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SOME ASPECTS OF FLOWS THROUGH POROUS MEDIA

A Non-Darcy Approach

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Some Aspects of Flows Through Porous Media — A Non-Darcy Approach

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The basic law characterizing the flow of a viscous fluid in a porous medium is formulated empirically by Darcy [1], [2], [3]. It expresses that the pressure gradient pushes the fluid in the porous medium against the body force on it exerted by the fluid. Observing that this drag force is proportional to the seepage velocity ($\vec{V}$) and the viscosity coefficient (μ) of the fluid, Darcy's law can be expressed as

$$\vec{V} = -\frac{k}{\mu}\nabla p \qquad (1)$$

together with the continuity equation:

$$\varepsilon \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0 \qquad (2)$$

where p is the fluid pressure, ρ the fluid density in the medium, k the permeability and ε the porosity (i.e., the ratio of the pore-volume to the total volume) of the medium. The conditions proposed on the boundary of the medium are (i) the continuity of the pressure, (ii) the continuity of the normal mass flux and (iii) the tangential velocity just outside the medium is zero.

This law, for most practical purposes, is considered to be quite satisfactory (particularly at low speeds and is in good agreement with the results of several experiments which are, of course, modelled on one dimensional flows. However, the speed in the medium is not always small, as a consequence of which, convective forces may be developed in the fluid. Furthermore, the internal stress generated in the fluid due to its viscous nature produces distortions in the velocity field. It is also observed that when the porosity is very large, the viscous stress penetrates into the body through the pores and seems to produce a flow near the surface even ~~of~~ in the absence of pressure gradient. All these factors strongly influence the flow pattern in the medium. The pressure gradient in the medium and the fluid velocity need not be unidirectional in flows of general character. Darcy's law is thus inadequate to describe the flow, more particularly, in situations with large speeds or near surfaces which are either permeable or rigid. Hence considerations for a non-Darcy description for the viscous flows in porous media is warranted.

Bevers and Joseph [4] accepted the eq.(1) but proposed an empirical slip boundary condition at the interface ~~of~~ between the porous medium and a clear medium, the shear stress in the later is proportional to the difference in the tangential velocities in the two media. Saffman [5], employing a statistical method, derived a general governing equation for the flow in a porous medium which takes into account the viscous stress. Furthermore, he suggested a boundary condition similar to that of Joseph and Bevers (loc cit.). Taylor [6] and Richerdson [7] carried out exact analysis of the situation, assuming a different model together with Joseph - Bever condition on the interface. Another modification has been suggested by Brinkman [8]:

$$0 = -\nabla p - \frac{\mu}{k}\vec{V} + \mu\nabla^2\vec{V} \qquad (3)$$

wherein the term $\mu\nabla^2\vec{V}$ is intended to account for large distortions of the velocity profiles near the boundary of the medium. Yamamoto and Yoshida [9] examined basic field equations incorporating the convective terms. Recently Yamamoto and Iwamura [10] investigated a variety of flows in porous media employing a more general field equations taking into account the fluid inertia and the viscous stress as well. We describe ~~these~~ equations briefly hereunder.

The fluid medium is regarded as an essemblage of small identical particles fixed in space. Let σ be the radius of each of these spheres and N the volume-density number. The porosity of the medium will be $\epsilon = 1 - \frac{4}{3}\pi\sigma^3 N$ which is almost unity as in the case with many filtering materials. This is because $\sigma^3 N \to 0$ as $N \to \infty$ eventhough σN is finite. Since the viscous drag on a sphere of radius σ is $6\pi\mu\sigma\vec{V}$, the total force exerted by the assembly of the spheres on unit volume of the fluid will be $6\pi\sigma\mu\vec{V}N$ (= a finite quantity) and this will be the body force on the fluid.

If V^* is a characteristic velocity of the flow, the Reynolds number $R_\sigma = \rho\sigma V^*/\mu$ will be very small, even though the reference velocity V^* is not so. The characteristic length L of the macroscopic flow is in general much larger than σ and so the Reynolds number $R_L = \rho L V^*/\mu$ for the flow in the porous medium is not necessarily so small as to neglect the fluid inertia.

