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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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REMARKS ON A PROBLEM OF THE THEORY OF LUBRICATION

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Remarks on a problem of the theory of lubrication

G. Cimatti, Nov. ~~Sept.~~ 1976.

1. The physical problem

The aim of this lecture is an illustration of a mechanical problem which can be stated in the form of a variational inequality.

The unknown function in the problem is the pressure distribution p in a thin film of lubricant (a viscous fluid of viscosity μ) contained in the narrow clearance between two circular cylinders Γ_a and Γ_b of equal length L. The axes of Γ_a and Γ_b are taken to be parallel though distinct and it is assumed that the motion of Γ_b is a rotation around its axis with constant angular speed ω.

Let us introduce the following symbols

R : radius of the journal bearing Γ_a

r_1 : radius of the shaft Γ_0 $\quad (r > r_1)$

e : distance between the axis of the bearing and the axis of the shaft.

ε : eccentricity ratio, defined by $\dfrac{e}{r-r_1}$. $0 \le \varepsilon \le \alpha < 1$

~~U relative velocity of the bearing surfaces~~

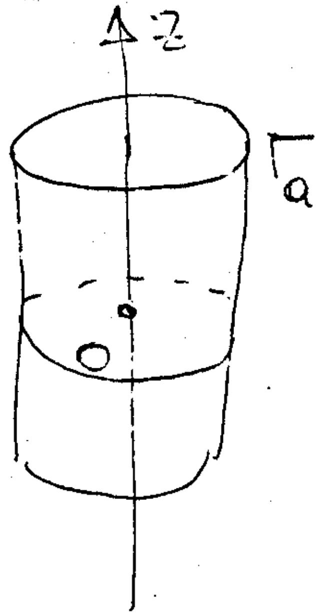
Then, in a suitable dimensionless frame, the equation of the problem is the Reynolds equation [8], [9] which reads

$$(1.1)\quad \frac{\partial}{\partial \theta}\left[(1+\varepsilon\cos\theta)^3 \frac{\partial}{\partial \theta} p\right] + \frac{\partial}{\partial y}\left[(1+\varepsilon\cos\theta)^3 \frac{\partial}{\partial y} p\right] = -\varepsilon \sin\theta .$$

The angle θ is measured from the position of maximum clearance situated on the line of centre (see fig 1),

fig 1

y is taken in the axial direction as in fig 2.



$$y = \frac{z}{r}$$

$$b = \frac{L}{2r}$$

fig 2.

Clearly we need to add to equation (1.1) boundary conditions, which can be assumed to be

(1.2) $\quad p(\theta, b) = p(\theta, -b) = 0,$

if the pressure at the free ends of the bearing is athmospheric. Moreover we have an obvious condition of periodicity

(1.3) $\quad p(\theta, y) = p(\theta + 2\pi, y).$

In order to see why the problem described by (1.1), (1.2), (1.3) is not satisfactory from the physical point of view, let us say a few words about this elliptic boundary value problem.

Let us introduce the following open domains of R^2 and spaces of functions

$$Q = \{(\theta, y);\ -\infty < \theta < +\infty,\ |y| < b\}$$

$$R = \{(\theta, y);\ 0 < \theta < 2\pi,\ |y| < b\}$$

$$C^\infty_\#(\bar{Q}) = \{v \in C^\infty(\bar{Q});\ v(\theta, y) = v(\theta + 2\pi, y)\}$$

$$C^\infty_{\#0}(\bar{Q}) = \{v \in C^\infty_\#(\bar{Q}),\ \text{vanish near } \partial Q\}.$$

Next we define the norm

$$\|v\|_{H^1} = \left(\int_R |v|^2 + \int_R |\text{grad}\, v|^2\right)^{1/2}.$$

Then we define

$$H^1_\# = \overline{C^\infty_\#(\bar{Q})}, \text{ where the closure is taken with } \|\ \|_{H^1}$$

$$H^1_{\#0} = \overline{C^\infty_{\#0}(\bar{Q})} \quad \text{" \quad " \quad " \quad " \quad " \quad "}$$

Problem (1.1), (1.2), (1.3) can now be stated as follows

(1.5) $p \in H^1_{\#0}$, $\int_R d^3\, \text{grad}\, p \cdot \text{grad}\, v = \varepsilon \int_R \sin\theta\, v, \quad \forall v \in H^1_{\#0}$

Lemma 1.1

There exists a unique solution $p \in C^\infty(\overline{Q})$ of problem (1,5). Moreover we have

(1.6) $p(\theta, y) = - p(2\pi - \theta, y)$

(1.7) $p(0,y) = p(\pi, y) = p(2\pi, y) = 0$

(1.8) $p > 0$ in R_1, $p < 0$ in R_2, with

$R_1 = \{(\theta, y);\ 0 < \theta < \pi,\ |y| < b\}$ $R_2 = \{(\theta, y);\ \pi < \theta < 2\pi,\ |y| < b\}$

Proof

Existence and uniqueness follows immediately by using the Lax-Milgram lemma. The regularity follows from standard results. (1,6) and (1,7) are obvious; (1.8) follows from the maximum principle. □

Now the property (1.8) of the solution to problem (1.5) is totally unrealistic: negative pressure cannot be supported by the lubricant fluid and where the film yields to slight sub-atmospheric pressure, a bubble full of air is formed and we have a region of cavitation; we notice that the shape of this region is unknown. Hence a much more realistic approach is given in the next section in terms of a free boundary problems.

