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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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NUMERICAL APPROXIMATIONS FOR SOME SINGULAR INTEGRAL EQUATIONS

(Part II)

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3. Numerical approximations of Problem (1) by finite elements in $\mathbb{R}^3$.

Chavent (this session) has shown how to solve an interior Diritchlet problem in two variables via finite element method. I first shall give some recalls and comments on that method. And I shall give also some comments for the exterior problem. We shall them show how the representation obtain in the last chapter lead to another method of approximation of both problems.

How do you approximate the interior problem (1.1) by finite elements methods?

– The general process is to built a subspace V_h of $H_0^1(\Omega)$ and to solve them

$$\sum_{i=1}^{3} \int_\Omega \frac{\partial w_h}{\partial x_i} \frac{\partial v_h}{\partial x_i}\, d\omega = \sum_{i=1}^{3} \int_\Omega \frac{\partial \varphi}{\partial x_i} \frac{\partial v_h}{\partial x_i}\, d\omega \quad ; \quad \forall v_h \in V_h .$$

and the approximate solution is then

$$u_h = w_h + \varphi$$

In fact we have no need of φ because we built a subspace $\tilde{V}_h$ of $H^1(\Omega)$ and we solve

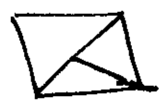
$$\sum_{i=1}^{3} \int_\Omega \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_i}\, d\omega = 0 \quad ; \quad \forall v_h \in \tilde{V}_h \cap H_0^1(\Omega)$$

and $u_h|_\Gamma = u_\Gamma$.

The classical way of building such $\hat{V}_h$ is to use finite elements. In our case we built first a "triangulation" of Ω. This is possible if Ω is a polyhedra. If not we built a triangulation of Ω_h approximating Ω.

A triangulation in R^3 is constituted of tetrahedras which faces are triangulars. We recall that, the same in two dimensions, two adjacent tetrahedras must have in common

either a face (total)
either an edge (total)
either a vertex



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We shall call $\mathcal{C}_h$ such a "triangulation" on each tetrahedra T

A finite element is them given by two things

(α) a set of degrees of freedom which are for examples: value of functions in certains nodes, or values of derivatives..

(β) a linear space of functions of finite dimension (the same as the numbers of degrees of freedom). In general these functions are polynomials. We call P that space.

We shall restrict us here to the cases when the degrees of freedom are values of the functions. So that they are defined by the giving of the list of points where they are choose, says $\Sigma = \bigcup_{i=1}^{N} a_i$; $a_i \in T$.

We need also the property that

(γ) There is a unique $p \in P$ such that
$$p(a_i) = \alpha_i \quad , i = 1, N \; ; \; \alpha_i \in R .$$

The space V_h is then defined by

$$V_h = \{ u_h \in C^0(\Omega_h) \; ; \; u_h|_T \in \mathbb{P} \}$$

<u>examples</u>

- The simplest case is with

  $\mathbb{P} = P_1$ polynomials of degree 1

  so that $\dim(P_1) = 4$

  We choose Σ the vertex of the pyramida which is also 4 points.

  The property (γ) is fulfilled when the pyramida is not contain in a surface of dimension 2.

  (It is not flat).

  It is easy to see then that the degrees of freedom of the space V_h are the values of u_h on the nodes of the triangulation.

  The linear system that we obtain is symmetric definite positive and the matrix is very sparse.

- The next finite element in this family consist of choosing

  $\mathbb{P} = \mathbb{P}_2$ polynomials of degree 2.

  $\dim(\mathbb{P}_2) = 10$

  and the middle

of the edges (6).

We can verify that the condition (γ) is also fulfilled

exercice Compute in terms of the barycentric coordinates in the tetrahedra, the "basis functions" defined on each tetrahedra by

$$\varphi_i(a_j) = \delta_{ij}, \quad i,j = 1,..10, \ \varphi_i \in \mathbb{P}_2 .$$

The remark on that methods, that works well, is that we needs a great deal of unknows to get a reasonnable accuracy for our solution. This fact is related to the space $\mathbb{R}^3$. Also it is quite difficult to manage such a computer program because it is quite difficult to build a triangulation of the domain Ω.

What about the exterior problem ?