Further, if it assumed that the medium is completely saturated with the viscous liquid which, when set in motion, generates viscous stress

The fundamental equations characterizing the flow of a viscous incompressible fluid in a saturated porous medium can thus be written as

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = -\nabla p - \frac{\mu}{k}\vec{V} + \mu\nabla^2\vec{V} \qquad (4)$$

together with the continuity equation:

$$\operatorname{div}\vec{V} = 0 \qquad (5)$$

It is interesting to notice that these equations are the same as the Navier-Stokes equations ~~for~~ when the flow is influenced by a special type of body force $\left(= \frac{\mu}{k}\vec{V}\right)$.

The equations (4), in the limit as $k \to \infty$ reduce to the N-S equations of motion which are valid for a clear region (i.e., with $N = 0$). The appropriate boundary conditions can be derived by applying the conservation laws of mass and linear momentum to a control volume stradding the boundary or the interface with another medium (which we suppose in the further discussion ~~to~~ be a clear medium).

The mass conservation law yields the continuity of normal velocity

$$\vec{V}_p\cdot\hat{n} = \vec{V}_c\cdot\hat{n}\ (= V_0 \text{ say}) \qquad (6)$$

and from the momentum conservation, we get

$$(\vec{T}_p - \vec{T}_c)\cdot\hat{n} = R_L(p_p - p_c) \qquad (7)$$

and

In the above equations, $\vec{T}$ stands for the viscous stress-Vector, $\hat{n}$ the unit normal and $\hat{\tau}$ the unit tangent ~~on~~ the boundary. Also the subscripts P and C stand for the quantities in the porous and clear media respectively. Furthermore, the continuity of the tangential velocity in the pores at the surface

$$\vec{V}_p \cdot \hat{\tau} = \hat{V}_c \cdot \hat{\tau} \qquad (9)$$

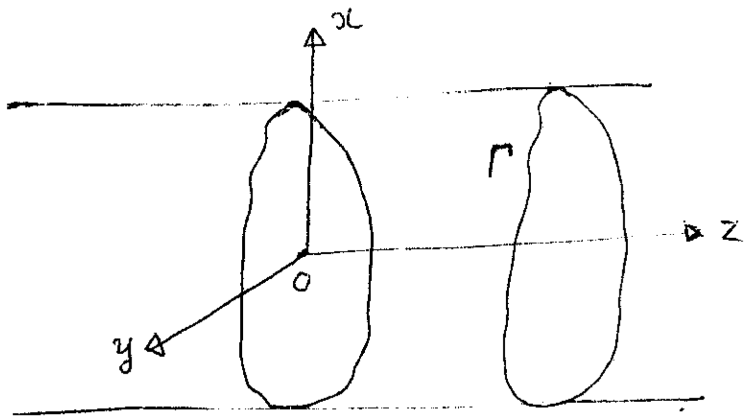
is another reasonable assumption because of the viscosity of the fluid. As a consequence of (9) ~~which~~, the condition (8) ~~can~~ reduces to

$$\vec{T}_p \cdot \hat{\tau} = \vec{T}_c \cdot \hat{\tau} \qquad (10)$$

which speaks about the continuity of the tangential component of the stress vector on the boundary. The flows of viscous fluids in a porous medium ~~are~~ satisfy the equation (4) in the medium and the conditions (6), (7), (9) and (10). Some simple flows are presented in what follows, to demonstrate the applications of the above basic equations.

A **Steady flow in a straight ~~pipe~~ pipe:**

Let us consider the steady flow of a viscous liquid in a straight porous pipe whose boundary is impermeable under the influence of a constant pressure gradient c.



Choosing a system of rectangular Cartesian coordinates $O(xyz)$ with z coordinate measured parallel to the tube-axis, the rectilinear flow through the tube may be represented by the velocity field $[0, 0, w(x,y)]$.