2. The free boundary problem

To find a function $p(\theta, \vartheta)$ and an open set $\Omega \subset Q$ such that

(i) $$p(\theta, b) = p(\theta, -b) = 0$$

(ii) equation (1.5) is satisfied in Ω

(iii) $$p = \frac{dp}{dn} = 0 \quad \text{on } \partial\Omega \cap Q$$

(iv) $$p \geq 0 \quad \text{everywhere}.$$

We note that (iii) are the conditions of transition on the free boundary.

The possibility of studying the free boundary problems (i), (ii), (iii), (iv) with the techniques described by H. Lewy and G. Stampacchia in [1] has been noticed by many people [2], [3], [4].

We state now the variational inequality of the problem. Let

$$\mathbb{K} = \{ v \in H^1_{\#0},\ v \geq 0 \},$$

the pressure distribution satisfies

(2.1) $p \in \mathbb{K}$, $\int_R d^3\, \mathrm{grad}\, p \cdot \mathrm{grad}(v-p) \geq \varepsilon \int_R \sin\theta\,(v-p)$, $\forall v \in \mathbb{K}$.

Lemma 2.1

There exists a unique solution $p \in C^1(\bar{Q})$ to (2.1)

Proof

This lemma follows from the well-known results of [1], [5]. □

Corollary

Where $p > 0$ eq. (1.1) is satisfying.

We shall call the "coincidence set" I of (2.1) region of cavitation.

3. Properties of the Region of Cavitation

We state now a certain number of properties of the region of cavitation whose proofs can be found, in [6], [7].

Lemma 3.1

Cavitation occurs always i.e $I \neq \emptyset$.

Lemma 3.2

We have $p > 0$ for $0 \leq \theta \leq \pi$, $|y| < b$ i.e

$I \subset R_2$

Proof (Hint)

It follows easily from the maximum principle

Lemma 3.3

The solution is symmetric with respect to the θ-axis. Moreover we have

$$p_y \leq 0 \text{ in } \overline{R_3} \quad \text{and} \quad p_y \geqq 0 \text{ in } \overline{R_4},$$

where $R_3 = \{(\theta, y);\ 0 \leq \theta \leq 2\pi,\ |y| < b\}$

$R_4 = \{(\theta, y);\ 0 < \theta < 2\pi,\ |y| < b\}$.

Corollary If $p(\theta_0, y_0) = 0$, $0 \leq y_0 < b$, then $p(\theta_0, y) = 0$, $y_0 \leq y \leq b$.

Let us consider now, ~~together~~ together with (1.5), the variational inequality

$$(3.1)\quad p_1 \in \mathbb{K}, \int_R \operatorname{grad} p_1 \cdot \operatorname{grad}(v - p_1) \geq \int_R -\sin\theta\,(v - p_1), \ \forall v \in \mathbb{K}.$$

In the following theorem, we show that the solution of (3.1) gives an approximation for the solution of (1,5).

<u>Theorem</u> 3.1

<u>Let</u> p <u>be the solution</u> to (1,5) <u>and</u> p_1 <u>the solution to</u> (3.1). <u>Then we have</u>

$$(3.2)\qquad \| p - \varepsilon p_1 \|_{H^1} \leq C\varepsilon^2,$$ <u>where</u> C <u>does not depend on</u> ε.

<u>Proof</u> (Hint)

We put ~~v~~ $v \equiv 0 \in \mathbb{K}$ in (3.1) and obtain

$$(3.3)\qquad \| \operatorname{grad} p \|_{L^2} \leq C\varepsilon.$$

By ~~xxxx~~ using (3.3), we have, after easy manipulations (3.2). □

Lemma 3.4

The solution p_1 of (2.1) is symmetric ~~with~~ with respect to the lines

$$\theta = \frac{\pi}{2} \qquad \text{and} \quad \theta = \frac{3}{2}\pi .$$

Hence $p_{1\theta}(\frac{\pi}{2}, y) = p_{1\theta}(\frac{3}{2}\pi, y) = 0 \ , |y| \leq b.$

~~Proof~~

~~[illegible]~~

Lemma 3.5

We have $p_{1\theta} \leq 0$ in $\{(\theta, y); \frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi, |y| \leq b\}$

$p_{1\theta} \geq 0$ in $\{(\theta, y); 0 \leq \theta \leq \pi/2, \frac{3}{2}\pi \leq \theta \leq 2\pi, |y| \leq b\}$

