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TO MECHANICS

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BOUNDARY VALUE PROBLEMS FOR PERIODIC HETEROGENEOUS MEDIA AND APPLICATIONS TO COMPOSITE MATERIALS

"Compléments"

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Boundary value problems for periodic heterogeous medium. Complements

The first part of the lecture notes concerning this subject beeing the exact copy of a document relatif to an Euromech. are rather · disciplines. This second part which seems to repeat · some · pages of the preceding one, gives the details of the proofs.

Compliment 1 : Precisions about the introduction of the representative cell P

Taking the same notations than in p. 3, it is clear that a point M of the ~~ba~~ composite body is characterize by

i) its position $\underline{x}$ in Ω.

ii) its position in the cell $\mathcal{C}_{\underline{h}}$ to wich it belows

[$\underline{h} = h_1, h_2 \dots h_N$; this index · indicates that this cell occupies the number of order h_i in the x_i direction).

This last position beeing itself characterize by this of the corresponding $\underline{y}$ in the representative cell P.

So, let recapitulate changing slightly the notations.

- The Physical coefficients are periodic, of periode ε_i in the x_i direction.

$$c(\underline{x} + \underline{h}\,\varepsilon) = c(\underline{x})$$

$$M = \max_{i=1..N} \varepsilon_i \quad ; \quad y'_i = \frac{x_i}{\varepsilon} \quad ; \quad p_i = \frac{\varepsilon_i}{\varepsilon}$$

So, the homothetie $H_\varepsilon = (0, \frac{1}{\varepsilon})$ associates at

$$\underline{x} \in \mathcal{C}_{\underline{h}} \quad \text{the element} \quad \underline{y}' \in P'_{\underline{h}} = H_\varepsilon(\mathcal{C}_{\underline{h}})$$

Then we carry $P'_{\underline{h}}$ a the origine by the translation.

$$T_{\underline{h}} = -(\underline{h} - 1)\,\underline{p} = [-(h_1-1)p_1, \dots\dots, -(h_N-1)p_N].$$

So this translation associates at

$$\underline{y}' \in P'_{\underline{h}} \quad \text{the point} \quad \underline{y} = \underline{y}' - \underline{(h-1)\,p} \in P.$$

and now $c(\underline{x}) = c(\varepsilon \underline{y}') = a(y') = a(y)$

Complement 2 : Details about the multiple scale method

a) Formal working.

If we consider one particle of the periodic heterogeneous body, two things will be important to characterize it:

- its position $\underline{x}$ in Ω.
- its position in the basic cell to wich it belongs; this one can be represented by the corresponding $\underline{y}$ in P

In the other hand, we are interested in the eventual limit of u_ε, solution of (3)(4)(5), when ε goes to 0 keeping P fixed.

The two preceding remarks leads us to look for u_ε under the form of an asymptotic expansion of the type:

$$(30) \qquad u_\varepsilon(\underline{x}) = w_0(\underline{x},\underline{y}) + \varepsilon\, w_1(\underline{x},\underline{y}) + \varepsilon^2 w_2(\underline{x},\underline{y}) + o(\varepsilon^2)$$

where $y_i = \dfrac{x_i}{\varepsilon} + \text{constant}$; $\underline{x},\underline{y} \in \Omega \times P$:

The w_i being "P. periodicals" in $\underline{y}$

If we set generally $\varphi(\underline{x},\underline{y}) = \varphi^*(\underline{x})$, we have

$$\frac{\partial \varphi^*}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial \varphi}{\partial y_i} \quad \text{and then}$$

$$A^\varepsilon u_\varepsilon = -\frac{\partial}{\partial x_i}\left\{ a_{ij}(y)\frac{\partial}{\partial x_j}\left(w_0^*(\underline{x}) + \varepsilon\, w_1^*(\underline{x}) + \varepsilon^2 w_2^*(\underline{x}) + o(\varepsilon^2)\right)\right\} = f(\underline{x})$$