We can think of using the same method. But the domain is not bounded. So we need a bound. We give us for example, a big parallelipódic B. And we change the problem by adding an artificial boundary conditions such that

$u|_{\partial B} = 0$ (or something known is we we know

$$u(x) \simeq \frac{1}{4\pi} \int_{\Gamma} q(x)\, d\gamma(x) \times \frac{1}{\mathrm{dist}(x, \Gamma)}$$

when $|x| \to \infty$.)

So that the same process as for the interior domain, lead to a system of linear equation.
The same remark is that the system to solve is very big and the triangulation is quite difficult to build.

4. Numerical approximations of problem (1) using the integral representation.

If we consider the bilinear form (2.19) the formulation of the problem in the unknown q is

(4.1) $$b(q, q') = \langle u_0, q' \rangle \quad ; \forall q' \in H^{-\frac{1}{2}}(\Gamma)$$

and once we know the value of the load q, the function u is given every where by

(4.2) $$u(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x)}{|x-y|} \, d\gamma(x) \quad .$$

If we consider now a finite dimensional subspace of $H^{-1/2}(\Gamma)$, call it V_h, we can consider the problem

(4.3) $$b(q_h, q'_h) = \langle u_0, q'_h \rangle \quad ; \forall q'_h \in V_h$$

and such an approximate load q_h give us an approximate u_h by the same formula:

(4.4) $$u_h(y) = \frac{1}{4\pi} \int_\Gamma \frac{q_h(x)}{|x-y|} \, d\gamma(x)$$

It can not be so easy to build a subspace V_h defined on Γ. So we are lead to introduce also Γ_h, an approximate surface of the surface Γ. We consider the approximate bilinear form

(4.5) $$b_h(q_h, q'_h) = \frac{1}{\ldots} \iint \frac{q_h(x)\, q'_h(y)}{\ldots} \, d\gamma(x) \, d\gamma(y)$$

We introduce a subspace V_h of finite dimension of functions defined on Γ_h. We approximate problem is then

$$
(4.6)\quad \left\{ \begin{array}{l} \text{find } q_h \in V_h \text{ such that :} \\ b_h(q_h, q'_h) = \displaystyle\int_{\Gamma_h} u_{0h} \cdot q'_h \, d\gamma \quad ; \ \forall\, q'_h \in V_h . \end{array} \right.
$$

Concerning the error estimates we have the theorem

Theorem 4.1

The error between q and q_h solution of (4.3) is bounded by

$$
(4.7)\qquad \| q - q_h \|_{H^{-1/2}(\Gamma)} \leq c \inf_{q'_h \in V_h} \| q - q'_h \|_{H^{-1/2}(\Gamma)} .
$$

and the difference between u and u_h given by (4.4) satisfies

$$
(4.8)\qquad \| u - u_h \|_{W^1(R^3)} \leq c \, \| q - q_h \|_{H^{-1/2}(\Gamma)}
$$

Proof. We have

$$
b(q - q_h, q'_h) = 0 \qquad \forall\, q'_h \in V_h .
$$

So that

$$
\beta \| q'_h - q_h \|^2_{H^{-1/2}(\Gamma)} \leq b(q'_h - q_h, q'_h - q_h) = \cancel{b(q - q_h, q'_h - q_h)} - b(q - q'_h, q'_h - q_h)
$$

$$
\leq c \, \| q - q'_h \|_{H^{-1/2}(\Gamma)} \, \| q'_h - q_h \|_{H^{-1/2}(\Gamma)}
$$

We have then

$$u(y) - u_h(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x) - q_h(x)}{|x-y|} \, d\gamma(x)$$

So that the inequality (4.8) is a consequence of the continuity of the mapping : $q \in H^{-1/2}(\Gamma) \longrightarrow u \in W^1(R^3)$.

The case of the problem (4.5) is more difficult because q_h is not defined on Γ but on Γ_h. To compare q and q_h we have to associate to q_h a function $\hat{q}_h$ defined on Γ. We need also to suppose that the bilinear form b_h is uniformly coercitive i.e

(4.9) $$b_h(q_h, q_h) \geq \beta \, \|q_h\|^2_{H^{-1/2}(\Gamma_h)} \quad . \beta > 0 .$$

Suppose also that the mapping $q_h \to \hat{q}_h$ is bijective and uniformly continuous and its inverse also. Then we have the theorem

<u>Theorem 4.2</u>.