If Γ is a typical cross-section of the tube perpendicular to the flow-direction and Ω is the region bounded by Γ, the velocity function w satisfies the equation:

$$\nabla^2 w - \frac{1}{K} w = +\frac{1}{\mu}\frac{\partial p}{\partial z} = -\frac{c}{\mu} \quad \text{in } \Omega \qquad (11)$$

along with

$$w = 0 \quad \text{on } \Gamma. \qquad (12)$$

Analogy with a vibrating membrane:

If we make the substitution:

$$\phi = w - \frac{Kc}{\mu} \; ; \; p^2 = -\frac{1}{K}, \qquad (13.1)$$

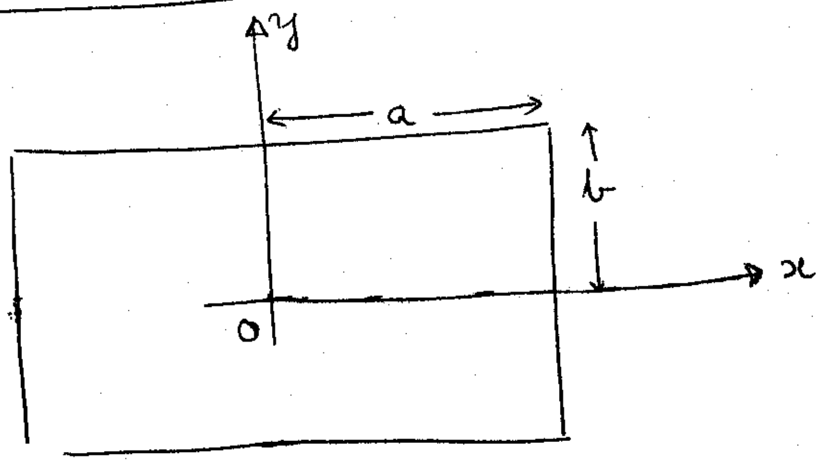
the above equations reduce to

$$\nabla^2\phi + p^2\phi = 0 \quad \text{in } \Omega \qquad (13.2)$$

This shows that the problem of steady flow under a constant pressure gradient in a straight porous pipe with an impermeable boundary is analogous to that of a vibrating membrane whose edge vibrates harmonically. The later problem has been studied extensively by Seth [11]

Explicit closed form solutions are given here-under for some commonly occuring cross-sections (Γ) of the tube.

1. Rectangular cross-section: $|x| = a$, $|y| = b$



The solution in this case is given by

$$w = \sum_{m,n=0}^{\infty} \beta_{mn} \cos \frac{\lambda_m x}{a} \cdot \cos \frac{\lambda_n y}{b} \qquad (14.1)$$

where

$$\lambda_m = \frac{(2m+1)\pi}{2} \qquad (14.2)$$

and

$$\beta_{mn} = \frac{4C}{\mu} (-1)^{m+n} \frac{1}{\lambda_m \lambda_n \left\{ \frac{\lambda_m^2}{a^2} + \frac{\lambda_n^2}{b^2} + \frac{1}{k} \right\}} \qquad (14.3)$$

It is quite interesting to recover from this solution, the flow between parallel plates $y=\pm h$ when $a \to \infty$, retaining b (= a constant). However, the flow field can be independently determined from the equation

$$\frac{d^2 w}{dy^2} - \frac{w}{K} = -\frac{c}{\mu} \qquad (15.1)$$

with

$$w\,(y=\pm h) = 0 \qquad (15.2)$$

These equations yield the velocity:

$$w = \frac{Kc}{\mu}\left[1 - \frac{\cosh y/\sqrt{K}}{\cosh h/\sqrt{K}}\right] \qquad (15.3)$$

from which the average flow rate can be obtained as

$$\bar{w} = \frac{cK}{\mu}\left[1 - \frac{\sqrt{K}}{h}\tanh\frac{h}{\sqrt{K}}\right] \qquad (15.4)$$

This, in the limit as $K \to \infty$, reduces to $\frac{ch^2}{3\mu}$ a result which is true for the flow in a clear medium.