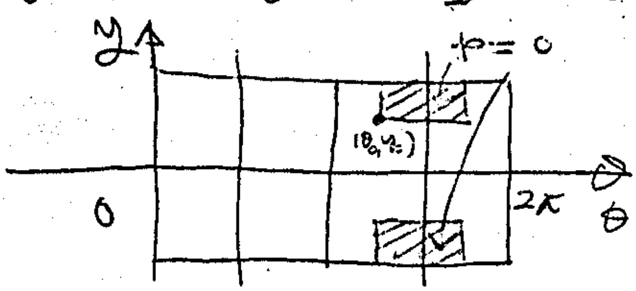
Proof (Hint)

The result is obtained by applying again the maximum principle. □

Corollary

If $p(\theta_0, y_0) = 0$ $\pi \leq \theta_0 \leq \frac{3}{2}\pi$, $y_0 \geq 0$, then

$p \equiv 0$ in $\{(\theta, y); \theta_0 \leq \theta \leq \frac{3}{2}\pi, y_0 \leq y \leq b\}$.



Let us define I_1 the coincidence set of the variational inequality (3.1) i.e

$$I_1 = \{(\theta, y);\ 0 \le \theta \le 2\pi,\ |y| < b,\ p_1(\theta, y) = 0\}.$$

Lemma 3.6

The set I_1 is connected

Hint of the Proof.

First of all it is necessary to show that there exists a point $(\theta_0, 0)$, $\pi < \theta_0 < 2\pi$ such that $p_1(\theta_0, 0) = 0$. Then the result follows easily.

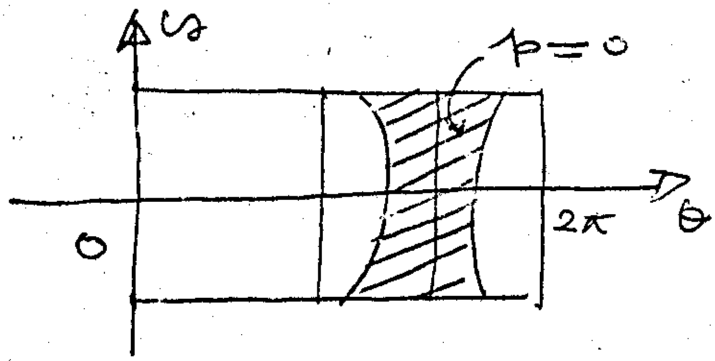
Lemma 3.7

The set $\partial I_1 \cap R$ (free boundary of the problem) cannot have components parallel to the coordinates axes.

We denote by $g(\theta)$ the greatest lower bound of the values of y (with $\frac{\pi}{2} < \theta \le \frac{3}{2}\pi$, and $0 \le y \le b$) for which $p(\theta, y) = 0$; we obtain that $g(\theta)$ is a strictly decreasing function of θ defined in a suitable interval $J \subset]\pi, \frac{3}{2}\pi[$ and

$\partial I_1 \cap \{(\theta,y) ; \frac{\pi}{2} \leq \theta \leq \frac{3}{2}, 0 \leq y \leq b\} = \{(\theta,y) ; y = g(\theta), \theta \in J\}$

The entire boundary of the region of cavitation can now be obtained by using the symmetries of the solution



Lemma 3.8

Let p_1' be the solution corresponding to $b' > b > 0$ and p_1 the solution corresponding to b. Define

$$I_1^* = \{(\theta,y) \in R , p'(\theta,y) = 0\}.$$

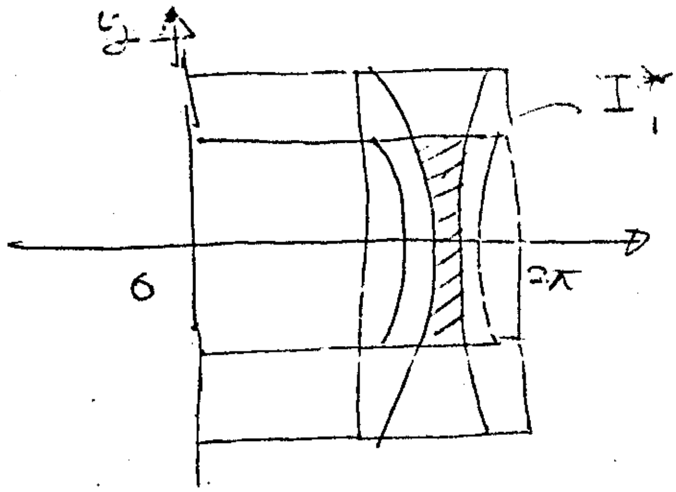
Then we have

(3.4) $$p'\big|_R \geq p$$ and

(3.5) $$I_1^* \subseteq I_1$$

Hint of the proof

It is necessary to use lemma 3.5.



4. Numerical Results

Both variational inequalities (2.1) and (3.1) have been solved numerically, by using the results of [10], [11].

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TAVOLA 1. $\varepsilon = 0,1.$

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3	–	0.01565
4	–	0.01920
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6	–	0.02630
7	–	0.02985
8	–	0.03340
9	–	0.03695

TAVOLA 2. $\varepsilon = 0,5.$

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9	–	0.33207

TAVOLA 3 $\varepsilon = 0,8.$

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