The error between q and $\hat{q}_h$ is given by :

(4.10) $$\left\{ \|q - \hat{q}_h\|_{H^{-1/2}(\Gamma)} \leq c \left[\inf_{q'_h \in V_h} \left(\|q - \hat{q}'_h\|_{H^{-1/2}(\Gamma)} + \sup_{w_h \in V_h} \left(\frac{\int_{\Gamma_h} u_{0h} w_h \, d\gamma - \int_\Gamma u_0 \hat{w}_h \, d\gamma}{\|\hat{w}_h\|_{H^{-1/2}(\Gamma)}} + \frac{b(\hat{q}'_h, \hat{w}_h) - b_h(q_h, w_h)}{\|\hat{w}_h\|_{H^{-1/2}(\Gamma)}} \right] \right. \right.$$

<u>Proof</u> We have for some $\gamma > 0$

$$\gamma \| \hat{q}_h - \hat{q}'_h \|^2_{H^{-1/2}(\Gamma)} \leq \beta \| q_h - q'_h \|^2_{H^{-1/2}(\Gamma_h)} \leq b_h(q_h - q'_h, q_h - q'_h)$$

$$= b_h(q'_h, q'_h - q_h) - b(\tilde{q}'_h, \hat{q}_h' - \hat{q}_h) + b(q, \hat{q}_h - \hat{q}_h')$$

$$-b(\tilde{q}_h, \tilde{q}_h - \tilde{q}_h') + \int_{\Gamma_h} u_{0h}(q_h - q'_h)\, d\gamma_h - \langle u_0, \hat{q}_h - \hat{q}_h' \rangle$$

We can then deduced our result by taking the infimum for each $q'_h \in V_h$ and using the triangular inequality.

■

<u>How do you build the space V_h and the surface Γ_h</u> ?

You have first to precise how the surface Γ is defined. It is defined by the giving of a finite number, call it p, of mappings, we shall call them ϕ_i.

The mapping ϕ_i maps an open domain $\Theta_i \subset R^2$ into R^3. It has the property of being bijective from Θ_i onto the image $\phi_i(\Theta_i)$. The inverse will be denoted by ϕ_i^{-1}. Locally the tangent vectors

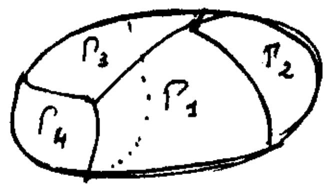
$$e_1 = \frac{\partial \phi_i}{\partial \xi_1} \qquad e_2 = \frac{\partial \phi_i}{\partial \xi_2}$$

are independant that mean that the tangential plan exist.

Functions ϕ_i are supposed to be "regular" and the mappings $\phi_i^{-1} \circ \phi_j$ are regular and bijective.

We suppose now that we choose in each domain $\mathcal{D}_i$ a subdomain D_i which is a polyhedra. Call

$$\Gamma_i = \phi_i(D_i)$$



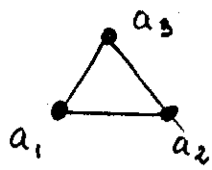
We suppose that

$$\Gamma = \bigcup_{i=1}^{p} \phi_i(D_i)$$

and $\Gamma_i \cap \Gamma_j$ are curves on Γ.

We can now "triangulate" each domain D_i. And suppose we had been given a finite element on each triangle. We can defined the "interpolate" ϕ_{ih} of the mapping ϕ_i. And we defined the surface Γ_h as the surface which mappings are the ϕ_{ih}. Take examples to make it clear.

<u>example 1</u> You choose the $\mathbb{P}_1$. element with nodes on the vertex. Suppose that triangle belongs to D_i.



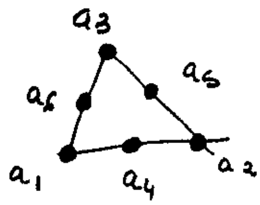
Then by property (γ) there is a unique mapping ϕ_{ih_T} such that each of its coordonates are affine in the variables (ξ_1, ξ_2) and such that

$$\phi_{ih|_T}(a_j) = \phi_i(a_j) \qquad j = 1, 2, 3.$$

If we suppose that the different triangulation on each D_i, call them $\mathcal{T}_{h_i}$, are compatible in the sense that the vertex on the image $\phi_i(D_i) \cap \phi_j(D_j)$ are the same, then the mapping ϕ_{oh} defines a closed surface.