2. Equilateral triangular cross-section:

When the boundary Γ is

$$y = a, \quad y = \pm\sqrt{3}x, \quad \dots \qquad (16.1)$$

the velocity field is given by

$$w = -\frac{cK}{\mu}\,\mathrm{Re.}\left[-1 + 2\cos\frac{py}{2}\cos\frac{\sqrt{3}px}{2} - \cos py - \cot\frac{pa}{2}\left(2\sin\frac{py}{2}\cos\frac{\sqrt{3}px}{2} - \sin py\right)\right] \qquad (16.2)$$

where

3. Circular Cross-Section:

Referring to a system of cylindrical coordinates (r, θ, z), the axis of z of which coincides with the tube-axis, the steady state velocity field $[0, 0, w(r)]$ down the tube satisfies

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - \frac{1}{K} w = -\frac{c}{\mu} \tag{17.1}$$

and

$$w(r = a) = 0 \tag{17.2}$$

where r is the tube radius. The solution of these equations can be seen to be

$$w = \frac{cK}{\mu}\left[1 - \frac{I_0(r/\sqrt{K})}{I_0(a/\sqrt{K})}\right] \tag{17.3}$$

where $I_0(\ )$ represents modified Bessel function of first kind.

For large values of K,

$$I_0(r/\sqrt{K}) \sim 1 + \frac{r^2}{4K} \tag{17.4}$$

and hence we have

$$w \sim \frac{c}{4\mu} \cdot \frac{1}{1 + \frac{a^2}{4K}} (a^2 - r^2) \tag{17.5}$$

The velocity profile in the porous medium with a large porosity coefficient (K) is the same as that in the clear medium with a pressure gradient reduced in the ratio $1 : \left(1 + \frac{a^2}{4K}\right)$.

For small values of K,

$$I_0(r/\sqrt{K}) \sim \frac{1}{(2\pi r/\sqrt{K})^{1/2}} \exp(r/\sqrt{K}) \tag{17.6}$$

and so (17.5) reduces to

$$- (a - r)/\sqrt{K} \ ] \tag{17.7}$$

the second term of which damps out more rapidly as K decreases faster. The non-Darcy effect is thus more predominent in a thin layer of thickness δ:

$$\delta = O(\sqrt{K}) \quad \ldots \quad (17.8)$$

around the tube-wall and the classical Darcy-flow is realized only in a core close to the axis of the tube. This phenomenon is similar to that noticed by Sexl for the pulsating viscous flow in a similar tube [12].

4. Annular cross-section:

If Γ is composed of two concentric circles of radii a and b, the solution of the eq. (17.1) can be obtained as

$$w(r) = -\frac{cK}{\mu} \cdot \frac{\left[T_0\left(\frac{1}{\sqrt{K}}; a, r\right) + T_0\left(\frac{1}{\sqrt{K}}; r, b\right) + T_0\left(\frac{1}{\sqrt{K}}; b, a\right)\right]}{T_0\left(\frac{1}{\sqrt{K}}; a, b\right)} \quad \ldots \ldots \quad (18.1)$$

where

$$T_0\left(\frac{1}{\sqrt{K}}; x, y\right) = I_0\left(\frac{x}{\sqrt{K}}\right) K_0\left(\frac{y}{\sqrt{K}}\right) - K_0\left(\frac{x}{\sqrt{K}}\right) I_0\left(\frac{y}{\sqrt{K}}\right) \ldots (18.2)$$