can be written

$$(31) \qquad \left(\varepsilon^{-2}A_1 + \varepsilon^{-1}A_2 + A_3\right)\left(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots\right) = f.$$

with

$$A_1\varphi = -a_{ij}\frac{\partial^2\varphi}{\partial y_i \partial y_j} - \frac{\partial a_{ij}}{\partial y_i}\frac{\partial \varphi}{\partial y_j} = -\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial\varphi}{\partial y_j}\right)$$

$$A_2\varphi = -a_{ij}\left[\frac{\partial^2\varphi}{\partial x_i \partial y_j} + \frac{\partial^2\varphi}{\partial x_j \partial y_i}\right] - \frac{\partial a_{ij}}{\partial y_i}\cdot\frac{\partial\varphi}{\partial x_j}$$

$$= -\frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial\varphi}{\partial x_j}\right] - a_{ij}\frac{\partial^2\varphi}{\partial x_i \partial y_j}.$$

$A_3\varphi^* = \ldots \; \partial^2\varphi$

By identification in (31) for the different orders on ε, we obtain. ccessively

(32) ε^{-2}: $\quad A_1 w_0 = 0$

(33) ε^{-1}: $\quad A_1 w_1 + A_2 w_0 = 0$.

(34) ε^{0}: $\quad A_1 w_2 + A_2 w_1 + A_3 w_0 = f$.

Let explicit (32)

$$-\frac{\partial}{\partial y_i}\left[a_{ij}(y)\frac{\partial w_0}{\partial y_j}\right] = 0.$$

if we multiply by w_0 and integrate by parts over P, we obtain. hanks to the "P-periodicatility" of a_{ij} and w_0 in y: (*)

$$0 = \int_P \frac{\partial}{\partial y_i}\left[a_{ij}\frac{\partial w_0}{\partial y_j}\right] w_0\, dy = -\int_P a_{ij}\frac{\partial w_0}{\partial y_j}\frac{\partial w_0}{\partial y_i}\, dy \leq -\beta \int_P |\mathrm{grad}\, w_0|^2 dy.$$

that implies.

$$\frac{\partial w_0}{\partial y_i} = 0 \quad \text{almost everywhere in } P.$$

and then

$$(35) \qquad \boxed{w_0(\underline{x}, y) = u(\underline{x})}$$

u is still unknown.

Let explicit (33)

$$(36) \qquad -\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial w_1}{\partial y_j}\right) = -A_2 w_0^{*} = \frac{\partial a_{ij}}{\partial y_i}\frac{\partial u}{\partial x_j}.$$

Let introduce the space

$W = \{\varphi \mid \varphi \in H^1(P),\ P\text{-periodicable}\}$.

$\dot{W} = \dfrac{W}{\mathcal{R}}$ where $\mathcal{R}$ is the equivalence relation defined by.

$$\varphi\, \mathcal{R}\, \psi \iff \varphi - \psi = \text{constant}$$

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fact $\displaystyle\int_{\partial P} a_{ij}\frac{\partial w_0}{\partial y_j} w_0 n_i\, ds = 0$. because a_{ij}, w_0, $\dfrac{\partial w_0}{\partial y_i}$ take.

then $\left(\int_P |\operatorname{grad} \varphi|^2 dy.\right)^{1/2}$ can be choosen as a norm on $\dot{W}$

$\frac{\partial u}{\partial x_j}$ depending exclusively of $\underline{x}$ we can introduce $\underline{\chi}$ by.

$$w_1(\underline{x}, \underline{y}) = -\chi^j(\underline{y}) \frac{\partial u}{\partial x_j}:$$

in a such a way that χ^j be "P-periodicable" and satisfies.

$$(37) \quad \begin{cases} A_1 \chi^j = -\dfrac{\partial a_{ij}}{\partial y_i} \\ \chi^j \in W \end{cases} .$$