The image of each triangle is also a triangle. We can see that the image of the nodes are on Γ. So that Γ_h is a polyhedra with vertex on Γ and which faces are triangles.

<u>example 2</u> The element is the P_2 element



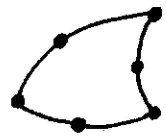
$\phi_{ih|T}$ will be here the unique mapping which coordinates are polynomials of degree 2 in the variables (ξ_1, ξ_2) and such that

$$\phi_{ih|T}(a_j) = \phi_i(a_j) \qquad j = 1, 2, \dots, 6 .$$

We can check also that such a ϕ_{oh} is continuous on D_i.

If the same compatibility condition is verified by the triangulation $\mathcal{T}_{h_i}$, then the mappings ϕ_{ih} defines us a closed surface. The image of each triangle is a curved surface. The nodes of this surface (the image of the a_j

each image of a triangle look like a curve triangular



<u>remark</u> If Γ is a polyhedra then Γ_h is in fact Γ, if we have choose the nodes carefully.

<u>How do you build the space V_h</u> ?

We remark that we have no need of continuity to have a function in $H^{-1/2}(\Gamma_h)$. So that our first choice will be

<u>example 1</u>

$$V_h = \{ q_h \text{ defined on } \Gamma_h, \quad q_h \circ \Phi_{ih}^{-1}|_T \in \mathbb{P}_0 \}$$

The load <u>is constant on each</u> triangle of Γ_h.

<u>example 2</u>

We can choose a more sophisticate approximation

$$V_h = \{ q_h \text{ defined on } \Gamma_h, \; q_h \in C^0(\Gamma_h),$$
$$q_h \circ \Phi_{ih}^{-1}|_T \in \mathbb{P}_1 \}$$

The element in V_h are functions on Γ_h <u>continuous</u> and having linear variations on each triangle (but in the variables ξ_1, ξ_2)

The structure of the linear system to solve.

We consider here the simplest case where Γ_h is defined using P_1 polynomials and the space V_h is the space of constant loads on each triangle.

If there is N triangles, V_h is of dimension N.

The unknowns are the q_i, $i = 1 \dots N$, q_i is the load of T_i. The system has the form

(4.11) $$\sum_{j=1}^{N} b_{ij}\, q_j = u_i \qquad i = 1, \ldots, N.$$

(4.12) $$b_{ij} = \frac{1}{4\pi} \int_{T_i} \int_{T_j} \frac{d\gamma(x)\, d\gamma(y)}{|x-y|}$$

(4.13) $$u_i = \int_{T_i} u_{0h}(x)\, d\gamma(x)$$

So the b_{ij} are a kind of coefficient of self-influence of the triangle T_i on T_j.

We know also by

$$b_h(q_h, q_h) \geq \beta \, \|q_h\|^2_{H^{-1/2}(\Gamma_h)}$$

that the matrix is symmetric and definite positive.

We shall prove later that its conditioning number is as $\frac{1}{h}$ where h is the maximum diameter of the triangles. Also the matrix has positive coefficients (non zero)

The computation of the coefficients b_{ij} is not so easy because of the singularity of the integrand. But it appears that

* when T_i and T_j are far enough we can use any numerical formula

* when T_i and T_j are adjacent or when $T_i = T_j$

the function

$$\varphi(y) = \frac{1}{4\pi} \int_{T_j} \frac{d\gamma(x)}{|x-y|}$$

can be computed analytically.

The second integration can be made numerically because $\varphi(y)$ is a quite regular function of y.

In the case of example 2 the structure of the matrix is the same. But the coefficients are more complex to compute.

comparison with finite element method.

* The first remark is that the number of unknowns is not of the same scale.

* We solve the exterior problem without using an artificial boundary.

* We can prove also that the conditionning number for the the system obtain by finite element is $\frac{1}{h^2}$, which is worse.

the matrix of the system is very difficult to compute and take a lot of time of computation in the integral method.
Also the matrix is <u>full</u> and requires a lot of memories.

* the geometry of triangulation is of the same order of difficulty, except if the mappings defining Γ are quite simple (which is often the case)