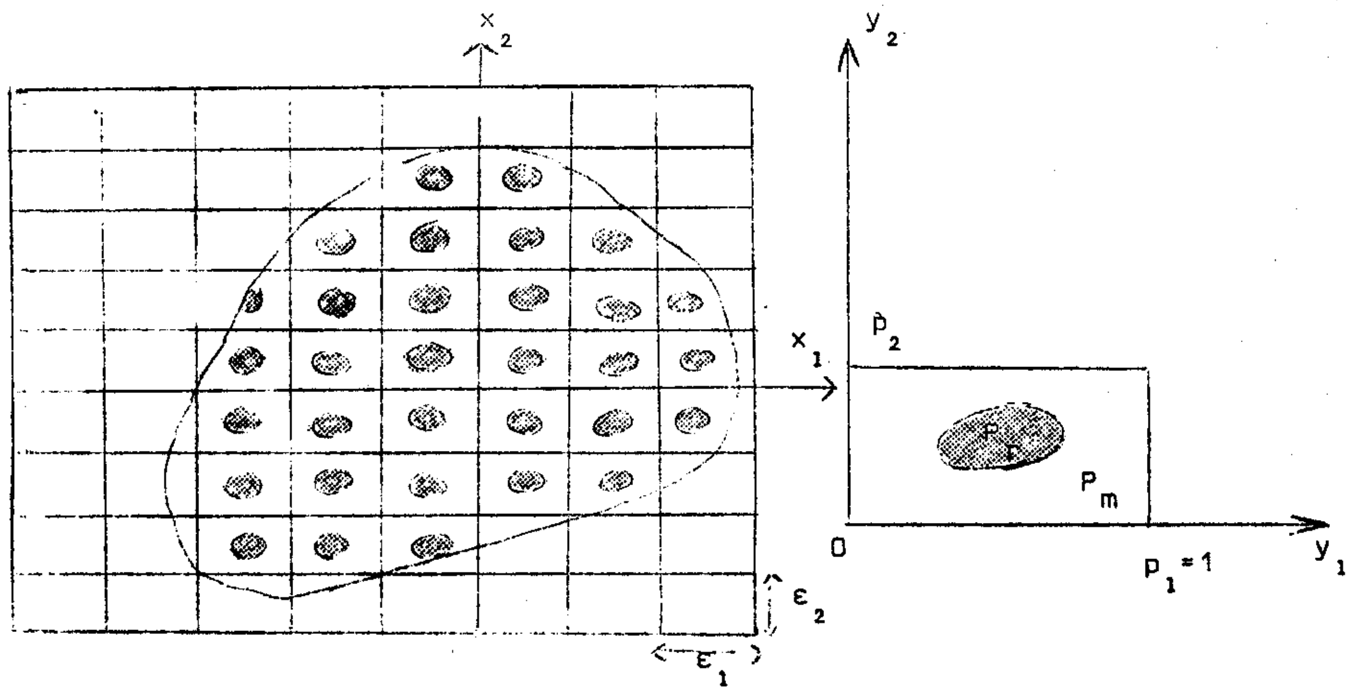
and

$$a(\underline{y} + \underline{kp}) = a(\underline{y})$$

a, defined on the rectangular parallelepiped.

$$P = [0, p_1] \times \ldots \times [0, p_N] \subset R^N$$

is said P- periodic. P is the image of the representative cell of the medium.

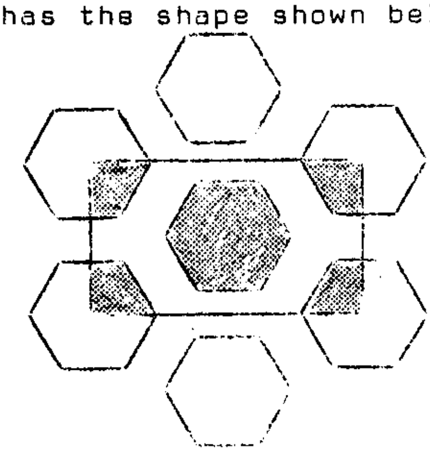


Example : for a composite material with 2 components we shall have

$$a(\underline{y}) = \begin{cases} C_r \text{ on } P_r & \text{(image of the reenforced part of the cell)} \\ C_m \text{ on } P_m & \text{(image of the matrix part of the cell)} \end{cases}$$

The constants C_r and C_m being respectively the values of the considered coefficient for the reenforcement and the matrix.

It is important to point out that the basic cell of the medium is not necessarily simple ; in the case of an hexagonal array, the representatived cell has the shape shown below



II - BOUNDARY VALUE PROBLEM :

Suppose now that we formulate a boundary problem for a periodic heterogeneous medium ; we have a partial differential equation (or system) with variable coefficients and, some boundary (eventually initial) conditions . Even if we can prove "existence and uniqueness" of the solution, we are generally unable to obtain it by numerical coefficients ; we are then looking for an approached solution and that is the topic of the present theory.

To be clearer, let us choose, as example, a simple mathematical model obtained by the formulation of several physical situations ; after having given a quick description of this model, we shall explicite a mechanical corresponding problem.

1. Mathematical model :

Let consider the differential operator

$$A^{\varepsilon} = - \frac{\partial}{\partial x_i} \left[a_{ij} \left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j} \right] \tag{3}$$

...ere a_{ij}, P. *periodic bounded coefficients*, are such that :

$$\exists \beta > 0 \quad ; \quad a_{ij}(y)\ \xi_i \xi_j \geq \beta\ |\xi|^2 \quad \forall \xi \in R^N \text{ and } \forall y \in P \tag{4}$$

(The a_{ij} are not necessarily symmetrical in i and j)

Suppose now that we have to resolv the following Dirichlet problem :

Problem ($\mathcal{P}\varepsilon$) *: Find* u_ε *such that :*

$$\begin{cases} A^{\varepsilon} u_\varepsilon = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

Ω, *bounded domain in* R^N

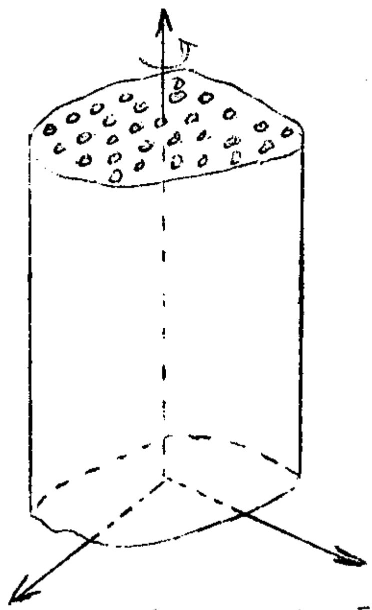
f *given in* $L^2(\Omega)$; (*that is to say* $\int_\Omega |f|^2 dx < \infty$)

We know by the general theory of partial differential equations that, thanks to the property of coercivity, there exists u_ε, unique solution of (5). (*See Lax milgram theorem lecture notes 23/1*)

If ε is small by comparaison with the size of Ω, it is impossible to compute u_ε; so it is natural to wonder if u_ε tends toward a limit u when ε *goes to* 0 and then, if u can be considered as a good approximation of u_ε as soon as ε is sufficiently small.

2. Torsion of fibers reenforced bars :

If we formulate the standard problem of torsion for a cylindrical fibers reenforced bars, it is known (cf lecture notes 23-25 p 35) that



The stress tensor is such that

$$(16)\quad \begin{cases} \sigma_{ij} = 0 \text{ exept} \\ \sigma_{13} = \dfrac{\partial\theta}{\partial x_2} = \sigma_{31} \text{ and} \\ \sigma_{23} = -\dfrac{\partial\theta}{\partial x_1} = \sigma_{32} \end{cases}$$

θ, called "stress function" being defined on the cross section Ω of the bar and verifying the following value problem

$$(7)\quad \begin{cases} \dfrac{\partial}{\partial x_i}\left[\dfrac{1}{\mu(x)}\dfrac{\partial\theta}{\partial x_i}\right] + 2\alpha = 0 & \text{on } \Omega \\ \theta = 0 & \text{on } \partial\Omega \end{cases}$$

Where : μ is the shear modulus. (μ_f on the fibers, μ_m on the matrix)

α the torsion angle imposed at the terminal section

We shall suppose here than the distribution of fibers is in such a way that the period ε is the same in the 2 directions ox_1, ox_2. In this case, if n is the number of fibers by unity of area (easy to count) it is not difficult to show that roughly

$$\varepsilon = \frac{1}{\sqrt{n}}$$

Then, with the same change of coordinate as above ;

$$y_i = \frac{x_i}{\varepsilon}$$

We have :

$$\mu(\underline{x}) = a(\underline{y}) = \begin{cases} \mu_f \text{ on } P_f \\ \end{cases}$$

This problem is actually a $(\mathcal{P}_\varepsilon)$ one with :

$$a_{ij}(\underline{y}) = \frac{\delta_{ij}}{a(\underline{y})} \qquad (\delta_{ij}\text{, kronecker symbol})$$

$$f = 2\alpha\text{, constant}$$

$$\beta = \frac{1}{\sup\limits_{P} a(\underline{y})} = \frac{1}{\mu_f} \qquad (\text{if, as it is realistic, } \mu_f > \mu_m)$$

Remark 1 : What means here the fact to make ε tend toward 0 ?

It implies that we consider a sequence of cylinders of same shape, same components, with an increasing number of fibers but a constant degree of reenforcement; that is to say : if V_f is the total volume of fibers and V that of the cylender

$$\frac{V_f}{V} = \text{constant}$$

This condition is very logic on a physical point of view and very comfortable for the mathematics, because it ensures the fixity of P_f and P_m when ε is discreasing

Remark 2 : A lot of others problems are entering in the $(\mathcal{P}_\varepsilon)$ scope after an eventual translation of the solution ; for example, the diffusion equation in steady cases :

. f can be 0 and $u_\varepsilon = g \neq 0$ given on the boundary (problem submitted by D. VAN DEN ASSEM) it is sufficient then to choose the new unknown function

$$u_\varepsilon^* = u_\varepsilon - g^*$$

(where g^* is an arbitrary, sufficiently regular, given function equal to g on $\partial\Omega$) to obtain a $(\mathcal{P}_\varepsilon)$ problem with

$$f = f^* = \frac{\partial}{\partial x_i}\left[a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial g^*}{\partial x_j}\right]$$

. f can be periodic (problem submitted by B. SCHULZ during the colloquium) when every inclusion can be considered as a source of heat, we have still a $(\mathcal{P}_\varepsilon)$ problem at the condition to replace in the second member, f by

$$\bar{f} = \frac{1}{V(P)} \int f(y)\, dy$$

III - THEORETICAL RESULTS :

1. Statement :

With the above mentioned hypothesis about a_{ij} and f, we can prove than :

If u_ε is the unique solution of :

$$(\mathcal{P}_\varepsilon) : A^\varepsilon u_\varepsilon = f \quad \text{in } \Omega \quad ; \quad u_\varepsilon = 0 \quad \text{on} \partial\Omega,$$

then $\lim_{\varepsilon \to 0} u_\varepsilon = u$ (in a more or less strong sense),

where u is the unique solution of

$$(\mathcal{P}) : \mathcal{A} u = f \quad \text{in } \Omega \quad ; \quad u = 0 \quad \text{on } \partial\Omega$$

and $\mathcal{A}$ the differential operator defined by

$$\mathcal{A} = - q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \qquad (8)$$

q_{ij} being here constant coefficients defined by the way mentioned in the following paragraphe.