A_1 is coercive on $\dot{W}$; in fact:

$$\int_P (A_1\varphi)\varphi\, dy = \int_P a_{ij}\, \varphi_{,j}\, \varphi_{,i}\, dy \geq \beta \|\varphi\|^2_{\dot{W}}$$

then we can apply the Lax-Milgram theorem and conclude. at the existence of χ^j at a constante (that is to say at a function of $\underline{x}$) in W. Hence the form of $W_1(\underline{x}, \underline{y})$

$$(38) \quad \boxed{W_1(\underline{x}, \underline{y}) = -\left(\chi^j(\underline{y}) \frac{\partial u}{\partial x_j} + \tilde{w}_1(x)\right)}$$

At this point, we have still to determine u and $\tilde{w}_1$

Let explicite (34)

$$A_1 w_2 = f - A_2 w_1 - A_3 w_0$$

is an equation of the type.

$$\begin{cases} A_1 \phi = F \\ \quad \phi \text{ "P-periodicable" in } y. \end{cases}$$

or in its weak formulation.

$$(39) \quad a_P(\phi, \psi) = (F, \psi) \qquad \forall\, \psi \in \dot{W}$$

with by definition:

which is a Bilinear· coercive on $\dot{W}$.

Then (39) admits one unic solution on $\dot{W}$ if and only if

$$(41) \quad \int_P F(y)\, dy = 0 .$$

because the set of constant (on function of x) is the neutral element of $\dot{W}$

(41) is exactly $\int_P [f(x) - A_2 w_1(x,y) - A_3 u(x)]\, dy = 0$.

that is to say. in denoting by $V(P)$ the volume of P.

$$V(P) f = \int_P (A_2 w_1 + A_3 u)\, dy .$$

Let remark again that.

$$\int_P A_2 \varphi\, dy = -\int_P a_{ij} \cdot \frac{\partial^2 \varphi}{\partial x_i \partial y_j}\, dy$$

$\forall\ \varphi(x, y)$ "P. periodicalle" in y.

Then, taking account of (35) and (38) we obtain.

$$V(P) f = \int_P \left\{ a_{ij} \frac{\partial \chi^k}{\partial y_j} \frac{\partial^2 u}{\partial x_k \partial x_i} - a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\} dy .$$

$$= \int_P a_{ij} \left(\frac{\partial \chi^k}{\partial y_j} - \delta_{kj} \right) \frac{\partial^2 u}{\partial x_k \partial x_i}\, dy .$$

from Whence.

$$f = - q_{ik} \frac{\partial^2 u}{\partial x_k \partial x_i} \quad \text{if we define } q_{ik} \text{ by}.$$

$$(42) \quad \boxed{q_{ik} = \frac{1}{V(P)} \int_P a_{ij} \frac{\partial}{\partial y_j} (y_k - \chi^k)\, dy.}$$

in conclusion, the q_{ij} being determined by (37) and (42), it is

$$(43) \quad - q_{ij} \frac{\partial^2 u}{\partial x_h \partial x_i} = f \quad \text{on } \Omega \ ; \ u \in H_0^1(\Omega)$$

because for $\varepsilon = 0$, u has to satisfy the same boundary conditions as u_ε

<u>Remark 9</u> : It is time now to realize what we got:

By the preceding formal workings, we obtained:

$$u_\varepsilon(\underline{x}) = u(\underline{x}) - \varepsilon \left[\chi^j(\underline{y}) \frac{\partial u}{\partial x_j} + \widetilde{w}_1(\underline{x})\right] + \varepsilon^2 w_2(\underline{x}, \underline{y}) + o(\varepsilon^2)$$

where χ^j and u are determined by the successive solutions of (37) (42) and (43), $\widetilde{w}_1(\underline{x})$ and $w_2(\underline{x}, \underline{y})$ by the following identifications of (31)