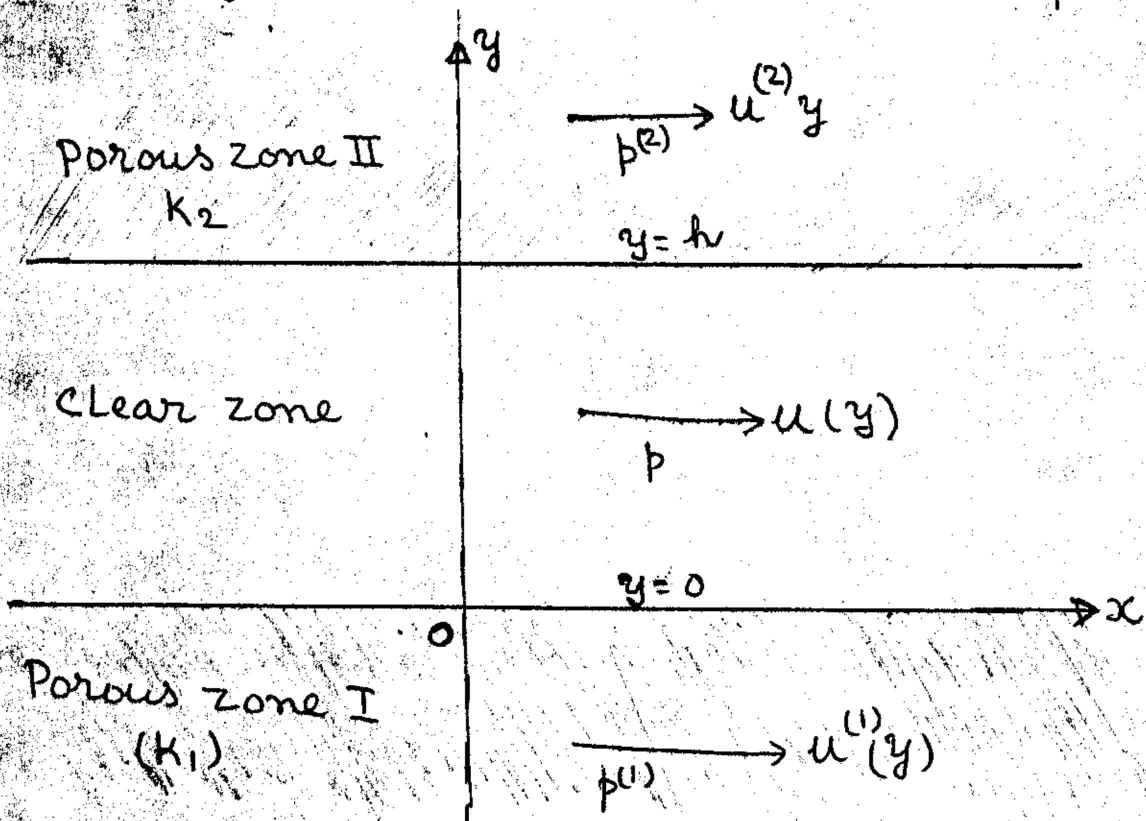
and $I_0(\ )$ and $K_0(\ )$ are modified Bessel functions of ~~where~~ the first and second kinds. The flow (17.3) in a circular tube of radius a can be deduced from (18.1) in the limit as $b \to 0$.

Flow between parallel plates: This can be realized from (18.1) as both a and $b \to \infty$, keeping $a - b = 2h$, a finite quantity equal to the distance by which the two plates are seperated. We employ cartesian coordinates $O(x, y, z)$ such that the xz plane is midway between the two plates. The two plates can be represented by $y = \pm h$. The transition from the cylindrical and cartesian coordinates can be effected by the transformation

$$r = b + h + y \qquad (18.3)$$

B. <u>Steady parallel flow sandwiched between two porous zones:</u>

Let us examine the steady parallel flow of a viscous incompressible fluid in the (clear) region between $y = 0$ and $y = h$ sandwiched between two porous materials occupying the regions $y < 0$ and $y > h$ and having porosity coefficients K_1 and K_2 respectively. Let the flow be induced by a constant pressure gradient (parallel to the plates) along the axis of x, say and $a^{(1)}$ and $a^{(2)}$ be the pressure gradients induced in the two porous zones respectively. The bounding walls $y=0$ and $y=h$ are assumed to be impermeable.



The velocity field $u(y)$ satisfies the following equations:

In the porous zone I : $\quad y \in]-\infty, 0[$

$$0 = -\frac{\partial p^{(1)}}{\partial x} - \frac{\mu}{K_1} u^{(1)} + \mu \frac{d^2 u^{(1)}}{dy^2} \quad ; \qquad (19)$$

In the clear zone : $\quad y \in]0, h[$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} \quad \ldots \qquad (20)$$

In the porous zone II :

$$0 = -\frac{\partial p^{(2)}}{\partial x} - \frac{\mu}{K_2} u^{(2)} + \mu \frac{d^2 u^{(2)}}{dy^2} \quad \ldots \qquad (21)$$

In these equations u and p stand for the velocity and pressure fields and the letters with superscripts (1) and (2) stand for the corresponding quantities in the two porous zones respectively. The applied pressure gradient in the clear zone is a constant:

$$-\frac{\partial p^{(1)}}{\partial x} = a , \qquad (22.1)$$

and the induced pressure gradients in the two zones

$$-\frac{\partial p^{(1)}}{\partial x} = a^{(1)} \quad \text{and} \quad -\frac{\partial p^{(2)}}{\partial x} = a^{(2)} \qquad (22.2)$$

are constants to be determined.