In fact, u is the first term of an asymptotic development,

$$u_\varepsilon (x) = u(x) - \varepsilon \left[\chi_i \left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_i} + \tilde{w}(x) \right] + O(\varepsilon^2), \qquad (9)$$

whose the first order term can be also compute without too much difficulties.

2. Determination of the operator $\mathcal{A}$:

We have, first of all, to define an admissible functional space to work on ; in this way, let remind the definition of the first Sobolev space relatively to a domain Ω (*See for more precisions lecture notes*

$$H^1(\Omega) = \left\{ \phi / \phi \in L^2(\Omega), \frac{\partial y}{\partial x_i} \in L^2(\Omega) \right\}$$

where

$$L^2(\Omega) = H^\circ(\Omega) = \left\{ \phi / \int_\Omega |\phi|^2 dx < \infty \right\}$$

Then the admissible space can be introduce

(10) $$W = \left\{ \psi / \psi \in H^1(P) \text{ and } \psi \text{ "P- periodicable"} \right\}$$

The locution "P- periodicable" means that the functions of W must be capable of prolongation, in a P-periodic way, to the whole physical space R^N : they have for that to take equal values on two opposite faces of P.

Because, no values are imposed on ∂P for the functions of W, they can be defined only at a constant, more or less, and then, we need to introduce the quotient space.

(11) $$\dot{W} = \frac{W}{C} \quad \text{, i e "W modulo constants"}$$

The norm

(12) $$\|\psi\|^2_{\dot{W}} = \int_P |grad\ \psi|^2 dx$$

makes of $\dot{W}$ an Hilbert space ; that is to say, a complete metric space with the scalar product

(13) $$<\phi,\psi>_{\dot{W}} \int_P grad\phi\ grad\psi\ dx$$

Let us still introduce the bilinear form

(14) $$a_p(\phi,\psi) = \int_P a_{ij}(y) \frac{\partial\phi}{\partial y_i} \frac{\partial\psi}{\partial y_j} dy \ ;$$

the trivial inequality

$$a_p(\psi,\psi) \geq \beta \|\psi\|^2_{\dot{W}} \qquad \forall \psi \in \dot{W}$$

insures then the existence and uniqueness of the solution χ_i of

(15) $$\chi_i \in \dot{W} \quad ; \quad a_p(\chi_i - y_i, \psi) = 0 \qquad \forall \psi \in \dot{W}$$

and now q_{ij} is obtained by

(16) $$q_{ij} = \frac{1}{Vol P} a_p (\chi_j - y_j, \chi_i - y_i)$$

Remark 3 : It is clear, right-now, that the q_{ij} can be interpreted like some "homogenized" or "effective" coefficients for the considered (mechanical, thermical, electrical...) problem. These constant coefficients are in fact independant on the shape of Ω and on the boundary conditions ; they are essentially dependant on :

The representative cell of the heterogeneous medium
The differential operator

Remark 4 : It is instructive to have a look on what are χ_i and q_{ij} in the above mentioned problem of torsion :

. χ_1 is the solution of the following transmission problem

$$\begin{cases} \Delta\,\chi_1 = 0 & \text{in } P_f \\ \Delta\,\chi_1 = 0 & \text{in } P_m \\ [\![\chi_1]\!]_{\partial P_f} = 0 & \\ \left[\!\!\left[\frac{1}{\mu}\frac{\partial\chi_1}{\partial n}\right]\!\!\right]_{\partial P_f} = \left[\!\!\left[\frac{1}{\mu}\right]\!\!\right] n_1 \Big|_{\partial P_f} & ;\ \vec{n},\ \text{outward unitary normal to } \partial P_f \end{cases}$$

χ_1 taking equal values on two opposites sides of P.

($[\![\phi]\!]_\Gamma$ means "jump of the quantity ϕ" accross the curve or surface Γ)

(We have not yet interpreted the physical meaning of χ_1)

. Then, the "homogenized coefficients" are given by :

$$q_{ij} = \left[\frac{1}{\mu_f}\frac{V_f}{V} + \frac{1}{\mu_m}\frac{V-V_f}{V}\right]\delta_{ij} - \frac{1}{2\mu_f}\int_{P_f}(\chi_{i,j}+\chi_{j,i})dy - \frac{1}{2\mu_m}\int_{P_m}(\chi_{i,j}+\chi_{j,i})dy$$

We notice that the first part of this coefficient represents exactly the law of mixtures ; the second part being a corrective term depending on μ_f, μ_m, the respective shapes and volumes of P_f and P_m and also on the differential operator by the intermediary of χ

3. Advantages of the homogenization :

They are almost obvious

a) While the solution u_ε of $(\mathcal{P}_\varepsilon)$ is generally impossible to compute as soon as ε is small, u can be obtained by the successive resolutions of several simple problems independant of ε :

The χ_i are solutions of elliptic equations on the very simple domain P

The q_{ij} are obtained by integration on the same domain P

Lastly, u is given by a Dirichlet problem on Ω but now with constant coefficients.

b) The q_{ij} have to be compute only once for a whole family of problems corresponding to the same differential operator.

c) The approximation u is available as soon as ε is small and we are able to compute the corrective term

$$u_\varepsilon(x) - u(x) = \varepsilon \left\{ \chi_i \left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_i} + \tilde{W}(x) \right\} + O(\varepsilon^2)$$

In fact the partially corrected solution

$$\hat{u}(x) = u(x) - \varepsilon \chi_i \left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_i}$$

is already much better than u(x) and, in numerous cases where P posses-ses some characters of symmetry, $\tilde{W}(x) \equiv 0$

IV NUMERICAL RESULTS OBTAINED BY J.F. BOURGAT.

1. Description of the worked out computings :

We suppose that we are looking for the stresses in a cylin-drical fibers reenforced bar B, subjected to a torque, the data for this problem being choosen as following.

. Ω, square cross section of B with sides of unit lengh

. fibers of glass ($\mu_f = 4.10^6$ psi) or Boron ($\mu_f = 2,5.10^7$ psi) with square cross section such that:

. matrix epoxy : $\mu_m = 2{,}2.10^5$ psi

We obtained :

. The direct computing of θ_ε for $\varepsilon=\frac{1}{2}$, $\varepsilon=\frac{1}{4}$ and $\varepsilon=\frac{1}{8}$ with more and more difficulties in a such way that we cannot hope to obtain directly θ_ε for $\varepsilon > \frac{1}{8}$

. The solution θ of the associated $(\mathcal{S})$ problem

. The corrected solutions :

$$\hat{\theta}_\varepsilon(x) = \theta(x) + \varepsilon \left\{ \chi_i\left(\frac{x}{\varepsilon}\right) \frac{\partial u}{\partial x_i} + \hat{W}(x) \right\}$$

with $\hat{W}_1 \equiv 0$ here because the symetries of the basic cell P.

The following given curves are the respective sections of the surfaces $\theta_\varepsilon(x)$, $\theta(x)$ and $\hat{\theta}_\varepsilon(x)$ by a diagonal plane (or a "parrallèle to the sides" one) perpendiculaire to the domain Ω (cf figure 1 page 13)

2. Some comments about the following results :

2.1. For the "glass-epoxy" composite :

a) The homogenized coefficient obtained is $\mu = 2{,}68.10^5$ psi while the law of mixture would have given $\bar{\mu} = 2{,}46.10^5$psi

b) For $\varepsilon=\frac{1}{2}$, which is not small (cf figures 2 and 5) the difference between θ_ε and θ is relatively big, particularily in the neighbourhood of the fibers ; however, the corrected term $\hat{\theta}_\varepsilon$ is already much better if we remember that, in fact, the stress tensor is given by the components of *grad* θ_ε

