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Numerical approximations for some singular integral equations

Part I

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Introduction

We ~~shall~~ consider here the problem of finding in $\mathbb{R}^3$ the electric field created by a conductor which potential is know. We shall denoted by Ω the conductor, by Γ its boundary and by Ω' the exterior domain. Then the equation are

find u a function in $\mathbb{R}^3$ (the potential of the electric field) such that

$$(1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \Delta u = 0 & \text{in } \Omega' \\ u|_\Gamma = u_0 & , u_0 \text{ being the given potential of the conductor.} \end{cases}$$

In a first theoretical part, we shall give some theorems on the uniqueness and existence of ~~the~~ solution. We shall give also a representation of the solution by means of an integral operator.

In a second part, we shall indicate how we can approximate this solution of this integral equation, using finite element method on Γ. We shall then give error estimates between the exact solution and the approximate one.

1. The problem of the electric field.

The system of equation (1) is in fact decoupled into two problems of Diritchlet non. homogeneous.

The first one is an interior problem

$$(1.1)\qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\Gamma} = u_0 \end{cases}$$

The second one is an exterior problem

$$(1.2)\qquad \begin{cases} \Delta u = 0 \\ u|_{\Gamma} = u_0 \end{cases}$$

We have the following lemma

<u>lemma 1</u> Let $u_0 \in H^{1/2}(\Gamma)$, then the problem (1.1) admit a unique solution in the space $H^1(\Omega)$

(see M^lle Lanchon [ ] for the definition of these spaces)

<u>Proof</u>. We know that we can find a function $\varphi \in H^1(\Omega)$, such that.

$$\varphi|_{\Gamma} = u_0$$

Then the problem (1.1) is equivalent to

$$(1.3)\qquad \begin{cases} \Delta(u-\varphi) = -\Delta\varphi \\ (u-\varphi)|_{\Gamma} = 0 \end{cases}$$

And (1.3) is a Diritchlet problem with homogeneous

... $\Delta\varphi \in H^{-1}(\Omega)$ So that for the coerciveness of the associate bilinear form (Poincaré's Inequality)

$$a(u,v) = \sum_{i=1}^{3} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, d\omega$$

we know that $u-\varphi$ exist and is unique. It is the same for u.

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If we consider now the exterior problem, the same process can be used. Let $\varphi \in H^1(\Omega')$ with compact support in $\mathbb{R}^3$ be such that

$$\varphi|_\Gamma = u_0$$

The problem (1.2) is then equivalent to

$$(1.4) \quad \begin{cases} \Delta(u-\varphi) = -\Delta\varphi \\ (u-\varphi)|_\Gamma = 0 \end{cases}$$

The problem (1.4) is a variational problem in $u-\varphi$, which can be written as

find $v \in V$ such that

$$\begin{cases} \displaystyle\sum_{i=1}^{3} \int_{\Omega'} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \, d\omega = \sum_{i=1}^{3} \int_{\Omega'} \frac{\partial \varphi}{\partial x_i} \frac{\partial w}{\partial x_i} \, d\omega \; ; \; \forall w \in V \\ v|_\Gamma = 0 \quad \text{and} \quad w|_\Gamma = 0 \end{cases}$$

The only thing that ~~missed~~ is : what is the Hilbert space V. The answer is :

proposition 1.1. Let us consider the following Hilbert space

$$W_0^1(\Omega') = \left\{ u \;\middle|\; \frac{1}{(1+r^2)^{1/2}} u \in L^2(\Omega') \;,\; \frac{\partial u}{\partial x_i} \in L^2(\Omega'),\ i=1,2,3\right.$$

The bilinear form

$$a(u,v) = \sum_{i=1}^{3} \int_{\Omega'} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, d\omega$$

is continuous on $W_0^1 \times W_0^1$ and satisfies the coercivness condition

$$\| u \|^2_{W_0^1(\Omega')} = \left| \frac{u}{(1+r^2)^{1/2}} \right|^2_{L^2(\Omega')} + \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i} \right|^2_{L^2(\Omega')}$$

(1.5) $$a(u,u) \geq \frac{1}{10} \| u \|^2_{W_0^1(\Omega')}$$

The result is also true for $\Omega' = R^3$.

