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NUMERICAL METHOD FOR SOLVING THE PROBLEM OF ELASTO-PLASTIC TORSION OF CYLINDERICAL BARS

Ref: H. Lanchon, SMR/23/18, 22 and 25

Translated from French by M.M. Gupta

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Numerical method for solving the problem of Elasto-plastic torsion of Cylinderical bars.

Refer: H. Lanchon <u>SMR/23/ 18, 22 and 25.</u>

translated from French by M. M. Gupta.

The problem of a homogeneous, elasto-plastic cylinderical bar is as follows [see page 18 of SMR/23/22]:

Find θ_α such that

$$\text{Pb.}\ (B_1)\ \begin{cases} \theta_\alpha \in K ; \quad J_\alpha(\theta_\alpha) \le J_\alpha(\phi) \qquad \forall\ \phi \in K ; \\ \sigma_{ij} = 0 \quad \text{except} \quad \sigma_{13}^{\alpha} = \dfrac{\partial \theta_\alpha}{\partial x_2} = \sigma_{31}^{\alpha} \ ; \ \sigma_{23}^{\alpha} = -\dfrac{\partial \theta_\alpha}{\partial x_2} = \sigma_{23}^{\alpha} , \end{cases}$$

where

1) $$J_\alpha(\phi) = \int_\Omega |\text{grad}\,\phi|^2 dx - 4\mu\alpha \int_\Omega \phi\, dx \quad , \quad \phi \in k$$

2) $$k = \left\{ \phi \,/\, \phi \in H_0^1(\Omega),\ |\text{grad}\,\phi| \le 1 \text{ a.e. on } \Omega \text{ and } \text{grad}\,\phi = 0 \text{ on } \Omega_\ell ,\ \ell = 1, 2, \cdots, q \right\} .$$

Ω is a bounded domain in R^2 with boundary $\partial\Omega$; Ω_ℓ are subdomains of Ω, entirely contained inside Ω such that
$$\bar{\Omega}_i \cap \bar{\Omega}_j = \phi \quad \text{for } i \ne j .$$

For numerical solution, we cover the region Ω with a square mesh of size h ($h > 0$). Denote the node points as follows:

$$M_{ij} = (x_i, y_j) = (ih, jh)$$

$$M_{i+\frac{1}{2}, j+\frac{1}{2}} = (x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = \left((i+\tfrac{1}{2})h, (j+\tfrac{1}{2})h\right)$$

The points M_{ij} are marked on the figure by circles "o" and the ... marked by crosses "x".

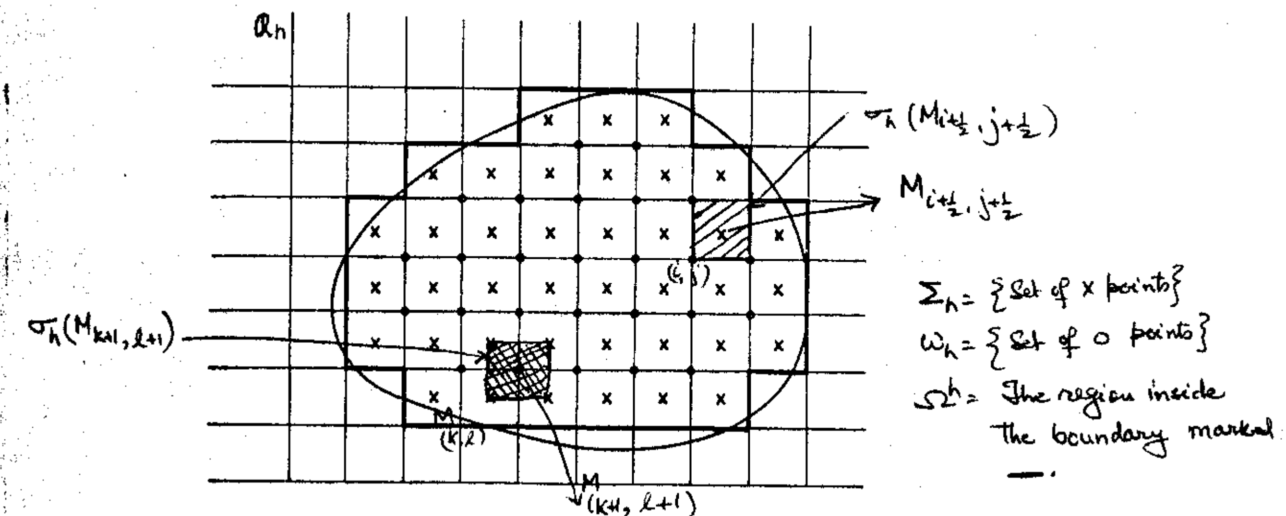

Fig. 4. — Discrétisation du problème.