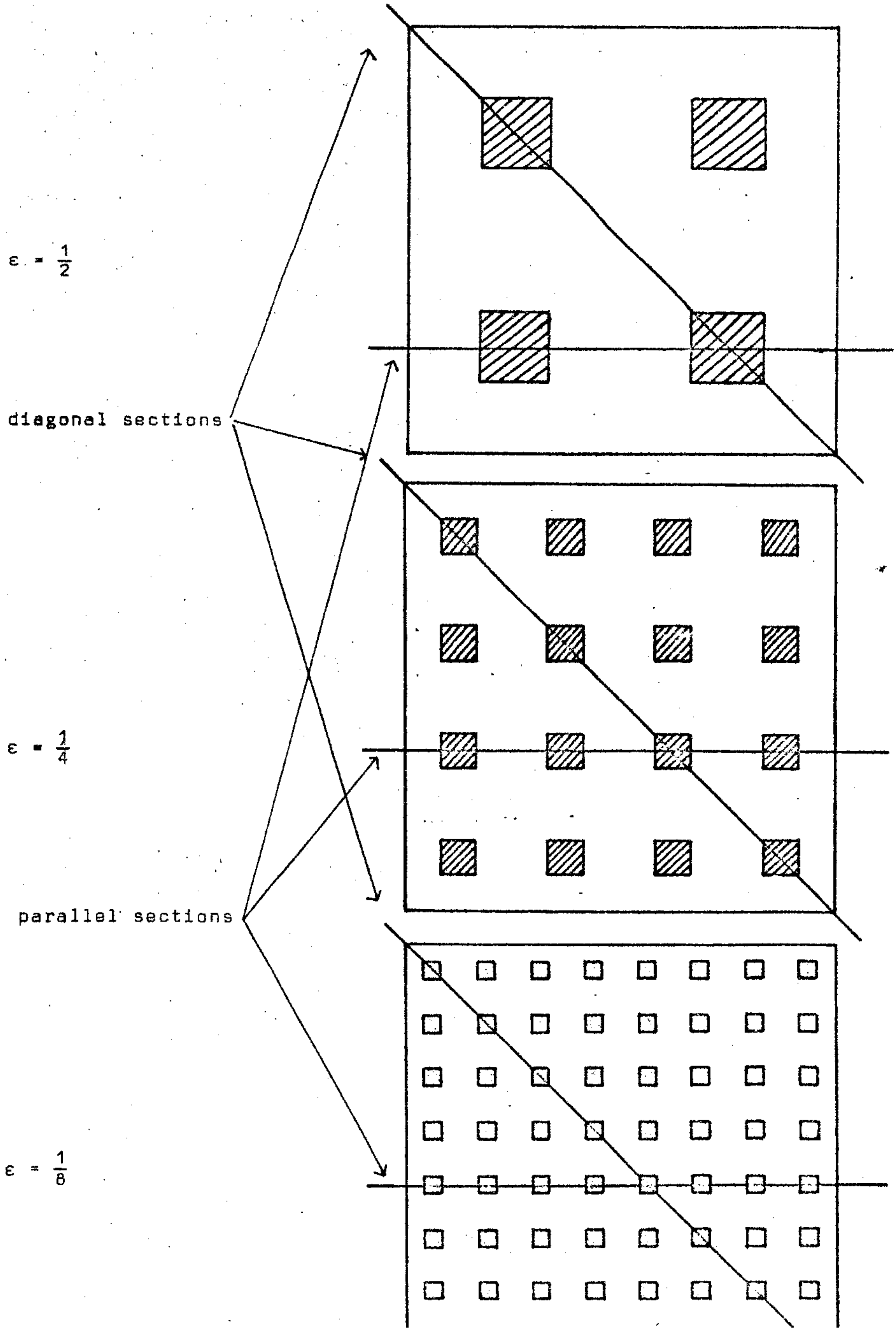
c) When ε is becoming smaller (cf figures 3,4,6,7) the results are better and better, specially for the slopes of θ_ε and $\hat{\theta}_\varepsilon$

2.2. For the "Boron-Epoxy" composite :

a) The homogenized coefficient is $\mu = 2{,}75.10^5$ in place of $\bar{\mu} = 2{,}47.10^5$ by the law of mixtures.

b) We cannot present the curves for $\varepsilon=\frac{1}{2}$ because the tops reached by the solution θ_ε on the fibers are too much high, but, for $\varepsilon=\frac{1}{4}$ (cf figures 8,10) which is not yet so small, the solutions θ and $\hat{\theta}_\varepsilon$ give already very good approachs of θ_ε and still better of *grad* θ_ε. For $\varepsilon=\frac{1}{8}$, θ_ε is very difficult to reach directly but the homogenized results seem to be satisfiying.

GURE 12 : Distribution of heterogeneities in the different treated cases :



Remark 5 : The scale of the curves in ordinate is not significant because, in fact, θ_ε, θ and $\tilde{\theta}_\varepsilon$ are proportional to α ; we have here choosen $\alpha\mu_m=5$ to obtain curves of sufficient size.

Remark 6 : To be able to compare our results to the experimental ones given in the book [7] p. 61, we made again the same computing for $\frac{V_\delta}{V} = 0{,}70$ and we obtained

$\mu = 9{,}65.10^5$ for the Glass-Epoxy in place of $\overline{\mu} = 6{,}5.10^5$ by the mixtures law.

$\mu = 1{,}2.10^6$ for the Boron-Epoxy in place of $\overline{\mu} = 7{,}2.10^5$ by the mixtures law.

Remark 7 : The "parallele sections" (cf figures 4,5,6,9 and 10) allow to determine directly $\sigma_{32} = -\frac{\partial\theta}{\partial x_1}$ which is nothing else in each point, as the slope of the curves obtained. Because the symmetries of the domain choosen, its possible to deduct of that, $\sigma_{31}=\frac{\partial\theta}{\partial x_2}$ at the symmetric points with respect to the diagonales of Ω.

FIGURE 2 : $\varepsilon = \frac{1}{2}$. COMPOSITE "GLASS-EPOXY". (diagonal sections)

$$\left.\begin{array}{l}\mu_f = 4.10^6 \\ \mu_m = 2{,}2.10^5\end{array}\right\} \quad \mu = 2{,}68.10^5 \quad \text{for } \frac{V_f}{V} = \frac{1}{9}$$

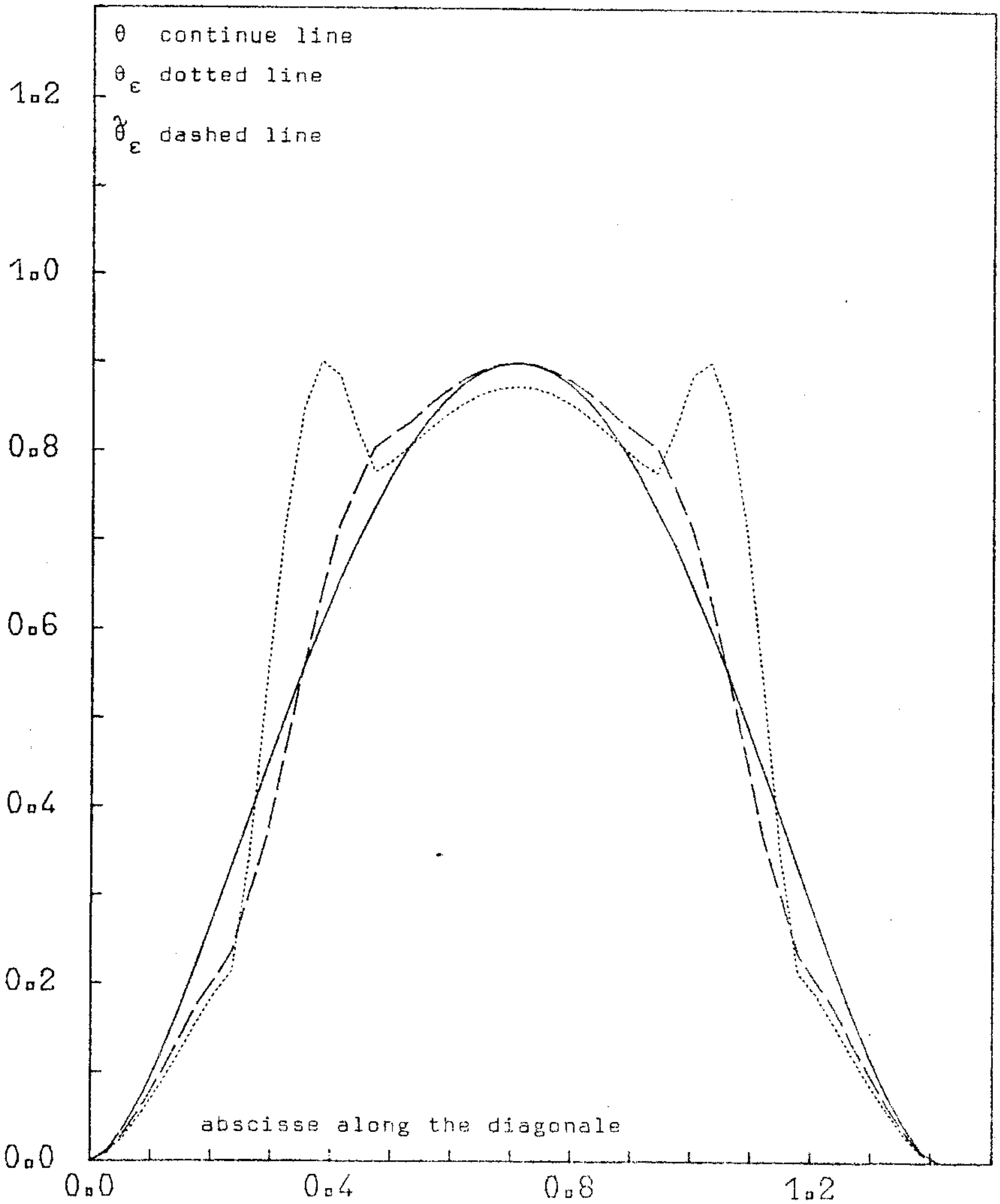


FIGURE 3 : $\varepsilon = \frac{1}{4}$. "GLASS-EPOXY". (diagonal sections)