We can then expect for a first step, that u is a good approximation of u_ε when ε is small and wonder if that is interesting on a numerical point of view as well as of a study of the global behaviour of the solution u_ε. To answer these questions, it is necessary

a) to study "existence and uniqueness" for (43).
b) to study the converging of u_ε toward u when $\varepsilon \to 0$.
c) to evaluate, if possible, the error $v_\varepsilon = u_\varepsilon - u$.
d) to have a look at the nature of the preliminary computing (37) and (42)

b) <u>Study of the homogenized problem (43)</u>

<u>Lemma 1</u> : $\boxed{q_{ij} = \frac{1}{V(P)} \ a_P(\chi^j - y_j, \chi^i - y_i)}$ (44)

This lemma is useful to obtain q_{ij} on a more symetric form and to be then able ~~to~~ to study more easily the coercivity of the operator

$A = - q \ \partial^2$

To obtain this form of q_{ij}, we have first to modify slighty the definition (42)_ in fact:

$$q_{ij} = \frac{1}{V(P)} \int_P a_{kl}(\underline{y}) \frac{\partial}{\partial y_l} (y_j - \chi^j)\, \delta_{ki}\, dy.$$

$$= \frac{1}{V(P)} \int_P a_{kl}(\underline{y}) \frac{\partial}{\partial y_l} (y_j - \chi^j) \frac{\partial y_i}{\partial y_l}\, dy.$$

$$= \frac{1}{V(P)}\ a_P(y_j - \chi^j, y_i).$$

In an other hand, if we multiply (37) by $\psi \in W$ and integrate by parts over P, we obtain.

$$\int_P a_{il}. \chi^j_{,l}\, \psi_{,i}\, dy = \int_P a_{ij}\, \psi_{,i}\, dy$$

$$= \int_P a_{il}\, \psi_{,i}\, \delta_{lj}.$$

(all the integrals on the boundary being zero because the properties of P. Periodicability of $\underline{a}$ and χ, ψ)

That gives us a kind of weak form for the problem (37).

$$(45) \quad \boxed{a_P(\chi^j, \psi) = a_P(y_j, \psi) \qquad \forall\ \psi \in W}$$

That implies in particular $a_P(\chi^j - y_j, \chi^i) = 0$.

hence the annonced form for the coefficients q_{ij} and also the following lemma.

<u>Lemma 2</u>. <u>The problem (43) is elliptic</u> <u>(or coercive)</u>

in fact: $$q_{ij}\, \xi_i\, \xi_j = \frac{1}{V(P)}\, a_P\left[\xi_j(\chi^j - y_j),\ \xi_i(\chi^i - y_i)\right]$$

$$= \frac{1}{V(P)} \int_P a_{lm}\, \xi_j (\chi^j - y_j)_{,m}\, \xi_i (\chi^i - y_i)_{,l}\, dy.$$

and we obtain:

$$q_{ij}\,\xi_i\,\xi_j \geqslant \frac{\alpha}{V(P)}\int_P |\operatorname{grad} v|^2\,dy \geqslant 0 \qquad \forall\ \xi \in R^N$$

Let show now that this inequality is strict. Let suppose then that for a $\underline{\xi} \neq 0$, $q_{ij}\,\xi_i\,\xi_j = 0$ that implies

$\operatorname{grad} v = 0$ almost everywhere on P. that is to say.

$v = \xi_i\,[\chi^i(\underline{y}) - y] =$ constante a. e on P:

Let $\vec{e}_i$ a new orthonormal basis such as $\vec{\xi} = \vec{e}_1$; in this new basis, the preceding equality can be written

$$\chi^1(\underline{y}) = \text{constante} + y_1 \quad \text{a. e on } P.$$

and that is impossible because χ^1 is "periodic" and y_1 is not. Whence:

$$q_{ij}\,\xi_i\,\xi_j > 0 \qquad \forall\ \xi \neq 0.$$

let then introduce

$$\beta = \inf_{|\xi|=1} q_{ij}\,\xi_i\,\xi_j$$

this inf is reach, thanks to the compacity of the sphere $|\xi| = 1$ in R^N, and therefore $\beta > 0$.. implies

$$q_{ij}\,\xi_i\,\xi_j \geqslant \beta\,|\xi|^2 \qquad \forall\ \xi \in R^N. \qquad \text{q. e. d.}$$

We can now by the Lax. Milgram theorem to the existence and uniqueness of u solution of (43)

c) <u>Study of the convergence of u_ε when $\varepsilon \to 0$ and estimate of the error</u>

It is time now to express precisely the complete result that we are going to obtain in this way

cf following page.