The appropriate boundary conditions for these equations are

$u^{(1)}$ is regular as $y \to -\infty$ $\qquad$ (23.1)

$u^{(2)}$ " $\quad y \to +\infty$ $\qquad$ (23.2) on $y=0$: $p^{(1)} = p$; $u^{(1)} = u$; $\frac{du^{(1)}}{dy} = \frac{du}{dy}$ (23.3)

and
on $y=h$: $p = p^{(2)}$; $u = u^{(2)}$; $\frac{du}{dy} = \frac{du^{(2)}}{dy}$ (23.4)

From (22.1)–(22.2), we get

$$p^{(1)} = -a_1^{(1)} x + p_0^{(1)} ;$$

$$p = -ax + p_0$$

and $$p^{(2)} = -a_2^{(2)} x + p_0^{(2)}$$

and on applying the continuity conditions (23.3)–(23.4) of pressure, we obtain that

$$a_1^{(1)} = a^{(2)} = a \text{ and } p_0^{(1)} = p_0^{(2)} = p_0 \quad (24)$$

which states the pressure gradients induced in the two porous zones are the same as the applied pressure gradient in the clear zone.

The velocity field is given by

$$u(y) = \begin{cases} \frac{aK_1}{\mu} + A^{(1)} e^{+y/\sqrt{K_1}} & \text{in the porous zone I} \\ -\frac{a}{2\mu} y^2 + A + By & \text{in the clear zone} \\ \frac{aK_2}{\mu} + B^{(2)} e^{-y/\sqrt{K_2}} & \text{in the porous zone II} \end{cases} \quad \ldots (25)$$

where $A^{(1)}$, A, B and $B^{(2)}$ are constants satisfying the equations:

$$\left.\begin{array}{l} \frac{aK_1}{\mu} + A^{(1)} = A ; \quad \frac{A^{(1)}}{\sqrt{K_1}} = B ; \\ -\frac{ah^2}{2\mu} + A + Bh = \frac{aK_2}{\mu} + B^{(2)} e^{-h/\sqrt{K_2}} \end{array}\right\} \quad (26)$$

The average velocity in the clear zone is given by

$$\bar{u} = \frac{1}{h}\int_0^h u(y)\,dy$$

$$= -\frac{ah^2}{6\mu} + \frac{Bh}{2} + A \quad \ldots\ldots \quad (27)$$

which can be compared with the corresponding result $\bar{u}_0 = \frac{ah^2}{12\mu}$ for the flow in the absence of porous zones.

$$u^* = \bar{u}/\bar{u}_0 = \frac{12\mu}{ah^2}\left[-\frac{ah^2}{6\mu} + \frac{Bh}{2} + A\right] \quad (28)$$

It is noticed that the porosities of the two zones have a strong influence in straightening the velocity profiles. Steep fall in the velocities is observed near the wall which is in contact with the porous zone of lower permeability coefficient (K) compared to the other wall. The velocity profiles and average velocity are illustrated for the range of values of K_1 and K_2:

K_1	·01	·02	·05	·1·0	·5	1·0	5·0	10·0
K_2	·01	·02	·05	·1	·5	1·0	5·0	10·0

C. Steady viscous flow in a sandwich with permeable walls:

Let us assume that the bounding walls $y = 0$ and $y = h$, in the problem discussed above, to be permeable so that a cross-flow through them is possible.

It can be noticed, as a consequence of the continuity equation and the boundary condition (6), that the cross-flow velocity is a constant, say V_0.