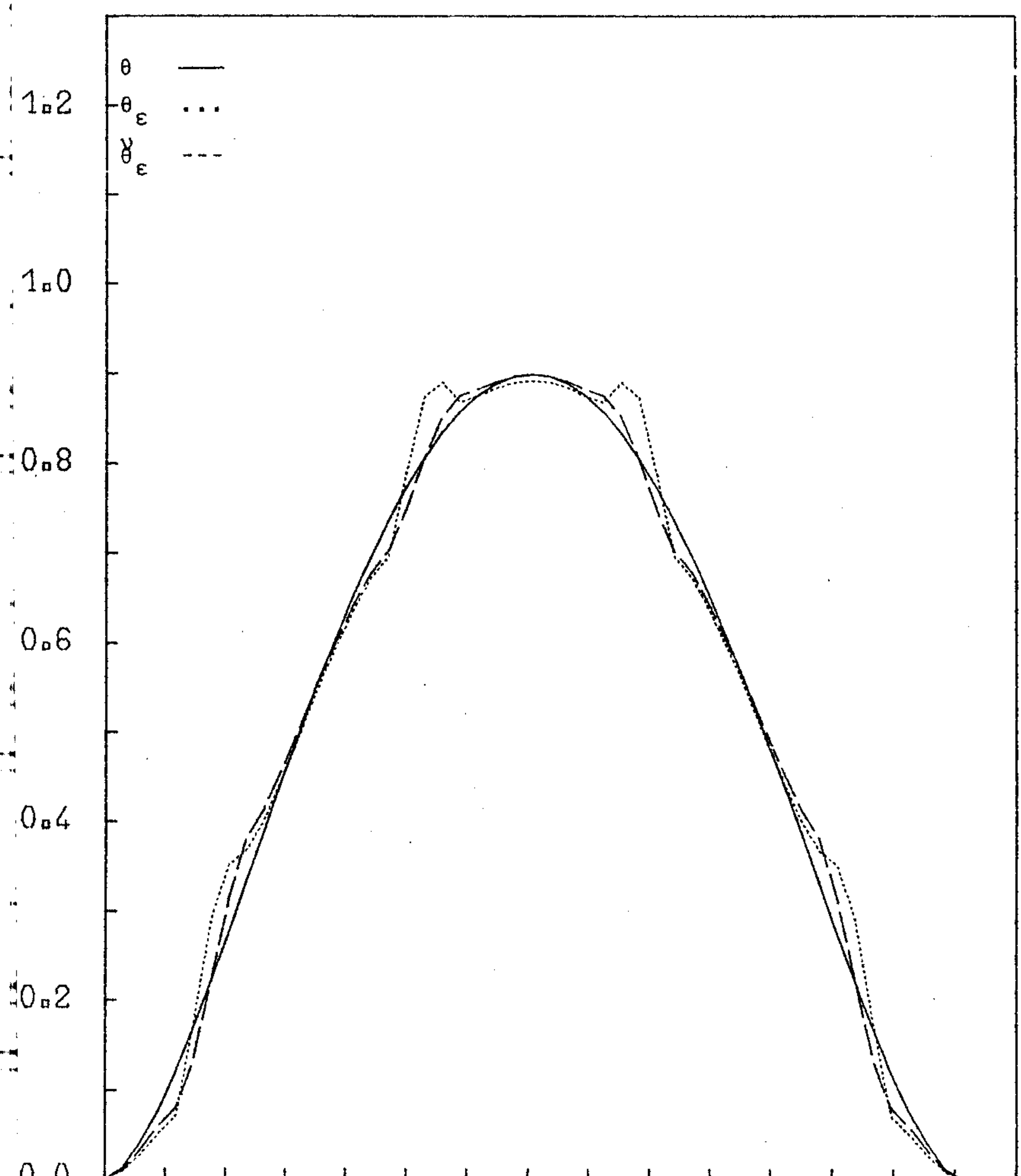


FIGURE 4 : $\varepsilon = \frac{1}{8}$. "GLASS EPOXY". (diagonal sections)

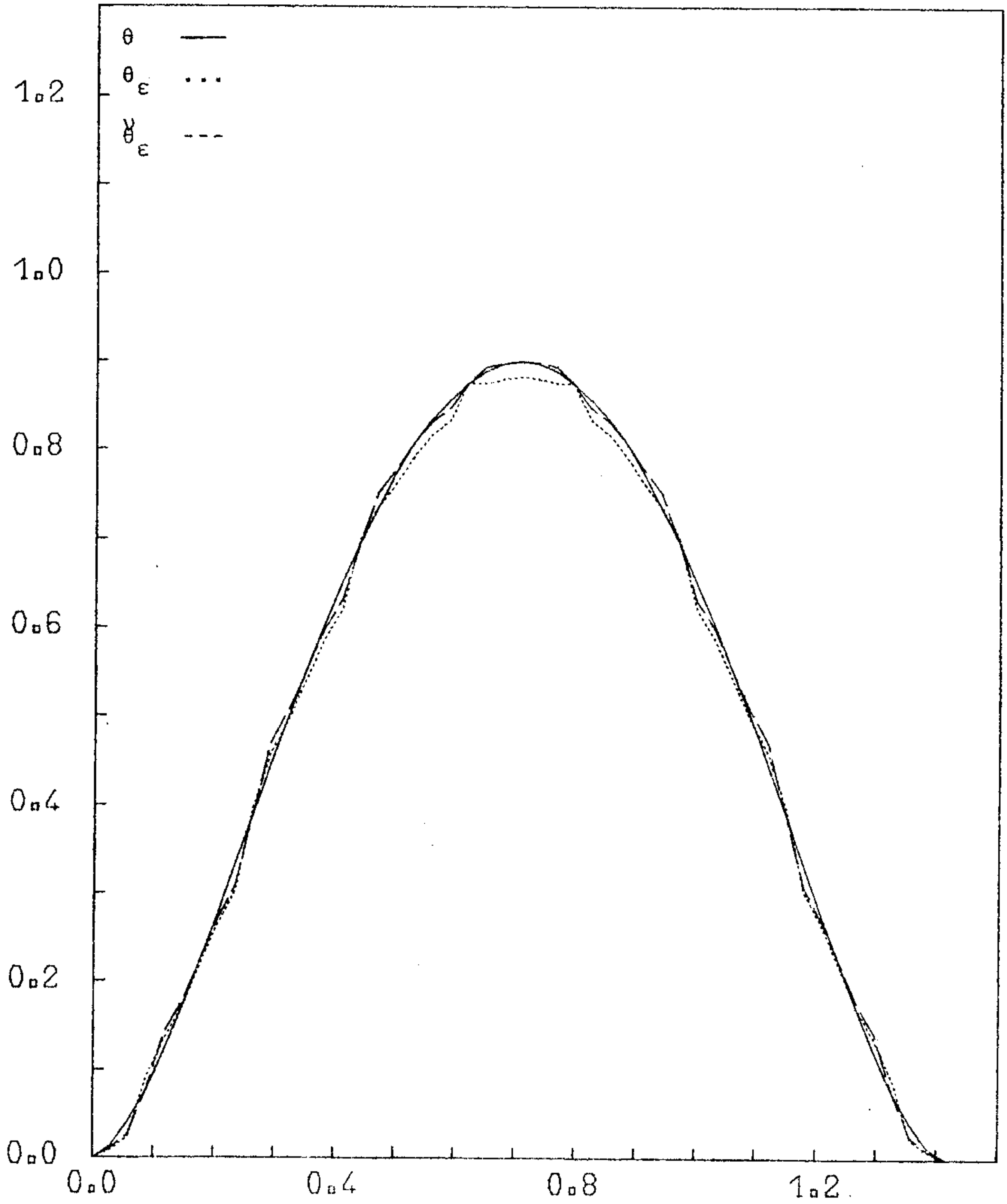


FIGURE 5 :: $\varepsilon = \frac{1}{2}$. "GLASS-EPOXY" . (parallèle sections)

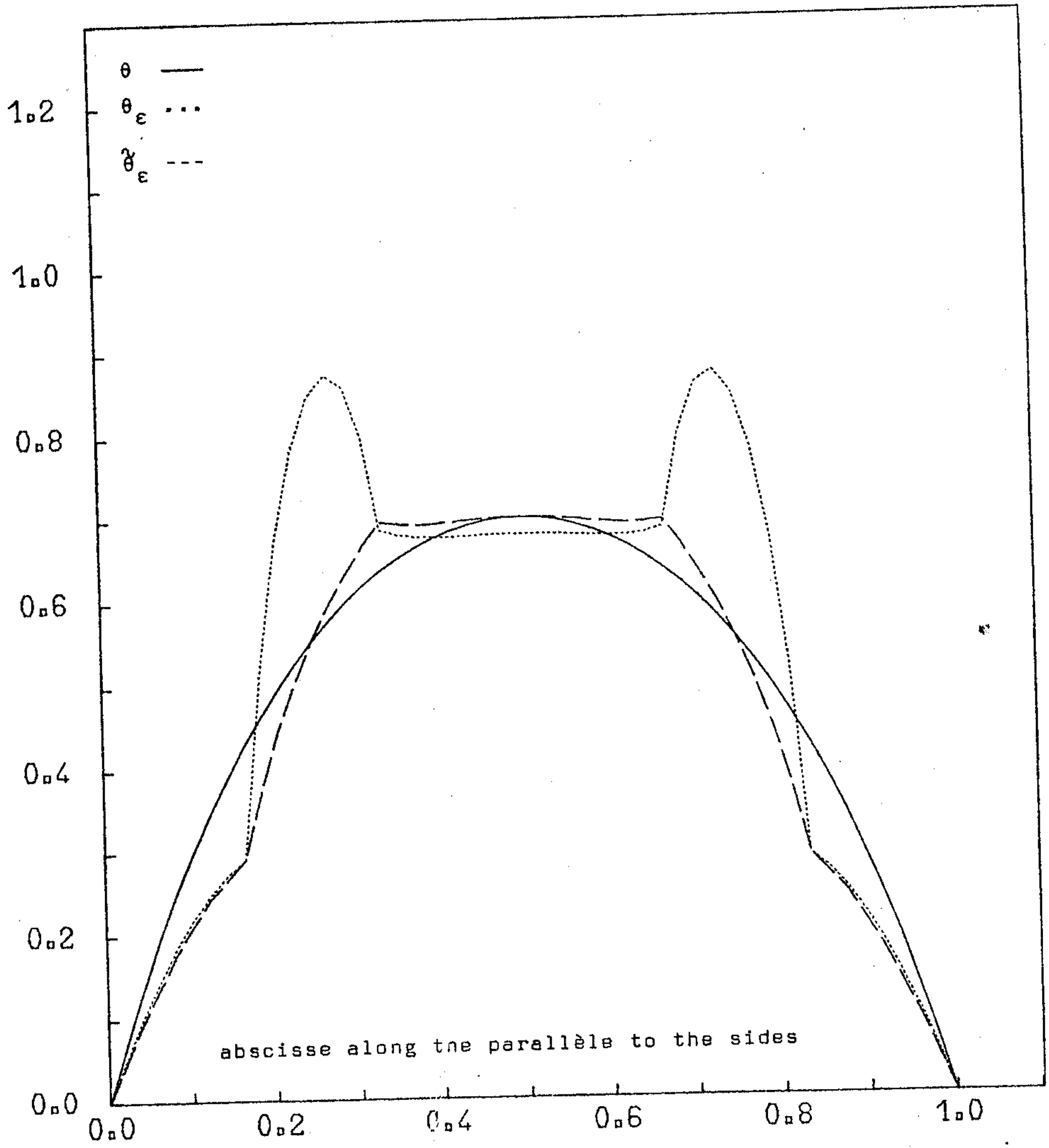


FIGURE 6 : $\varepsilon = \frac{1}{4}$. "GLASSE-EPOXY" (parallèle sections)