Proof

The fact that $W_0^1(\Omega')$ is a hilbert space is easy to check. The continuity of the bilinear form is trivial. So we have only to prove the inequality (1.5). We use a lemma

<u>lemma 2</u> (Hardy's Inequality) If $r f^2(r) \to 0$ as $r \to \infty$ and $r^2 \frac{df}{dr} \to 0$ as $r \to \infty$

we have

(1.6) $$\int_0^\infty (f(r))^2 \, dr \leq 4 \int_0^\infty \left| \frac{df}{dr} \right|^2 r^2 \, dr$$

<u>Proof</u> We integrate by parts the first term

$$\int_0^\infty (f(r))^2 \, dr = -2 \int_0^\infty f(r) \cdot \frac{df}{dr}(r) \cdot r \, dr + 0$$

We use Cauchy-Schwartz inequality:

$$\int_0^\infty (f(r))^2 dr \leq 2 \left(\int_0^\infty (f(r))^2 dr \right)^{1/2} \left(\int_0^\infty \left| \frac{df}{dr} \right|^2 r^2 dr \right)^{1/2}$$

We deduce easily (1.6) from the inequality above.

<u>Proof of proposition 1.1</u> (continued). If we consider

$$\tilde{u} = \begin{cases} u & \text{in } \Omega' \\ 0 & \text{in } \Omega \end{cases}$$

$$\|u\|_{W_0^1(\Omega')} = \|\tilde{u}\|_{W^1(R^3)}$$

$$a(u,u) = a(\tilde{u},\tilde{u}) \qquad \text{(a being defined trivially on } R^3 \text{)}$$

So that we have only to prove our result for $u \in W^1(R^3)$.

We admit that $\mathcal{D}(R^3)$ is dense in the space $W^1(R^3)$.

That allows us to suppose that $u \in \mathcal{D}(R^3)$. We introduce the spherical coordinates (r, θ, φ) with

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

We have

$$d\omega = \Psi(\theta,\varphi)\; r^2 dr \; d\theta \, d\varphi$$

(Ψ is a bounded regular fonction of θ and φ)

So that

$$\int_{R^3} \frac{|u(x)|^2}{1+r^2}\, d\omega = \int \Psi(\theta,\varphi) \left(\int_0^\infty \frac{|u(x)|^2}{1+r^2}\, r^2 dr \right) d\theta\, d\varphi$$

$$\int_{R^3} \left|\frac{\partial u}{\partial x_i}\right|^2 d\omega \geq \int_{R^3} \left|\frac{\partial u}{\partial r}\right|^2 d\omega = \int \Psi(\theta,\varphi) \left(\int_0^\infty \left|\frac{\partial u}{\partial r}\right|^2 r^2 dr \right) d\theta\, d\varphi$$

using our lemma 2.

$$\geq \frac{1}{4} \int \Psi(\theta,\varphi) \left(\int_0^\infty |u(x)|^2 dr \right) d\theta\, d\varphi \geq \frac{1}{4} \int_{R^3} \frac{|u(x)|^2}{1+r^2}\, d\omega$$

The inequality (1.5) follows immediately from the above inequality with $\alpha = \frac{1}{10}$.

▨

From our proposition and the Lax Milgram's theorem (see H. Lanchon [ ]) we deduce that the problem (1.4) and returning now to the problem (1.2) and (1) we can summarize our results in:

<u>Théorem 1.1</u> The problem (1) has a unique solution in the space $W^1(\mathbb{R}^3)$ if $u_0 \in H^{1/2}(\Gamma)$

$$W^1(\mathbb{R}^3) = \left\{ u \;\middle|\; \frac{1}{(1+r^2)^{1/2}}\, u \in L^2(\mathbb{R}^3) \,,\, \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^3) \underset{i=1,2,3}{} \right\}$$

<u>remark</u> If u_0 is more regular that $H^{1/2}(\Gamma)$, we can prove also that u is more regular. But we have to take care that u is more regular on Ω and on Ω', <u>but not globally</u>. In fact we shall see that the normal derivative of u is discontinuous when crossing Γ.