Let

R_h = Set of all mesh points in R^2 i.e. $\{M_{ij} = (ih, jh) \,/\, i,j \in Z\}$

Σ_h = Set of crosses "x" lying inside $\bar\Omega$ $\{M_{i+\frac12, j+\frac12} \in \bar\Omega\}$.

ω_h = Set of circles "o" lying inside $\bar\Omega$ $\{M_{i,j} \in \Omega\}$.

$\sigma_h(M_{ij})$ = The square cell, with centre at M_{ij}, formed by the four "cross" points $M_{i-\frac12, j-\frac12}$, $M_{i+\frac12, j-\frac12}$, $M_{i+\frac12, j+\frac12}$ and $M_{i-\frac12, j+\frac12}$.

$\sigma_h(M_{i+\frac12, j+\frac12})$ = The square cell, with centre at $M_{i+\frac12, j+\frac12}$, formed by the four "circle" points M_{ij}, $M_{i,j+1}$, $M_{i+1,j+1}$ and $M_{i+1,j}$.

$$\Omega^h = \bigcup_{M_{i+\frac12, j+\frac12} \in \Sigma_h} \overline{\sigma_h(M_{i+\frac12, j+\frac12})}$$

Note: Ω^h is the polygonal region enclosed by the thick lines in the above figure.

The region Ω^h is considered to be an approximation of Ω. The approximate solution Φ_h is to be computed on Ω^h.

Let V_h = Set of all real-valued functions Φ_h defined on the node points "o" of $\omega_h = \{\Phi_h \,/\, (\Phi_h)_{M_{ij}} = \Phi_{ij} \in R\}$.

The dimension of V_h = ...

The functional $J_\alpha(\phi)$ is approximated by

$$(3)\quad J_{\alpha h}(\phi_h)=\frac{h^2}{2}\sum_{M_{i+\frac12,j+\frac12}\in\Sigma_h}\left[\left(\frac{\phi_{i+1,j}-\phi_{ij}}{h}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i,j+1}}{h}\right)^2+\left(\frac{\phi_{i,j+1}-\phi_{i,j}}{h}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i+1,j}}{h}\right)^2\right]-4\mu\alpha\sum_{M_{ij}\in\omega_h}\phi_{ij}\quad \text{for all } \phi_h\in V_h;$$

with $\phi_{pq}=0$ if $M_{pq}\notin\omega_h$.

The above expression is obtained by discretizing & integrating $|\mathrm{grad}\,\phi|^2$ along the four sides of the square cell $\sigma_h(M_{i+\frac12,j+\frac12})$.

The set K is approximated as follows:

$$(4)\quad K_h=\Big\{\phi_h \,/\, \phi_h\in V_h,\ \frac12\left[\left(\frac{\phi_{i+1,j}-\phi_{ij}}{h}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i,j+1}}{h}\right)^2+\left(\frac{\phi_{ij+1}-\phi_{ij}}{h}\right)^2+\left(\frac{\phi_{i+1,j+1}-\phi_{i+1,j}}{h}\right)^2\right]\le 1\quad \forall\, M_{i+\frac12,j+\frac12}\in\Sigma_h;$$
$$\phi_{ij}=C_\ell\quad \forall\, M_{ij}\in\bar\Omega_\ell\cap\omega_h\Big\}.$$

Since, $\mathrm{grad}\,\phi=0$ in Ω_ℓ, we introduce constants C_ℓ such that $\phi=C_\ell$ inside Ω_ℓ.

The problem now is to minimize $J_{\alpha h}(\phi_h)$ subject to the condition (4).

The expression in square brackets in equation (3) is of the following form:

$$\frac{2}{h^2}\Big[\phi_{ij}^2+\phi_{i+1,j}^2+\phi_{i+1,j+1}^2+\phi_{i,j+1}^2-\phi_{ij}\,\phi_{i+1,j}-\phi_{i+1,j}\,\phi_{i+1,j+1}$$
$$+\quad+\quad+\quad+\ \Big]$$

When summed over all cells of Ω^h, this expression is equivalent to the quadratic form

$$(Av, v),$$

where A is a symmetric, positive definite matrix and v is the vector of unknowns $\{\Phi_{ij}\}$. The dimension of A = Number of "o" ~~elements~~ points in ω_h; $(\cdot,\cdot)$ represents the standard scalar product in R^N.

Thus the problem of minimizing (3) on the set (4) is equivalent to the following problem:

$$(5)\quad \begin{cases} \underset{v \in K}{\text{Minimize}} \left[(Av, v) - 2(b, v)\right], \\ \text{where } K = \{v \mid v \in R^N, \ g_j(v) \le \gamma_j, \ 1 \le j \le M\}. \end{cases}$$

The function $g_j(v)$, $1 \le j \le M$, is a positive semi-definite quadratic form on R^N. The problem (5) has a unique solution.