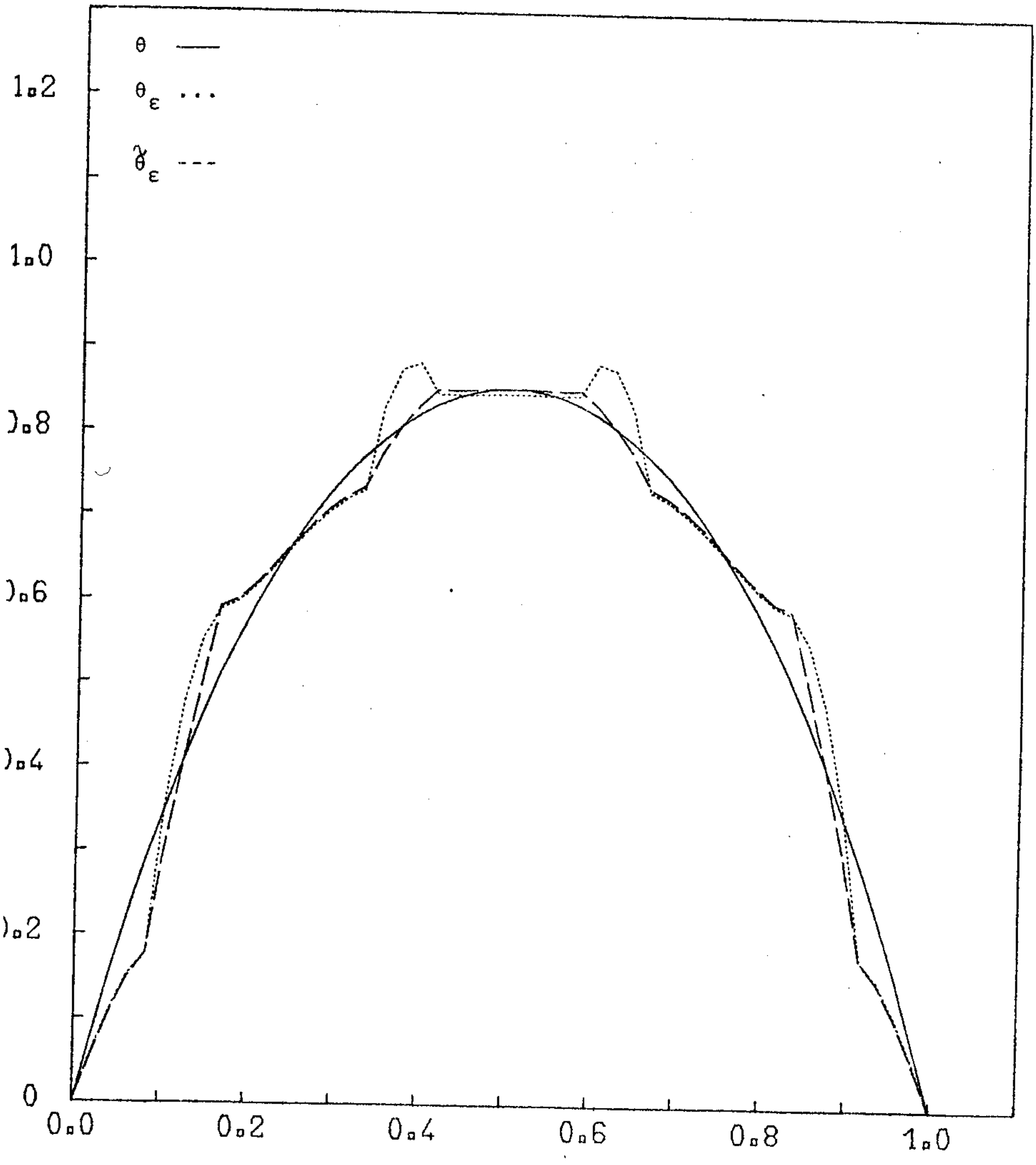


FIGURE 7 : $\varepsilon = \frac{1}{8}$. "GLASS-EPOXY" . (parallèle sections)

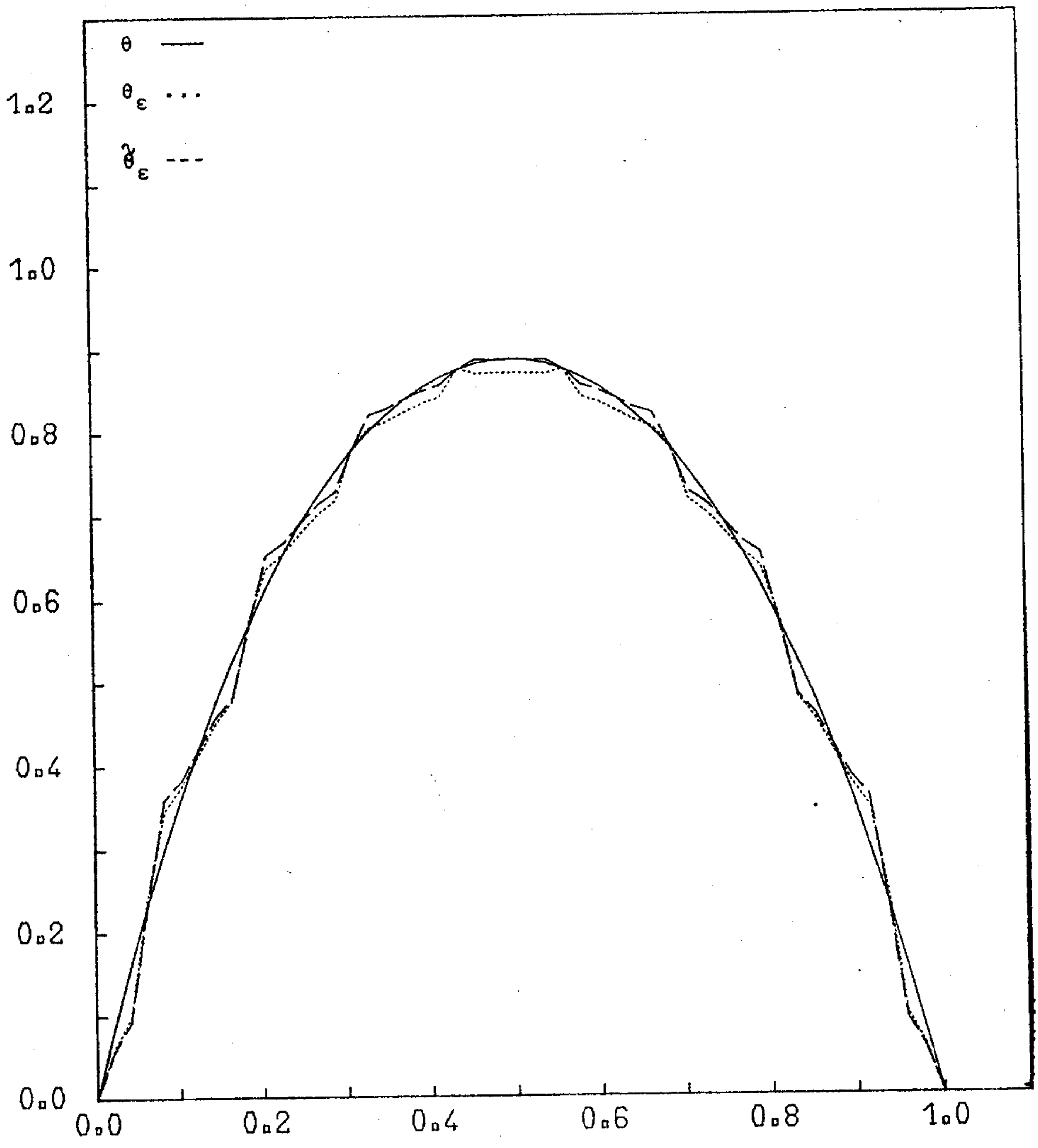


FIGURE 8 : $\varepsilon = \frac{1}{4}$. COMPOSITE "BORON EPOXY" . (diagonal sections)

$\left.\begin{array}{l}\mu_f = 2{,}5.10^7 \\ \mu_m = 2{,}2.10^5\end{array}\right\}$ $\mu = 2{,}75.10^5$ for $\frac{V_f}{V} = \frac{1}{9}$

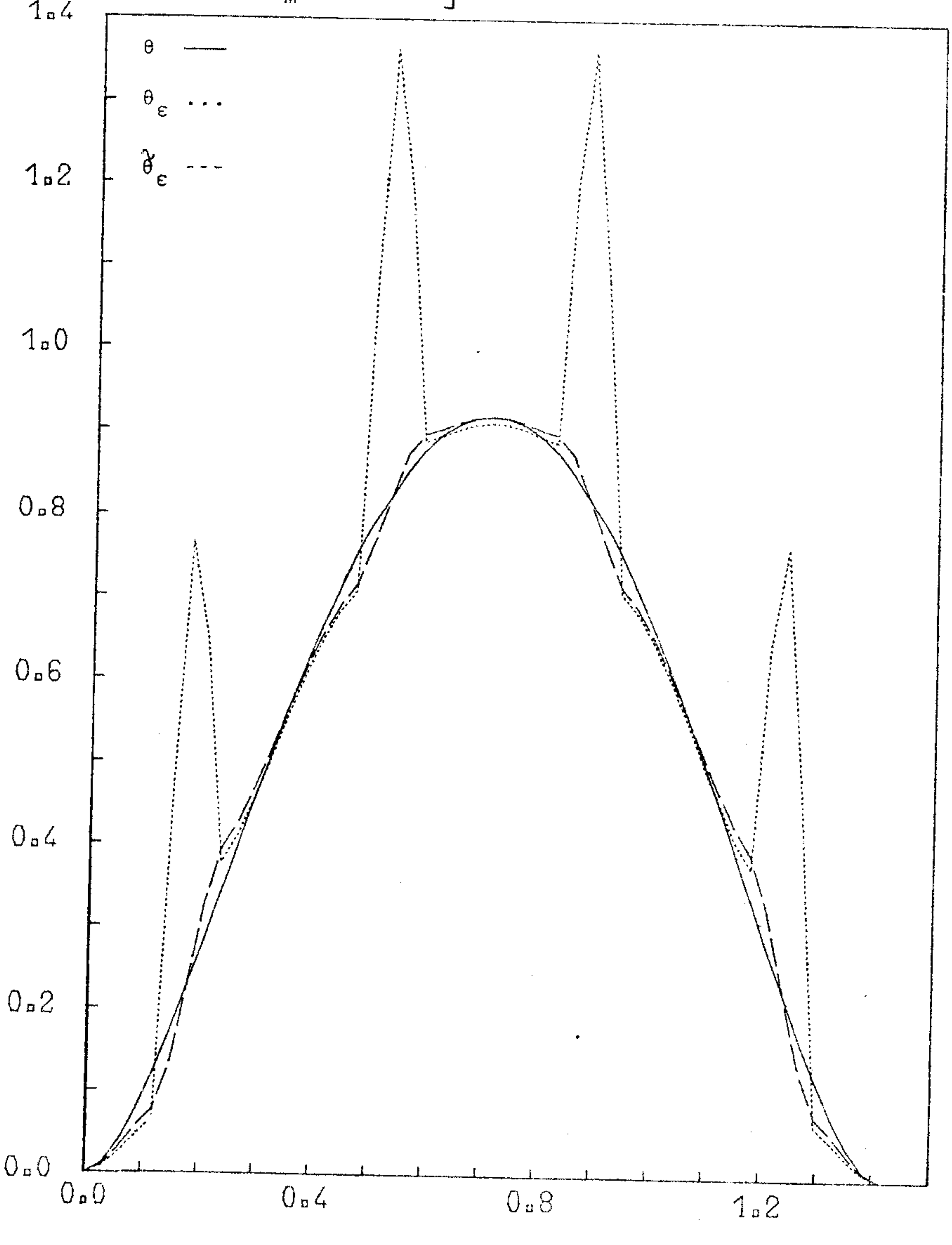


FIGURE 9 : $\varepsilon = \frac{1}{8}$. "BORON-EPOXY" . (diagonal sections)