__Theorem 1__: We suppose that the coefficient a_{ij} are $C^\infty(\bar\Omega)$ (that is obviously superabondant, but the regularity asked by the method is anyway too great for the applications)

Let A^ε be the differential operator defined by (3) and satisfying the hypothese of convexity (4)

Let $\mathcal{A}$ be the operator defined by

$$\mathcal{A} = -q_{ij}\frac{\partial^2}{\partial x_j \partial x_i}, \quad q_{ij} \text{ defined themselves by } (44).$$

Let consider now the 2 homogeneous Dirichlet problems

$$(P_\varepsilon) : \quad A^\varepsilon u_\varepsilon = f \quad \text{in } \Omega \ ; \ u_\varepsilon = 0 \quad \text{on } \partial\Omega .$$

$$(P) : \quad \mathcal{A} u = f \quad \text{in } \Omega \ ; \ u = 0 \quad \text{on } \partial\Omega .$$

Then u_ε converges toward u, uniformly in $C^0(\Omega)$ and

$$(46) \quad |u_\varepsilon(x) - u(x)| \leq c\,\varepsilon \qquad \forall\, x \in \Omega .$$

__Proof__: let introduce

$$v_\varepsilon(\underline{x}) = u_\varepsilon(\underline{x}) - u(x) - \varepsilon\, w_1(\underline{x}, \underline{y}) - \varepsilon^2 w_2(\underline{x}, \tfrac{x}{\varepsilon})$$

$$A^\varepsilon v_\varepsilon = f + (\varepsilon^{-2}A_1 + \varepsilon^{-1}A_2 + A_3)(-u - \varepsilon w_1 - \varepsilon^2 w_2)$$

but u satisfies $A_1 u = 0$. cf (32) and (35)

and also $A_2 u + A_1 w_1 = 0$. cf (33)

$f - A_3 u - A_2 w_1 - A_1 w_2 = 0$ cf (34).

therefore

$$A^\varepsilon v_\varepsilon = -\varepsilon\,(A_3 w_1 + A_2 w_2) - \varepsilon^2 A_3 w_2 .$$

the regularity of the coefficients guarantees this of w_1 and w_2 wich are therefore bounded on Ω. whence

$$\begin{cases} A_\varepsilon v_\varepsilon = O(\varepsilon) & \text{on } \Omega \\ v_\varepsilon = O(\varepsilon) & \text{on } \partial\Omega . \end{cases}$$

it seems that the estimate (46) comes from there. ? (good

Complement 6 : Details about the energy method

a) Hypothesis - weak formulation - statement

We assume $a_{ij} \in L^{\infty}(R^N)$ as a fonction of y $\forall$ i and j.
We consider a more general problem, that is to say a mixte problem.

$$(47) \quad \begin{cases} -\dfrac{\partial}{\partial x_i}\left[a_{ij}\left(\dfrac{x}{\varepsilon}\right)\dfrac{\partial u_\varepsilon}{\partial x_j}\right] = f \text{ in } \Omega \\ u_\varepsilon = 0 \quad \text{on } \Gamma_0 \subset \partial\Omega \text{ , } \Gamma_0 \text{ being of non zero measure.} \\ \dfrac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}} = 0 \quad \text{on } \Gamma_1 \subset \partial\Omega \end{cases}$$

with $\Gamma_0 \cap \Gamma_1 = \phi$; $\Gamma_0 \cup \Gamma_1 = \partial\Omega$

$$(48) \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}} = a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j} n_i \quad \text{on } \Gamma.$$

a_{ij} being still "P. periodical" and coercif (property (4))

We consider then the classic functional space (cf L.N 23/8 p 31)

$$(49) \qquad V = \{v \,/\, v \in H^1(\Omega),\ v = 0 \quad \text{on } \Gamma_0\}.$$

We have. $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$.