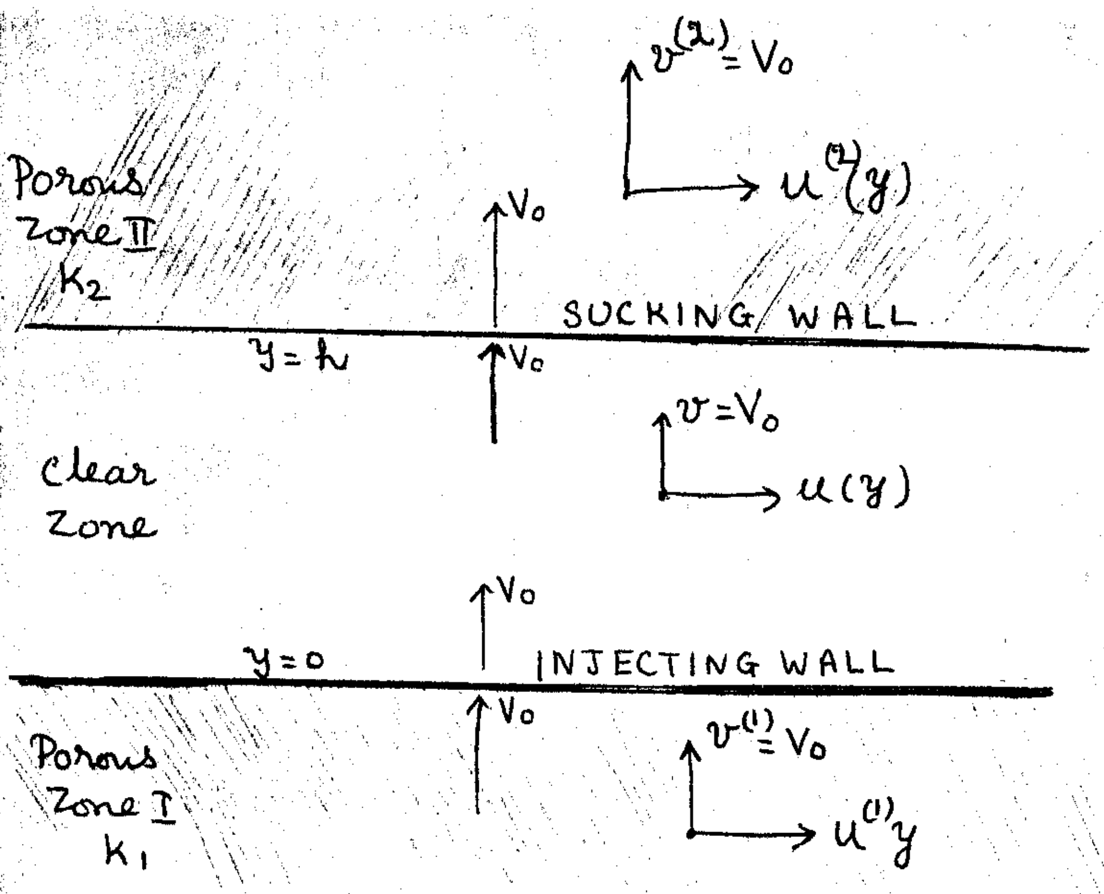
The flow is characterized by the equations:

$$\left.\begin{aligned} \rho V_0 \frac{du^{(1)}}{dy} &= -\frac{\partial p^{(1)}}{\partial x} - \frac{\mu}{K_1} u^{(1)} + \mu \frac{d^2 u^{(1)}}{dy^2} \\ 0 &= -\frac{\partial p^{(1)}}{\partial y} - \frac{\mu}{K_1} V_0 \;; \end{aligned}\right\} \text{in porous zone I} \quad \ldots (29)$$

$$\left.\begin{aligned} \rho V_0 \frac{du}{dy} &= -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} \\ 0 &= -\frac{\partial p}{\partial y} \;; \end{aligned}\right\} \text{in the clear region} \quad (30)$$

and

$$\left.\begin{aligned} \rho V_0 \frac{du^{(2)}}{dy} &= -\frac{\partial p^{(2)}}{\partial x} - \frac{\mu}{K_2} u^{(2)} + \mu \frac{d^2 u^{(2)}}{dy^2} \\ 0 &= -\frac{\partial p^{(2)}}{\partial y} - \frac{\mu}{K_2} V_0^{(2)} \end{aligned}\right\} \text{in the porous zone II} \quad (31).$$