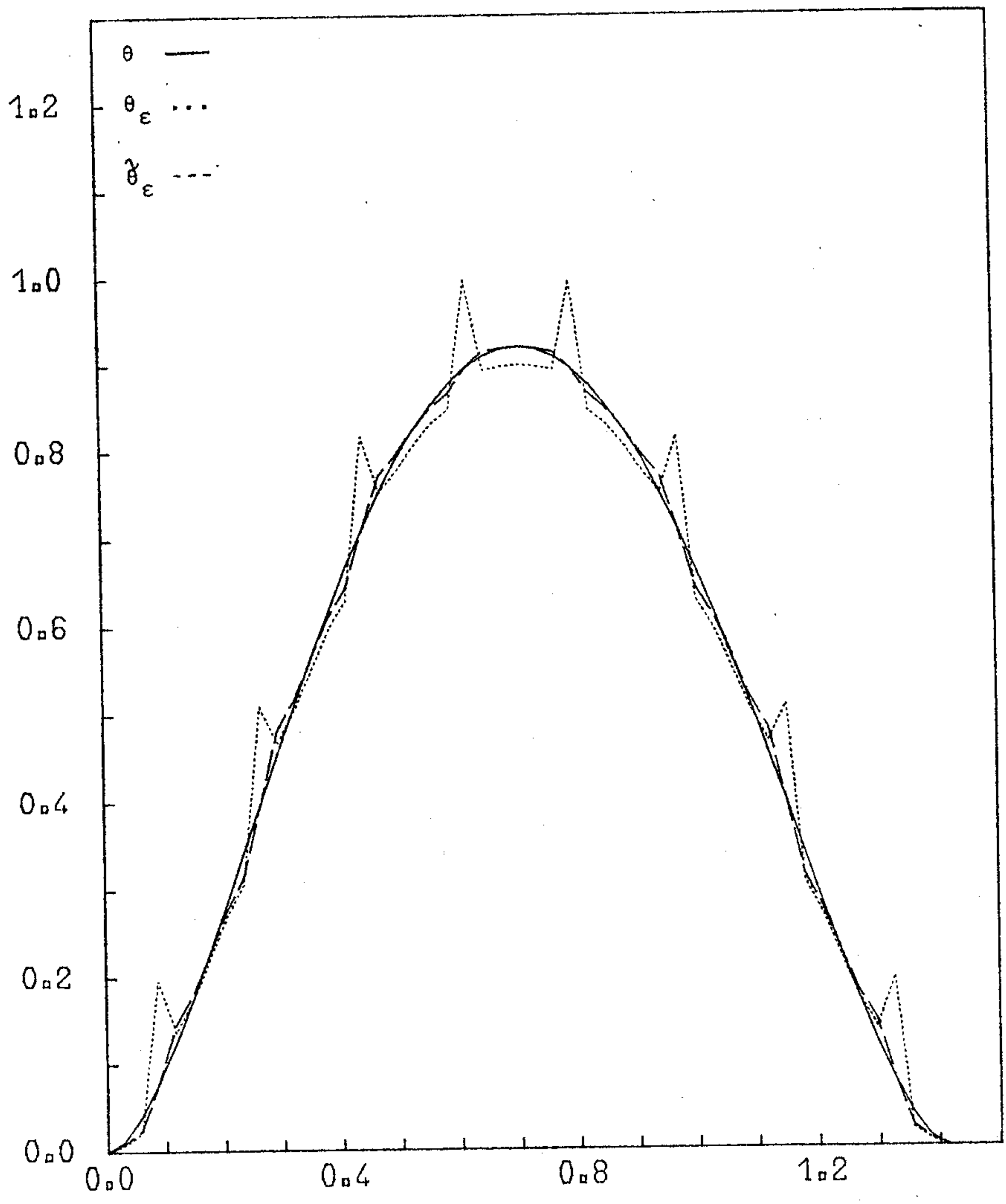


FIGURE 10 : $\varepsilon = \frac{1}{4}$. "BORON-EPOXY" . (parallèle sections)

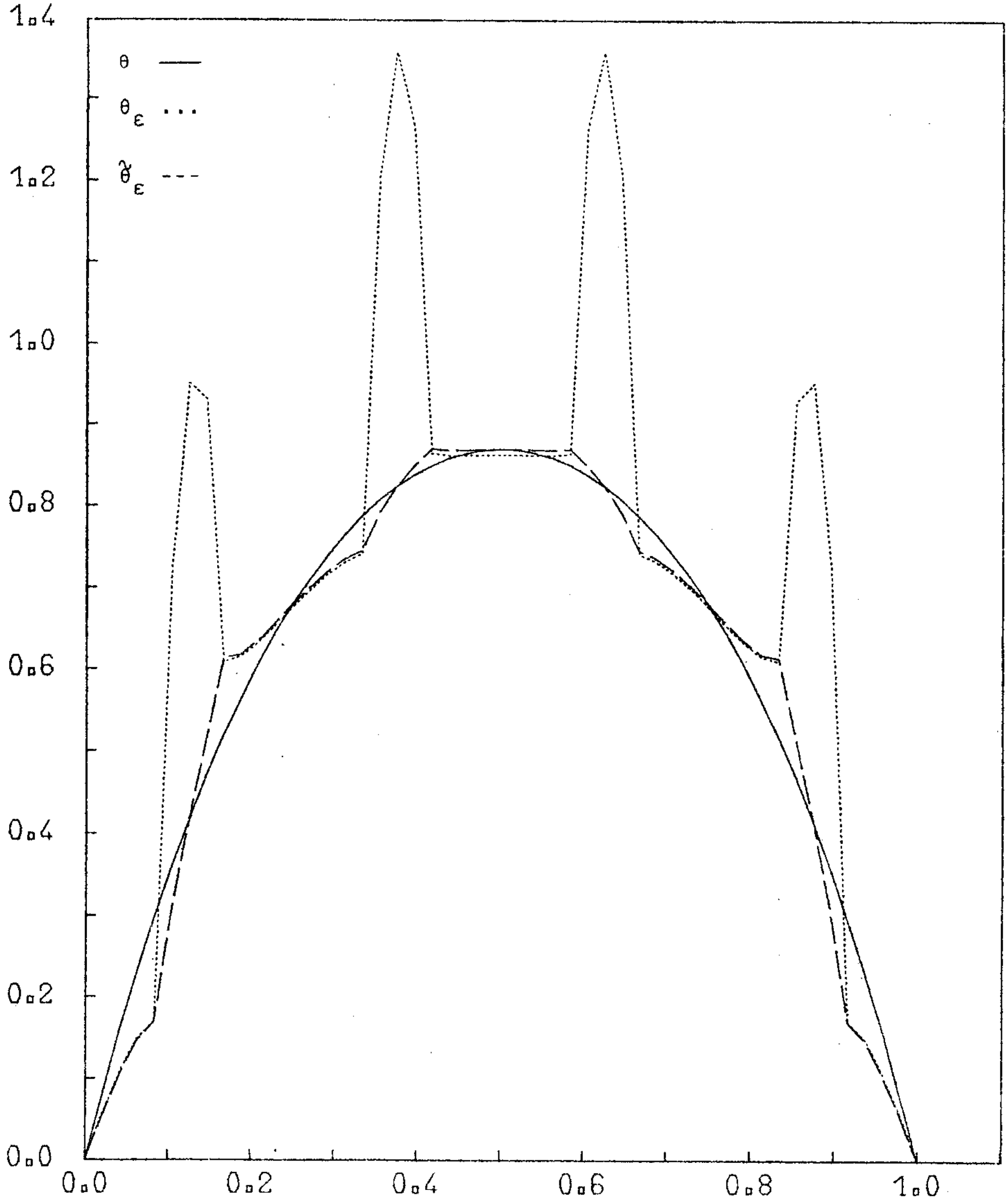
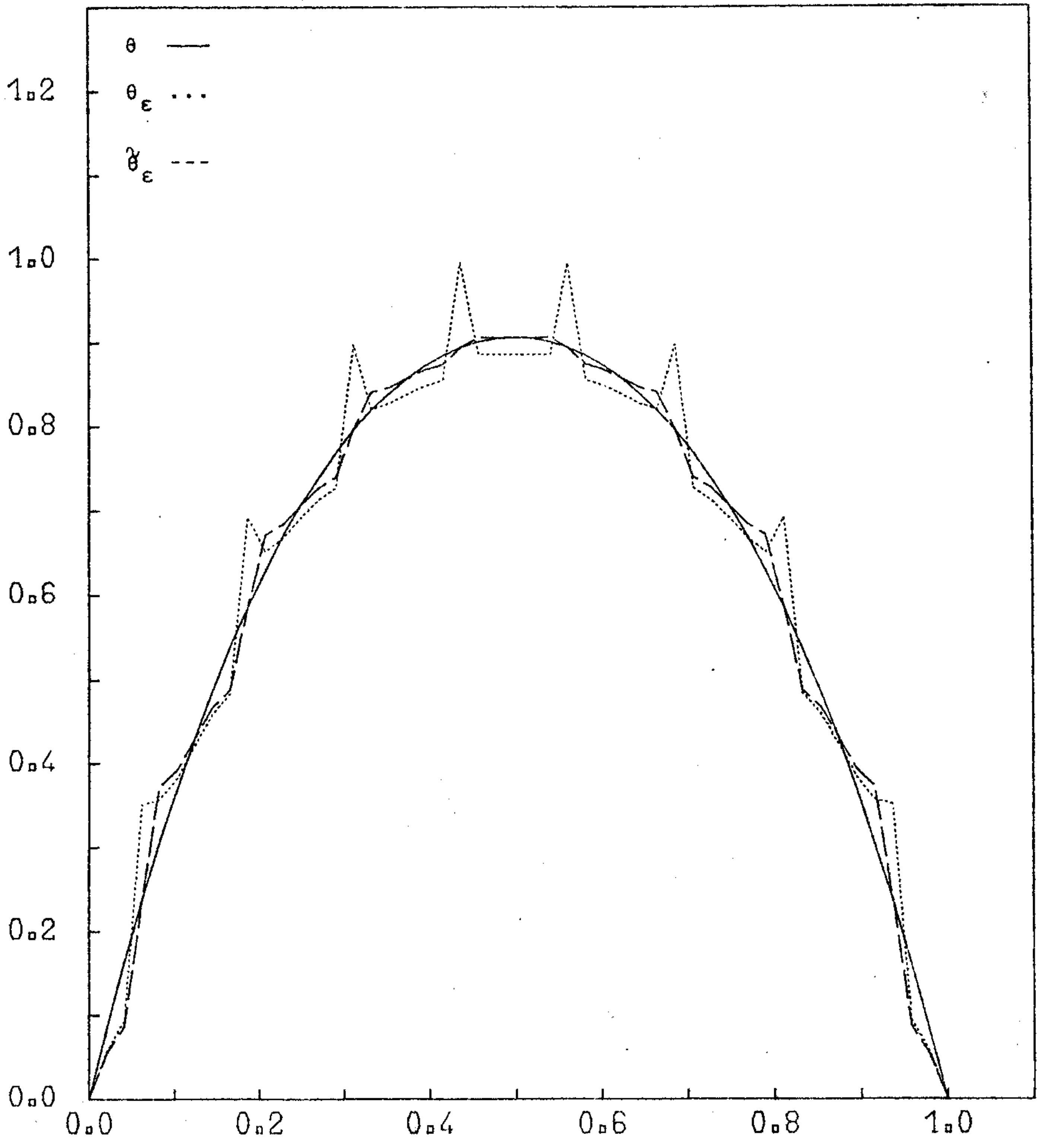