2. <u>The electric load of the conductor and the integral representation.</u>

Suppose, for a moment, that u is regular in Ω and in Ω' and let B_R a ball of radius R containing Γ strictly. We introduce the function.

(2.1) $$v_y(x) = \frac{1}{4\pi}\,\frac{1}{|x-y|}$$; $|\;|$ is the euclidian norm in $\mathbb{R}^3$.

<u>lemma 2</u> : except at the point y we have

(2.2) $$\Delta_x v_y = 0$$

<u>Proof</u>.

$$\frac{\partial v}{\partial x_i} = \frac{1}{4\pi} \frac{y_i - x_i}{|x-y|^3}$$

$$\frac{\partial^2 v}{\partial x_i^2} = \frac{1}{4\pi}\left(\frac{-1}{|x-y|^3} + \frac{3\,|y_i-x_i|^2}{|x-y|^5}\right)$$

Let $y \in \Omega$ and $B_\varepsilon(y)$ a ball of center y and radius ε contained in Ω. We choose the orientation of the normal on Γ exterior to Ω. We denote by $\left.\frac{\partial u}{\partial n}\right|_{\text{int}\,\Gamma}$ the derivate along the normal of the restriction of u to Ω. Then we have by Green's formula

$$0 = \int_{\Omega - B_\varepsilon(y)} (u\,\Delta v - v\,\Delta u)\,d\omega = \int_\Gamma \left(u\frac{\partial v}{\partial n} - v\left.\frac{\partial u}{\partial n}\right|_{\text{int}\,\Gamma}\right)d\gamma + \int_{\partial B_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)d\gamma$$

Let $\varepsilon \to 0$.

$$\int_{\partial B_\varepsilon} v\frac{\partial u}{\partial n}\,d\gamma \to 0 \qquad \left(= \frac{1}{4\pi}\int_{\partial B_\varepsilon} \frac{1}{\varepsilon}\frac{\partial u}{\partial n}\,d\gamma \;;\; d\gamma = \varepsilon\,\psi(\theta,\varphi)\,d\theta\,d\varphi\right)$$

$$\int_{\partial B_\varepsilon} u\frac{\partial v}{\partial n}\,d\gamma = \frac{1}{4\pi}\int_{\partial B_\varepsilon} u\;\psi(\theta,\varphi)\,d\theta\,d\varphi \to u(y)$$

So that:

(2.4) $$u(y) = -\int_\Gamma u\frac{\partial v}{\partial n}\,d\gamma + \frac{1}{4\pi}\int_\Gamma \left.\frac{\partial u(x)}{\partial n}\right|_{\text{int}\,\Gamma}\cdot\frac{1}{|x-y|}\,d\gamma(x)$$

If $y \in \Omega'$ the same process give

(2.5) $$0 = -\int_\Gamma u\frac{\partial v}{\partial n}\,d\gamma + \frac{1}{4\pi}\int_\Gamma \left.\frac{\partial u}{\partial n}(x)\right|_{\text{int}\,\Gamma}\frac{1}{|x-y|}\,d\gamma(x)$$

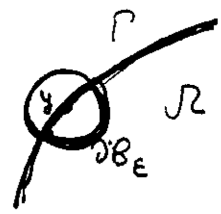
And if $y \in \Gamma$ we have to consider again B_ε and

$$0 = \int_{\Gamma - (B_\varepsilon \cap \Gamma)} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\Big|_{\text{int}\,\Gamma} \right) d\gamma + \int_{\partial B_\varepsilon \cap \Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\gamma$$

$$\frac{\partial v(x)}{\partial n_x}\Big|_\Gamma = \frac{1}{4\pi} \frac{(x-y \,.\, n_x)}{|x-y|^3}$$ $(\,.\,)$ is the euclidean scalar product.