Whence the weak formulation of (47)

$$(48) \quad \begin{cases} a_\varepsilon(u_\varepsilon, v) = (f, v) \qquad \forall v \in V \\ u_\varepsilon \in V \end{cases}$$

with :

$$(49) \quad a_\varepsilon(u, v) = \int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx \qquad \text{and}$$

$$(50) \qquad (f, v) = \int f(x)\, v(x)\, dx .$$

We can now establish the following statement

Theorem 2 : u_ε converges toward u in V weak, u bein solution of

$$(51)\quad \begin{cases} \mathcal{A}(u, v) = (f, v) & \forall v \in V \\ u \in V \end{cases}$$

$\mathcal{A}$ bein defined by.

$$(52)\quad \mathcal{A}(\varphi, \psi) = \int_\Omega q_{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_i}\, dx .$$

with the same constant coefficients q_{ij} as above.

b) Proof : we shall show successively

i) the existence of a subsequence, still called u_ε, such as.

$$u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u \quad \text{in } V \text{ weak}.$$

and :

$$\xi_i^\varepsilon = a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \xrightarrow[\varepsilon \to 0.]{} \xi_i \quad \text{in } L^2(\Omega) \text{ weak}.$$

ii) $\xi_i = \hat{q}_{ij} \dfrac{\partial u}{\partial x_j}$; $\hat{q}_{ij}$ being constant coefficients

iv) $\hat{q}_{ij} = q_{ij}$.

i) Existence of a subsequence weakly converging.

Let denote by $\| \ \|$ the norm in $H^1(\Omega)$ and $| \ |$ this of $L^2(\Omega)$.

$$\|v\|^2 = |v|^2 + |\operatorname{grad} v|^2 \quad \text{then}.$$

Taking account of the coercivity (4) and of the equivalence in V

$$\|v\|_0 = |\operatorname{grad} v|,$$

we obtain from (48).

$$m\beta \|u_\varepsilon\|^2 \leq \beta \|u_\varepsilon\|_0^2 \leq a_\varepsilon(u_\varepsilon, u_\varepsilon) \leq |f|\,|u_\varepsilon| \leq |f|\,\|u_\varepsilon\|$$

hence $$\|u_\varepsilon\| \leq \frac{1}{m\beta} |f|$$

and then the existence of a subsequence u_ε such that

(53) $$u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u \quad \text{in } V \text{ weak.}$$

Because a_{ij} is bounded.

(54) $$a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \xrightarrow[\varepsilon \to 0]{} \xi_i \quad \text{in } L^2(\Omega) \text{ weak.}$$

in consequence

$$a_\varepsilon(u_\varepsilon, v) = \left(\xi_i^\varepsilon, \frac{\partial v}{\partial x_i}\right) \xrightarrow[\varepsilon \to 0]{} \left(\xi_i, \frac{\partial v}{\partial x_i}\right) = (f, v) \quad \forall v \in V$$

and therefore $\forall v \in \mathcal{D}(\Omega)$ whence.

(55) $$f = -\frac{\partial \xi_i}{\partial x_i}$$

ii) identification of ξ_i

The idea of the following reasoning consists in choosing in (48) a special v related to the solution of the transposed problem in order to get rid of terms for wich the nature of the convergence in unknown.