FIGURE 11 : : $\varepsilon = \frac{1}{8}$. "BORON-EPOXY" . (parallèle sections)



V : SOME ELEMENTS ABOUT THE MATHEMATICAL TECHNIQUES EMPLOYED TO OBTAIN THE PRECEDING RESULTS :

1) Multiple scales method :

The solution u_ε has no reasons to be periodic itself because Ω is bounded and $\partial\Omega$ is no necessarely coinciding with the boundary of some basic cells. However, u_ε is obviously depending of the periode of distribution of the heterogeneities ; for these reasons we are going to look for a solution of the form :

(17) $$u_\varepsilon(\underline{x}) = W_0(\underline{x},\underline{y}) + \varepsilon W_1(\underline{x},\underline{y}) + \varepsilon^2 W_2(\underline{x},\underline{y}) + \sigma(\varepsilon^2)$$

where $y_i = \dfrac{x_i}{\varepsilon}$ and $\underline{x},\underline{y} \in \Omega \times P$

If we set generally : $\phi(\underline{x},\underline{y}) = \phi^*(\underline{x})$, we have

$$\frac{\partial\phi^*}{\partial x_i} = \frac{\partial\phi}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial\phi}{\partial y_i}$$ and then

$$A^\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i}\left\{ a_{ij}(\underline{y}) \frac{\partial}{\partial x_j}\left(W_0^* + \varepsilon W_1^* + \varepsilon^2 W_2^*\right)\right\} = f$$

can be written

(18) $$(\varepsilon^{-2} A_1 + \varepsilon^{-1} A_2 + A_3)(W_0 + \varepsilon W_1 + \varepsilon^2 W_2 \ldots) = f$$

with $$A_1 = -\frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial}{\partial y_j}\right]$$

$$A_2 = -\frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial}{\partial x_j}\right] - a_{ij}\frac{\partial^2}{\partial x_i \partial y_j}$$

$$A_3 = -a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

By identification in (18) for the different orders of ε we obtain successively

(19) ε^{-2} : $A_1 W_0 = 0$

(20) ε^{-1} : $A_1 W_1 + A_2 W_0 = 0$

(21) ε^{0} : $A_1 W_2 + A_2 W_1 + A_3 W_0 = f$

The relation (19) implies :

(22) $$W_0(\underline{x},\underline{y}) = u(\underline{x})$$

The relation (20) shows that $W_1(\underline{x},\underline{y})$ is necessarily of the form :

(23) $$W_1(\underline{x},\underline{y}) = - \left(\chi^j(\underline{y}) \frac{\partial u}{\partial x_j} + \tilde{W}(\underline{x})\right)$$

Where χ^j is the solution in $\dot{W}$ of

(24) $$A_1 \chi^j = - \frac{\partial a_{ij}}{\partial y_i}$$

whom the variational formulation is (15), and $\tilde{W}$ (x) definite by ulterior identification.

At the end, (21) needs, to have a solution, of a compatibility condition which, after some transformation, can be written

(25) $$f = - q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

where q_{ij} is given by (16).

Replacing in (17) W_0 and W_1 by (22) and (23) and, taking into account (25) we obtain the statement of the section III.

2) Energy method :

We considerhere a more general problem with mixted boundary conditions :

$$(26)\quad \begin{cases} u_\varepsilon = 0 \text{ on } \Gamma_1 \\ \dfrac{\partial u_\varepsilon}{\partial \nu_A \varepsilon} = a_{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u_\varepsilon}{\partial x_j}\, n_i = 0 \text{ on } \Gamma_2 \end{cases}$$

with $\Gamma_1 \cap \Gamma_2 = \phi$ and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$

In this case, the convenient functional space is

$$(27)\quad V = \left\{ v \,/\, v \in H^1(\Omega) \quad , \quad v = 0 \text{ on } \Gamma_1 \right\}$$

For sake of simplicity, we suppose in this paper that Γ_1 is of non zero measure in order that :

$$(28)\quad \|v\|^2 = \int_\Omega |grad\ v|^2 dx$$

defines a norm on V.

If we introduce on V the bilinear form,

$$(29)\quad a_\varepsilon(\phi,\psi) = \int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial \phi}{\partial x_j}\,\frac{\partial \psi}{\partial x_i}\,dx$$

it is easy to show that the variational formulation of $(\mathcal{P}_\varepsilon)$ is

$$\begin{cases} a_\varepsilon(u_\varepsilon, v) = (f, v) & \forall\, v \in V \\ u_\varepsilon \in Y \end{cases}$$

where $(f,v) = \int_\Omega f(x)\, v(x)\, dx$

Then we obtain successively by non trivial way that :

. u_ε is bounded in V thanks to (3) and therefore, u_ε converges weakly towards an u $\in$ V.

. $a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} = \xi_i^\varepsilon$ converges weakly toward $\xi_i \in L^2(\Omega)$

. Finally, $\xi_i = q_{ij} \frac{\partial u}{\partial x_j}$ where the constant coefficients q_{ij} are the same as above.

Remark 8 : The *multiple scales method* is practice because naturally constructive ; it allows in particular to obtain the first corrector. However, this method asks a great regularity for the coefficients. In the other hand, the *energy method* needs only of bounded coefficients but leeds to a weaker convergence and does not introduce naturally the corrector. So we can judge the very complementary character of these two mathematical techniques.

VI. ELEMENTS ABOUT NUMERICAL TECHNIQUES :

For solving the boundary value problems introduced in this paper, we use the finite elements method.

The domains P and Ω are divided into triangles and the spaces $H^1(\Omega)$ and $H^1(P)$ are approched by :

$$V_h = \left\{ v/v \in C^\circ(\Omega), \text{ or } v \in C^\circ(P) \text{ polynomes of degree 1 on each triangle} \right\}$$

The functions θ and θ_ε are approchead in the space

$$V_{oh} = \left\{ v \in V_h \;/\; V = 0 \quad \text{on} \quad \partial\Omega \right\}$$

and the function χ_i in the space

$$\dot{V}_h = \left\{ v \in V_h \;/\; v \text{ "P - periodicable"; } V = c \text{ at the four corners of P} \right\}$$

The triangulation of P, to compute χ_i, and the triangulation of Ω, to compute θ_ε, are such as the discontinuities of the coefficients a_{ij} coincide with the sides of the triangle.

Generally a few hundred triangles are sufficient to compute χ_i because the geometry of P is very simple while, to compute θ_ε several thousands are necessary in order to approximate the numerous discontinuities of a_{ij}

For θ which is solution of an operator whith constant coefficients, a few number of triangles is sufficient but for $\frac{\partial\theta}{\partial x_i}$, which appears in the first corrective term, we need about a thousand triangles.

For the elastic torsion example mentioned above, we took

288 triangles for computing χ_i

1152 triangles for computing θ

4608 triangles for computing θ_ε

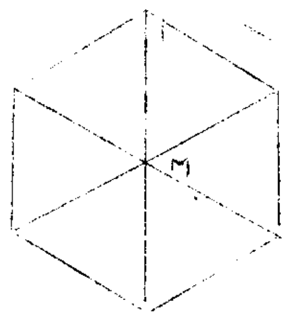
The linear systems obtained after approximation of the boundary value problems are solved by the overrelaxation method with optimal parameter.