We can remark that the boundary Γ being regular, that function of x is intégrable on Γ even if y is on Γ.

Let $\varepsilon \to 0$. We obtain



$$\int_{\partial B_\varepsilon \cap \Omega} v \frac{\partial u}{\partial n} d\gamma \to 0$$

$$\int_{\partial B_\varepsilon \cap \Omega} u \frac{\partial v}{\partial n} d\gamma \to \frac{1}{2} u(y)$$

$$\int_{\Gamma - (B_\varepsilon \cap \Gamma)} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\Big|_{\text{int}\,\Gamma} \right) d\gamma \to \int_\Gamma \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\Big|_{\text{int}\,\Gamma} \right) d\gamma$$

So that we obtain when $y \in \Gamma$

(2.6) $$\frac{u(y)}{2} = - \int_\Gamma u \frac{\partial v}{\partial n} d\gamma + \frac{1}{4\pi} \int_\Gamma \frac{\partial u}{\partial n}(x)\Big|_{\text{int}\,\Gamma} \frac{1}{|x-y|} d\gamma(x)$$

We cannot use directly Green formula in Ω'. So we use it in $\Omega' \cap B_R$. First we obtain

If $y \in \Omega$ (we denote by $\frac{\partial u}{\partial n}\Big|_{\text{ext}\,\Gamma}$ the normal derivative of u in Ω')

(2.7) $$0 = \int_\Gamma u \frac{\partial v}{\partial n} d\gamma - \frac{1}{4\pi} \int_\Gamma \frac{\partial u}{\partial n}(x)\Big|_{\text{ext}\,\Gamma} \frac{1}{|x-y|} d\gamma(x)$$

$$+ \int_{\partial B_R} \left(- u \frac{\partial v}{\partial n} + \frac{\partial u}{\partial n} v \right) d\gamma$$

If $y \in \Omega' \cap B_R$

(2.8) $$\ldots \int u \frac{\partial v}{\partial n} d\gamma \ldots \frac{1}{4\pi} \int \frac{\partial u}{\partial n}\Big| \ldots \frac{1}{\ldots} d\gamma(x) + \int \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\gamma$$

And if $y \in \Gamma$

$$(2.9) \qquad \frac{u(y)}{2} = \int_\Gamma u \frac{\partial v}{\partial n} \, d\gamma - \frac{1}{4\pi} \int_\Gamma \frac{\partial u}{\partial n}\Big|_{ext\,\Gamma} \frac{1}{|x-y|} \, d\gamma(x) + \int_{\partial B_R} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\gamma$$

By adding all these equalities we obtain finally for $y \in B_R$

$$(2.10) \qquad u(y) = \frac{1}{4\pi} \int_\Gamma \left(\frac{\partial u(x)}{\partial n}\Big|_{int\,\Gamma} - \frac{\partial u(x)}{\partial n}\Big|_{ext\,\Gamma} \right) \frac{1}{|x-y|} \, d\gamma(x)$$

$$+ \int_{\partial B_R} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\gamma$$

In fact we shall now prove the theorem of representation

<u>Theorem 2.1</u> If u is regular in Ω and Ω' and we denote by q the electric load of the conductor Γ:

$$(2.11) \qquad q = \frac{\partial u}{\partial n}\Big|_{int\,\Gamma} - \frac{\partial u}{\partial n}\Big|_{ext\,\Gamma}$$

we have

$$(2.12) \qquad u(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x)}{|x-y|} \, d\gamma(x)$$

<u>Proof</u>

Let

$$\tilde{u}(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x)}{|x-y|} \, d\gamma(x)$$

It is easy to see that

$$\Delta \hat{u} = 0 \quad \text{in } \Omega \text{ and in } \Omega'$$

By (2.10) we see that we have also

$u - \hat{u}$ is regular on crossing Γ so that

On the other hand $\hat{u} \in W^1(\mathbb{R}^3)$ can be easily proved if q is regular. So that by ~~Theorem~~ proposition 1.1:

$u - \hat{u} = 0$. (because $a(u-\hat{u}, u-\hat{u}) = 0$)

Let $H^{-1/2}(\Gamma)$ be the dual space of $H^{1/2}(\Gamma)$ which is also a Hilbert space. We shall denote by $\langle q, u \rangle$ the pairing between these spaces, which is the scalar product in $L^2(\Gamma)$ when q is in $L^2(\Gamma)$.