In order to solve (5), we form a Lagrangian

$$(6)\quad \mathcal{L}(v, \mu) = (Av, v) - 2(b, v) + \sum_{j=1}^{M} \mu_j\left(g_j(v) - \gamma_j\right).$$

This Lagrangian must be minimized with respect to v.

There exists $\lambda \in R^M$, $\lambda_j \ge 0$, $1 \le j \le M$ such that

$$(7)\quad \begin{cases} \dfrac{\partial \mathcal{L}}{\partial v_i}(u, \lambda) = 0, & i = 1, 2, \dots, N; \\ \lambda_j\left(g_j(u) - \lambda_j\right) = 0, & j = 1, 2, \dots, M \end{cases}$$

Such vectors λ are not necessarily unique. The existence of such λ is shown by [illegible] 1987

The problem (7) is solved by the following algorithm:

(8a) Start with arbitrary $\lambda = \lambda^0 \in R^M_+$ (i.e. $\lambda^0_j \geq 0$, $1 \leq j \leq M$).

Assuming $\lambda^n \in R^M_+$ is known, determine successive $u^n \in R^N$ and $\lambda^{n+1} \in R^M_+$ by solving

$$\frac{\partial \mathcal{L}}{\partial v_i}(u^n, \lambda^n) = 0 \quad , \quad 1 \leq i \leq N \; ; \tag{8b}$$

$$\lambda^{n+1}_j = \max\left(0, \lambda^n_j + \rho\left(g_j(u^n) - r_j\right)\right) \quad , \quad 1 \leq j \leq M \; ; \; \rho > 0 \tag{8c}$$

The equation (8b) represents a system of linear equations with a symmetric, positive-definite matrix when $\lambda^n \in R^M_+$. Given λ^n, the vector u^n is determined uniquely from (8b).

Glowinski [Rendi Conti de l'Université de Rome, 1971] has shown that, for sufficiently small ρ ($\rho > 0$), the solution u^n of (8a, b, c) converges to the solution u of the minimization problem (5).

<u>Note</u>. The problem of simply-connected domain is as follows:

B^*_1
$$\begin{cases} \text{Minimize } J_\alpha(\phi) \;, \; \phi \in K^* \\ \text{where } J_\alpha(\phi) \text{ is defined in } (1), \\ K^* = \{\phi \mid \phi \in H^1_0(\Omega), \; \operatorname{grad}\phi = 0 \text{ on } \Omega_\ell \;, 1 \leq \ell \leq q \; ; \\ \qquad |\phi(x)| \leq \theta^*_\infty(x) \text{ a.e. on } \Omega\}, \end{cases}$$

The set K^* maybe approximated by

(9) $$K^*_\infty = \{\phi_h / \phi_h \in V_h ; \ |\phi_{ij}| \le \Theta^*_\infty(M_{ij}) \quad \forall M_{ij} \in \omega_h ; \quad \phi_{ij} = C_\ell \quad \forall M_{ij} \in \bar{\Omega}_\ell \cap \omega_h \}.$$

This problem is of the type :

(10) $$\begin{cases} \underset{v \in K'}{\text{Minimize}} \left[(Av, v) - 2(b, v) \right] ; \\ K' = \{ v / v \in R^N , \ a_i \le v_i \le b_i \ \text{for } i = 1, 2, \dots, N \} \end{cases}$$

This problem has a unique solution u'.

This problem is easier to solve as the matrix A is ~~symmetric~~ positive definite with positive diagonal elements $(a_{ii} > 0)$. The solution u' is obtained by the method of successive over-relaxation with projection :

$$A = [a_{ij}], \ i,j = 1, \dots, N ; \quad b = (b_1, \dots, b_N)$$

(11a) Start with $u^0 \in K'$, arbitrary vector.

Assuming u^n is known, the elements u_i^{n+1} of u^{n+1} are obtained by

(11b) $$\bar{u}_i^{n+1} = \frac{1}{a_{ii}} \Big(b_i - \sum_{j<i} a_{ij} u_j^{n+1} - \sum_{j>i} a_{ij} u_j^n \Big),$$

$$u_i^{n+1} = \max\big(a_i , \min(b_i , u_i^n + \omega(\bar{u}_i^{n+1} - u_i^n)) \big)$$

The results of these computations were given in the Lanchon lecture notes SMR/23/25.

The computations were performed on IBM 360/91.

In the case of the third model (page 34 of SMR/23/25) of a circular section with two circular cavities, a mesh size of $h = \frac{1}{40}$ was chosen and the computation time was of the order of half a minute and depended upon the torsion angle α.

References.

R. Glowinski & H. Lanchon, Torsion elastoplatique d'une barre cylindrique de section multiconnexe, J. Mécanique, Vol. 12 (1973), pp. 152-171.

J.F. Bourgat, Thesis, Univ. Paris VI, 1971

M. Goursat, Thesis, Univ. Paris VI, 1971

F. Tomasset, Report I.R.I.A.

Translated & expanded from Glowinski & Lanchon by

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