The computing times obtained with a IBM 370168 are

$$\begin{cases} 1.5\text{s to compute the coefficients of the homogenized operator} \\ 4\text{ s to compute } \theta \\ 30\text{ s to compute } \theta_\varepsilon \end{cases}$$

On order to compute the first corrective term we approximate $\frac{\partial\theta}{\partial x_i}$ by

$$\frac{\partial\theta}{\partial x_i}(M) = \frac{1}{N_{v(M)}} \sum_{T \in v(M)} \left(\frac{\partial\theta}{\partial x_i}\right)_T$$



M is a node of the triangulation

$v(M)$ is the set of triangles whose M is a corner

$N_{v(M)}$ is the number of triangles of $v(M)$

$\left(\frac{\partial\theta}{\partial x_i}\right)_T$ is the value of the derivative of θ on the triangle T

We can use this formula because θ is very regular as a solution of a problem with constant coefficients and constant second

VII. CONCLUSIONS :

This method of homogenization, described here on a simple example, can be extended to numerous directions :

Neumann problem : for instance

$$\begin{cases} A^{\varepsilon} u_{\varepsilon} + a_0\left(\frac{x}{\varepsilon}\right) u_{\varepsilon} = f \text{ on } \Omega \\ \\ \frac{\partial u_{\varepsilon}}{\partial \nu_{A^{\varepsilon}}} = 0 \quad \text{on } \partial\Omega \end{cases}$$

(cf figure 11)

Variational inequalities : that is to say, when the solution has to be found in a convex subset of a vectorial space for instance

$$K = \left\{ \phi / \phi \in H_0^1(\Omega), \quad \phi \leqslant 0 \text{ in } \Omega \right\}$$

that is the case of $u_{\varepsilon} = -\theta_{\varepsilon}$ when θ_{ε} is a absolute temperature (cf figure 12)

It is interesting to note, on the figure 12, that the free boundary (unknown of the problem) between

$$\Omega_{\varepsilon}^{0} = \left\{ x \;/\; x \in \Omega \quad \text{s.t.} \quad u_{\varepsilon} = 0 \right\} \quad \text{and}$$

$$\Omega_{\varepsilon}^{-} = \left\{ x \;/\; x \in \Omega \quad \text{s.t.} \quad u_{\varepsilon} < 0 \right\} \; ,$$

is given with a good precision by the homogenized solution u.

Operators ***of*** *higher orders* of G. DUVAUT [5])

Evolution problems

Systems of partial differential equations : for instance **in elasticity** (cf G. DUVAUT [6]) or electromagnetism.

Operators ***with*** *coefficients depending more generally on* ε :

$a_{ij}\,(x, \frac{x}{\varepsilon})$ $a_{ij}(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon\eta})$ several physical examples are corresponding to theses cases.

Porous media

Mechanics of fluid suspensions.

Media with bubbles....etc

This list is, of course, boundless but, already now, we need of the help of Physicists to interpret the results and the different steps of the method, and also to suggest us the realistic problems in this way.

Some more recent news :

- J.F Bourgat has computed the second corrector, that is to say the term in ε^2 in the preceding example of torsion and improved a lot in the approach of θ_ε - an internal report is supposed to appear soon at the I.R.I.A with details about this problem.

- D. Cioranescu and J. Saint Jean Paulin (University Paris VI) have obtained the coefficient $\bar{\mu}$ of the "homogenized cylinder" corresponding, for the problem of Torsion, to a cylinder pierced of numerous cylindrical cavities of same direction (see description in lectures notes 2-3-25 paragraphe IV, 1) this result will appear later.

- J.L. Lions was supposed to give his lectures on this topic and teach during this term a course at the "college de France" in Paris in which a lot of other recent results are mentioned.

H.L Trieste 11.11.1976.

FIGURE 11 : Neumann problem . (diagonal sections)

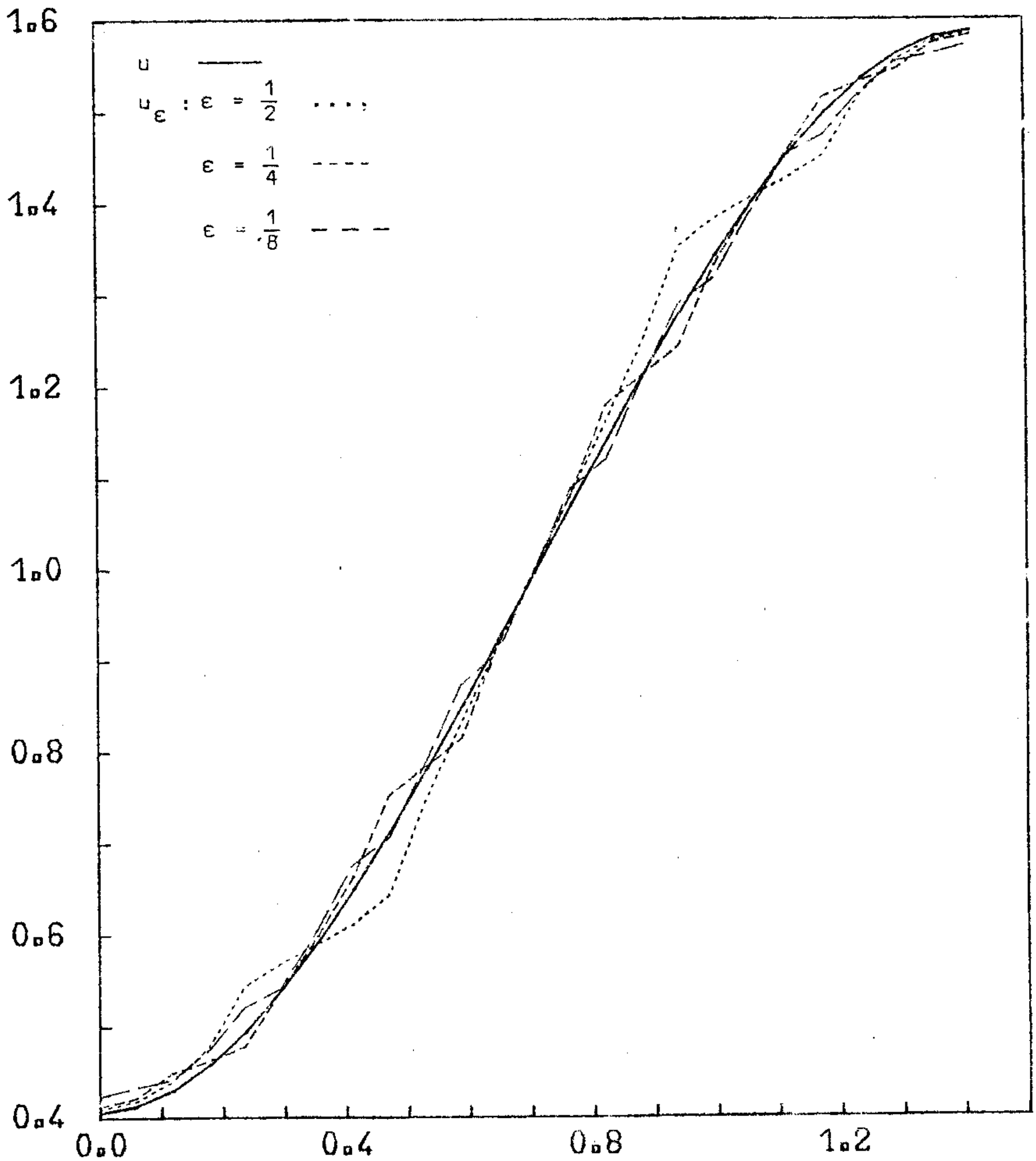
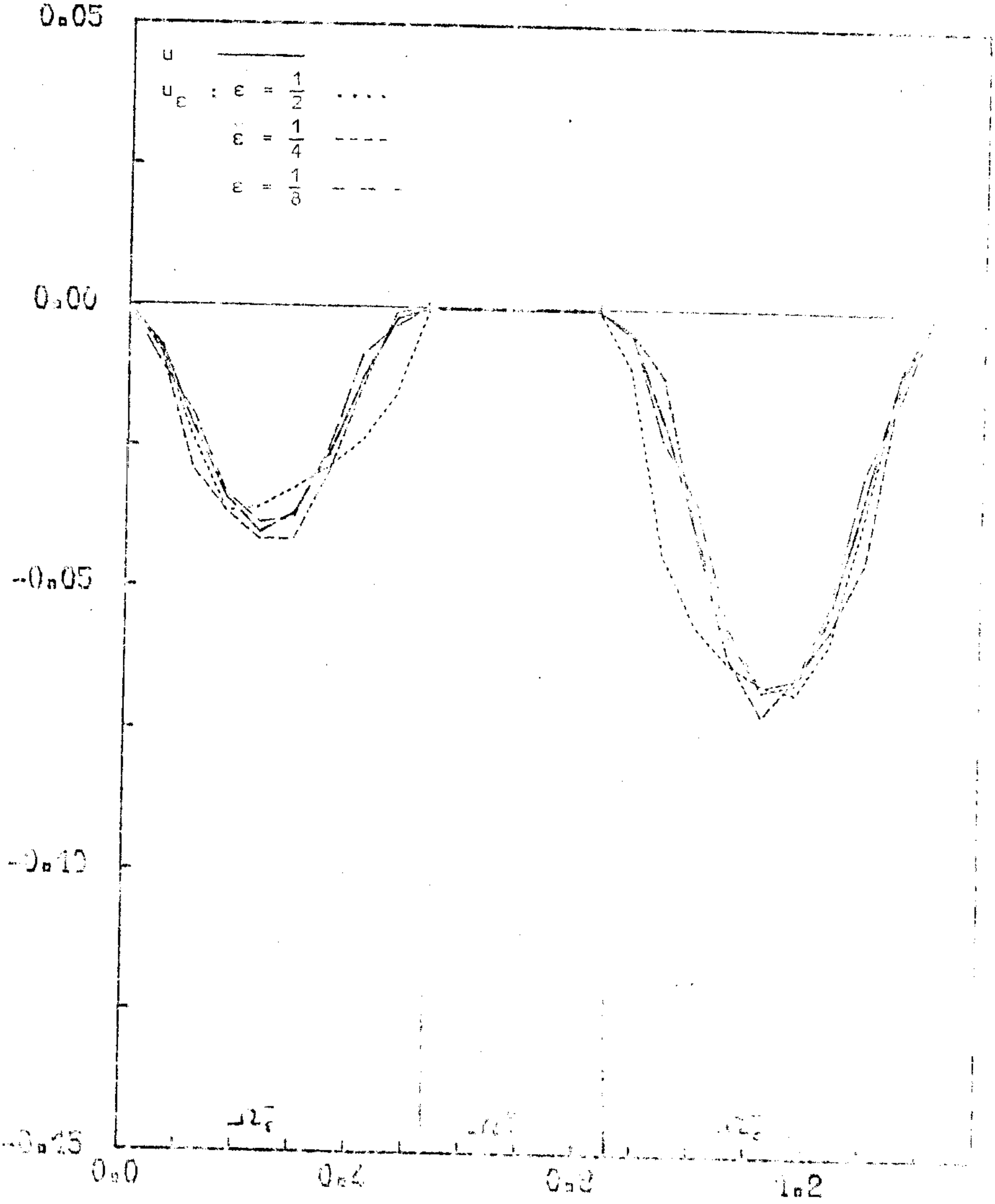


FIGURE 12 : Variational Inequalities . (diagonal sections)

$u_\varepsilon \leq 0$



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