Consider now the way we defined q. We have

(2.13) $$\left\langle \frac{\partial u}{\partial n}\Big|_{\mathrm{int}\,\Gamma}, v \right\rangle = \sum_{i=1}^{3} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, d\omega \quad ; \forall v \in H^1(\Omega)$$

and this formula proves that the application

$$u \in H^1(\Omega) \longrightarrow \frac{\partial u}{\partial n}\Big|_{\mathrm{int}\,\Gamma} \in H^{-1/2}(\Gamma)$$

is continuous

For the exterior domains we have also

(2.14) $$\left\langle \frac{\partial u}{\partial n}\Big|_{\mathrm{ext}\,\Gamma}, v \right\rangle = - \sum_{i=1}^{3} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, d\omega \quad ; \forall v \in W^1(\Omega') \text{ with compact support.}$$

and this formula proves that the application

$$u \in W^1(\Omega') \to \frac{\partial u}{\partial n}\Big|_{\mathrm{ext}\,\Gamma} \in H^{-1/2}(\Gamma)$$

is continuous. So combining with theorem 1.1 we obtain

Lemma 3.

The application

$$u_0 \in H^{1/2}(\Gamma) \longrightarrow q = \frac{\partial u}{\partial n}\Big|_{\mathrm{int}\,\Gamma} - \frac{\partial u}{\partial n}\Big|_{\mathrm{ext}\,\Gamma} \in H^{-1/2}(\Gamma)$$

is continuous

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Concerning the reverse application we can see by adding (2.13) and (2.14) that (v being chosen to have the same trace on Γ from in Ω and Ω')

(2.15) $$a(u,v) = \langle q, v_{|\Gamma} \rangle \quad ; \forall v \in W^1(R^3) .$$

But for q in $H^{-1/2}(\Gamma)$ the second member of (2.15) is a continuous linear form on $W^1(R^3)$ as a result of the trace theorem (see Mlle H. Lanchon [ ])

So that (2.15) has a unique solution for q given as a result of proposition 1.1.

And that u has a trace $u_0 \in H^{1/2}(\Gamma)$.

Furthermore we have the inequalities by continuity

(2.16) $$c\, |q|_{H^{-1/2}(\Gamma)} \leq |u_0|_{H^{1/2}(\Gamma)} \leq C\, |q|_{H^{-1/2}(\Gamma)}$$

and also

$$\langle q, u_0 \rangle = a(u,u) \geq \ldots \quad \alpha \|u\|^2_{W^1(R^3)}$$

(2.17) $$\langle q, u_0 \rangle \geq \beta \|q\|^2_{H^{-1/2}(\Gamma)} \quad ; \beta > 0 .$$

We summarize all these new results in the theorem

Theorem 2.2.

The application $q \to u_0 = u|_\Gamma$ defined by (2.15) is continuous and bijective from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$. Furthermore if q is regular it has the expression

(2.18) $$u(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x)}{|x-y|} \, d\gamma_x \quad ; \ \forall y \in \Gamma$$

(and $u(y)$ is defined also by this formula for $y \in R^3$)

This problem has also a variational formulation associate to the symmetric bilinear coercive following form:

$$b(q, q') = \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{q(x) \, q'(y)}{|x-y|} \, d\gamma_x \, d\gamma_y \quad ; \ \forall q, q' \in H^{-1/2}(\Gamma).$$

Remark If u_0 is more regular we have see (without proofs) that u is also more regular. It results that q is also more regular. In the reverse way, we can prove that q more regular implies u more regular and so for u_0. In fact we can prove that the problem (2.18) is an isomorphism between $H^s(\Gamma)$ and $H^{s+1}(\Gamma)$ for any s. (If Γ is sufficiently regular).