The spaces W and W being the same as above (cf. p.37) let introduce then the transposed operator A^* of A defined by

(56) $$A = -\frac{\partial}{\partial y_i}\left[a_{ij}(y)\frac{\partial}{\partial y_j}\right]$$

Let Π be a homogeous polynome of degree 1 wich will be

Then the following problem

$$(57)\quad \begin{cases} w \in H^1(P) \;;\quad A^* w = 0 \quad \text{in } P \\ w - \Pi \in \dot{W} \end{cases}$$

has an unique solution because it is reduced to look for $\theta = w - \Pi$ such that

$$(58)\quad \begin{cases} a_P^* (\theta, \psi) = - a_P^* (\Pi, \psi) \qquad \forall\, \psi \in \dot{W} \\ \theta \in \dot{W} \end{cases}$$

with, as above for a_P ; $\quad a_P^*(\varphi,\varphi) \geq \beta \|\varphi\|^2_{\dot{W}}$

$$w(y) = \Pi(y) + \theta(y)$$

is defined almost everywhere on P and also outside of P by a periodic extension of θ. Thanks to the homogeneity of Π we have

$$w\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon}\Pi(x) + \theta\left(\frac{x}{\varepsilon}\right).$$

Let introduce now:

(59) $\quad w^\varepsilon(x) = \varepsilon\, w\left(\frac{x}{\varepsilon}\right) = \Pi(x) + \varepsilon\theta\left(\frac{x}{\varepsilon}\right)$ which is solution of

(60) $\quad (A^\varepsilon)^* w^\varepsilon = -\dfrac{\partial}{\partial x_j}\left[a_{ij}\left(\frac{x}{\varepsilon}\right)\dfrac{\partial w^\varepsilon}{\partial x_i}\right] = \dfrac{1}{\varepsilon} A^* w = 0.\quad$ with $w^\varepsilon \in H^1(\Omega)$

of wich the weak formulation is

(61) $\quad a_\varepsilon^*(w^\varepsilon, \psi) = 0 \qquad \forall\, \psi \in H_1^0(\Omega)$;

<u>Remark 9</u> : this construction, rather artificial for everyone who has not invented it, allowed us to introduce this element w_ε

identifying ξ_i :

Let then $\varphi \in \mathcal{D}(\Omega)$, we are going to choose

$v = \varphi w_\varepsilon$ in (48)

$\psi = \varphi u_\varepsilon$ in (61)

in a such a way that sountraying this two equations we obtain.

$$(62) \quad a_\varepsilon(u_\varepsilon, \varphi w_\varepsilon) - a^*_\varepsilon(w^\varepsilon, \varphi u_\varepsilon) = \int_\Omega \xi_i^\varepsilon \frac{\partial \varphi}{\partial x_i} w_\varepsilon \, dx - \int_\Omega a_{ij} \frac{\partial \varphi}{\partial x_j} u_\varepsilon \frac{\partial w_\varepsilon}{\partial x_i} dx.$$

$$= (f, \varphi w_\varepsilon) \quad \forall \varphi \in \mathcal{D}(\Omega)$$

thanks to the disappearing of the termes $\int_\Omega \varphi \, a_{ij} \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial w_\varepsilon}{\partial x_i} dx$.

We can now operate the passage to the limite in (62), takinh account of.

- The weak converging of ξ_i^ε in $L^2(\Omega)$ already mentioned.
- The strong convergence of w_ε in $L^2(\Omega)$; in fact

$$\int_\Omega (w^\varepsilon - \Pi)^2 dx = \varepsilon^2 \int_\Omega \theta^2\left(\frac{x}{\varepsilon}\right) dx \xrightarrow[\varepsilon \to 0]{} 0. \quad \text{thanks to the}$$

periodicity of θ.