We suppose that an oblique pressure gradient

$$-\nabla p^{(1)} = (a^{(1)}, b^{(1)}) \qquad (32)$$

be imposed in the porous zone I which has the injecting boundary (i.e., $y=0$). This will be a quite practical assumption as this model can be employed in analyzing pumping of oil from oil wells. Let the pressure gradients induced in the other zones be

$$-\nabla p = (a, b) \qquad (33)$$

and $-\nabla p^{(2)} = (a^{(2)}, b^{(2)})$

respectively. From the second of the eqns (29)-(31), we obtain

$$\left.\begin{aligned} p^{(1)} &= -a^{(1)}x - b^{(1)}y + p_0^{(1)}, \\ p &= -a^{(1)}x + p_0 \qquad (b \equiv 0) \end{aligned}\right\} \qquad (34)$$

It can be noticed, on applying the conditions (7) that

cross-flow-velocity $= V_0 = \dfrac{K_1}{\mu} b^{(1)}$,

the transverse pressure gradient induced in the porous zone II $\Big\} = b^{(2)} = \dfrac{K_1}{K_2} b^{(1)}$ and

the pressure gradients down the plates in the clear zone and porous zone II are equal each to $a^{(1)}$.
i.e. $a = a^{(1)} = a^{(2)}$.

The velocity field, satisfying the boundedness conditions at $y = \pm\infty$, is given by

$$u(y) = \begin{cases} \dfrac{K_1 a^{(1)}}{\mu} + A^{(1)} e^{my} & \text{in zone I} \\[2ex] A + B e^{\rho V_0 y/\mu} + \dfrac{a^{(1)}}{\rho V_0} y & \text{in the clear zone} \\[2ex] \dfrac{K_2 a^{(2)}}{\mu} + B^{(2)} e^{-ny} & \text{in the zone II} \end{cases} \tag{35}$$

where

$$m = \frac{1}{2}\left[\frac{\rho V_0}{\mu} + \sqrt{\frac{\rho^2 V_0^2}{\mu^2} + \frac{4}{K_1}}\right] \tag{36}$$

and

$$n = \frac{1}{2}\left[\frac{\rho V_0}{\mu} - \sqrt{\frac{\rho^2 V_0^2}{\mu^2} - \frac{4}{K_2}}\right] \tag{37}$$

Also, the constants $A^{(1)}$, A, B, $B^{(2)}$ can be obtained as in the above case employing the conditions of continuity of tangential velocity and shear stress on $y = 0$ and $y = h$. The flow parameters have been analyzed for a wide spectrum of values $(\cdot 1, \cdot 5, 1, 2, 5)$ of the cross- ... $\rho V_0 h/\mu$ and K_1, K_2

Concluding remarks:

Darcy's law (1) is inadequate to describe viscous flows through porous media near solid boundaries/interfaces, permeable or not, and also at high speeds. The convective accelerations and viscous stresses play a dominating role in determining the flow pattern in a layer close to such surfaces. Of course, the flow at large distances is very much close to the Darcian flow. The transition from Non-Darcian to Darcian, however, is not sharp and can be investigated by adopting the technique of matched asymptotic expansions which is not discussed here.

The body force term $-\frac{\mu}{K}\vec{V}$ in eq. (4) is based on the assumption that each particle obstructing the flow in the medium is spherical, the drag force on which is proportional to the velocity $\vec{V}$ and the coefficient of viscosity μ if the fluid is Newtonian and the flow is slow. This term may have to be modified suitably if the basic cell structure obstructing the flow is altered or the fluid under investigation is non-Newtonian. Furthermore, there may be some inhomogenities present in the medium in which case the permeability coefficient K is not a constant, a case examined by St. Gheorghitza [13]. The field equations for a non-Newtonian flow in a non-homogeneous non-Darcian porous medium can now be written as

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = -\nabla p - \frac{1}{K}\vec{F} + \mathrm{Div}\,\overleftrightarrow{T} \qquad (38)$$

where

$\overleftrightarrow{T}$ = the deviatoric stress tensor characterizing the non-Newtonian nature of the fluid,

$\vec{F}$ = the drag coefficient on each cell (obstructing the motion) due to the seepage of the fluid

and

K = a function depending on the position, exhibiting the non-homogeneous character of the medium.

These equations will be inadequate when the fluid is a polar fluid (i.e., fluids with micro-structures sustaining couple stresses). In such a case, the eq.(38) is to be further modified and supplimented by the equation of the balance of angular momentum.

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