- The strong convergence of u_ε toward u in $L^2(\Omega)$; in fact, $u_\varepsilon \to u$ in $H^1(\Omega)$ weak and the injection of $H_1(\Omega)$ in $L^2(\Omega)$ is compact

- The fact that $\frac{\partial w^\varepsilon}{\partial x_i} = \frac{\partial \Pi}{\partial x_i} + \frac{\partial \theta}{\partial x_i} = c_i + \frac{\partial \theta}{\partial x_i}$ is P- periodic

as well as.

$$\hat{\chi}_i\left(\frac{x}{\varepsilon}\right) = a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial w^\varepsilon}{\partial x_j}.$$

The exists a "classical" result saying that.

if g est P periodic, $g\left(\frac{x}{\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{} \mathcal{M}_P(g)$ in $L^2(\Omega)$ weak.

with.

$$(63) \quad \mathcal{M}_P(g) = \frac{1}{\ } \int g(y)\, dy$$

So at the limit, (62) becomes.

$$\int_\Omega \xi_i \frac{\partial \varphi}{\partial x_i} \pi \, dx - m_p \left(a_{ij} \frac{\partial w}{\partial y_i} \right) \int_\Omega \frac{\partial \varphi}{\partial x_j} u \, dx = (f, \varphi \pi) \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

And then, takinh account of (55) we obtain.

$$(f, \varphi\pi) = \left(- \frac{\partial \xi_i}{\partial x_i}, \varphi \pi\right) = \left(\xi_i, \frac{\partial \varphi}{\partial x_i} \pi\right) + \left(\xi_i, \varphi \frac{\partial \pi}{\partial x_i}\right)$$

$$= \left(\xi_i, \frac{\partial \varphi}{\partial x_i} \pi\right) + m_p \left(a_{ij} \frac{\partial w}{\partial y_i}\right) \int \varphi \frac{\partial u}{\partial x_j} \, dx.$$

$$\forall \varphi \in \mathcal{D}(\Omega).$$

Whence

$$(64) \qquad \xi_i \frac{\partial \pi}{\partial x_i} = m_p \left(a_{ij} \frac{\partial w}{\partial y_i} \right) \frac{\partial u}{\partial x_j}.$$

But we have still the choice of π and because we are intended by ξ_i, it is naturae to choose simply. $\pi(x) = x_e$. Les denote by w_e the corresponding solution of (57) and then.

$$(65) \qquad \xi_e = m_p \left(a_{ij} \frac{\partial w_e}{\partial y_i} \right) \frac{\partial u}{\partial x_j} = \hat{q}_{ej} \frac{\partial u}{\partial x_j} \qquad \text{q. e. d.}$$

iii) <u>Comparison of $\hat{q}_{ej}$ with q_{ej} obtained by the first method.</u>

$$\hat{q}_{ej} = \frac{1}{V(P)} \int_P a_{ij}(y) \frac{\partial w_e}{\partial y_i} \, dy = \frac{1}{V(P)} \int_P a_{ij}(\underline{y}) \frac{\partial}{\partial y_i} \left[y_e + \theta_e(y) \right] dy.$$

$$= \frac{1}{V(P)} \, a_p^* \left(\theta_e + y_e, y_j \right)$$

So : $\quad V(P)\, \hat{q}_{ej} = a^*_p(\theta_e, y_j) + a^*_p(y_e, y_j)$.

Preceedingly, we founded (cf p 6)

$$V(P)\, q_{ej} = - a_p(\chi^j, y_e) + a_p(y_j, y_e)$$

But on the one hand:

and on the other hand.

$$a^*_p(\theta_e, y_j) = a_p(y_j, \theta_e) = a_p(\chi^j, \theta_e)$$ from (45) taking $\psi = \theta_e$

$$- a_p(\chi^j, y_e) = - a_p^*(y_e, \chi^j) = a_p^*(\theta_e, \chi^j) = a_p(\chi^j, \theta_e)$$
from (58) taking $\psi = \chi^j$.

Whence the conclusion.

$$q_{ej} = \hat{q}_{ej}$$

So, taking account of (65) and (55)

$$- q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

That is exactly the equation (51) because $u \in V$.

<u>Remark 10</u> : The results of the theorems 1 and 2 are consistent since the "homogenized solution" is the same. That insures the uniqueness of the set of the coefficients q_{ij}. The strong of the convergence depending of the regularity of the a_{ij}.
