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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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NUMERICAL APPROXIMATIONS FOR SOME SINGULAR INTEGRAL EQUATIONS

(Part III)

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5. Error estimates.

We shall divide this study into two parts. In the first we establish the error estimate when the geometry is exactly approached. In the second part we study the error estimates due to the approximation of the geometry of the surface Γ.

The case $\Gamma_h = \Gamma$.

By the theorem 4.1 we have only to estimate in this case the quantity

$$\inf_{q'_h \in V_h} \| q - q'_h \|_{H^{-1/2}(\Gamma)}$$

We shall give the proof in the simplest case when using polynomials of degree 0. We shall then indicate the general results, the proof being very similar.

Definition. We call $\Pi_h q$ the operator from $L^2(\Gamma)$ to V_h defined by:

(5.1) $$\Pi_h q|_T = \frac{1}{\text{mes}(T)} \int_T q(x)\, d\gamma$$

It is quite easy to verify that in fact (exercice)

$\Pi_h q$ is the projection of q onto V_h i.e

(5.2) $$\| q - \Pi_h q \| \leq \| q - q'_h \| \quad ; \quad \forall q'_h \in V_h$$

We have a first lemma

lemma 1. Let h be the maximum diameter of the triangles, then we have:

(5.3) $$\| q - \Pi_h q \|_{L^2(\Gamma)} \leq c\, h \, \| q \|_{H^1(\Gamma)} \quad , \text{ if } q \in H^1(\Gamma).$$

Proof.

It is sufficient to prove that for each triangle we have

(5.4) $$\int_T (q - \Pi_h q)^2 \, d\gamma \leq c h^2 \, \| q \|^2_{H^1(T)}$$

Let be $\hat{T}$ a reference triangle. We know (Chavent:) that if the triangle T is not flat there exist a mapping B invertible and a translation b such that

(5.5) $x = B \hat{x} + b$, maps $\hat{T}$ onto T. Let us call F_T this mapping.

Let us consider

(5.6) $$\hat{q}(\hat{x}) = q(B\hat{x} + b) \quad , \quad (\hat{q} = q \circ F_T).$$

We have: $$\hat{\Pi} \hat{q} = \frac{1}{mes(\hat{T})} \int_{\hat{T}} \hat{q} \, d\hat{\gamma}$$

(5.7) $$\int_T (q - \Pi_h q)^2 \, d\gamma = \int_{\hat{T}} (\hat{q} - \hat{\Pi} \hat{q})^2 \, d\hat{\gamma} \times \frac{mes(T)}{mes(\hat{T})}$$

Now we have a next lemma (in the lemma)

lemma 2 We have

(5.8) $$\int_{\hat{T}} (\hat{q} - \hat{\Pi} \hat{q})^2 \, d\hat{\gamma} \leq c \int_{\hat{T}} |\operatorname{grad} \hat{q}|^2 \, d\hat{\gamma}$$

prove that the quotient space of $\dot{H}^1(\hat{T})$ by the subspace of constant functions can be norm by the L^2 norm of the gradient.

Now using (5.7) and (5.8) we have

$$\int_T (q - \Pi_h q)^2 \, d\gamma \leq C \frac{\text{mes}(T)}{\text{mes}(\hat{T})} \int_{\hat{T}} |\text{grad}\, \hat{q}|^2 \, d\hat{\gamma}$$

But $\text{grad}\, \hat{q} = B^* \text{grad}\, q$.

So that

$$\frac{\text{mes}(T)}{\text{mes}(\hat{T})} \int_{\hat{T}} |\text{grad}\, \hat{q}|^2 \, d\hat{\gamma} \leq \|B\|^2 \int_T |\text{grad}\, q|^2 \, d\gamma$$

and you have seen with Chavent that we have :

$$\|B\| \leq \frac{h}{\hat{\rho}} .$$

That finish the proof of our lemma 1.

__Theorem 5.1__.

We have

(5.9) $$\|q - \Pi_h q\|_{L^2(\Gamma)} \leq C \sqrt{h} \, \|q\|_{H^{1/2}(\Gamma)} \quad ; \text{ if } q \in H^{1/2}(\Gamma).$$

and

(5.10) $$\|q - \Pi_h q\|_{H^{-1/2}(\Gamma)} \leq C \, h^{3/2} \, \|q\|_{H^1(\Gamma)} \quad ; \text{ if } q \in H^1(\Gamma) .$$

__Proof__

We have as a consequence of property of projector

$$\|q - \Pi_h q\|_{L^2(\Gamma)} \leq 2 \|q\|_{L^2(\Gamma)}$$

cannot be avoided here !)

$$\|q-\Pi_h q\|_{L^2(\Gamma)} \leq c\sqrt{h}\ \|q\|_{H^{1/2}(\Gamma)}$$

We have

$$\|q-\Pi_h q\|_{H^{-1/2}(\Gamma)} = \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle q-\Pi_h q, \varphi\rangle}{\|\varphi\|_{H^{1/2}(\Gamma)}} = \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle q-\Pi_h q, \varphi-\Pi_h\varphi\rangle}{\|\varphi\|_{H^{1/2}(\Gamma)}}$$

$$\leq \|q-\Pi_h q\|_{L^2(\Gamma)} \times c\sqrt{h} \quad , \quad \text{(by using (5.9))}.$$

$$\leq c\, h^{3/2}\ \|q\|_{H^1(\Gamma)} \quad \text{(using lemma 1)}$$

▨

So we have an error in $h^{3/2}$ on $q-q_h$.

what about the error in $L^2(\Gamma)$ norm. We have

<u>Theorem 5.2</u> If the angles of the triangles verify $\theta \geq \theta_0 > 0$, then

(5.11) $$\|q-q_h\|_{L^2(\Gamma)} \leq c\, h\ \|q\|_{H^1(\Gamma)}$$

<u>Proof</u> We write

$$q-q_h = q-\Pi_h q + \Pi_h q - q_h$$

$$\|q-q_h\|_{L^2(\Gamma)} \leq c\,h\ \|q\|_{H^1(\Gamma)} + \|q_h-\Pi_h q\|_{L^2(\Gamma)}$$

But for elements in V_h we have the "inverse inequality"

$$\|q_h'\|_{L^2(\Gamma)} \leq \frac{c}{\sqrt{h}}\ \|q_h\|_{H^{-1/2}(\Gamma)}$$

(we do not give the proof here)

So that

$$\text{''} \leq c\,h\ \|q\|_{\ } + \frac{c}{\ }\ \|q_h-\Pi_h q\|_{-1/2(\ )}$$

$$\leq c\, h \,\|q\|_{H^1(\Gamma)} + \frac{c}{\sqrt{h}}\left(\|q-q_h\|_{H^{-1/2}(\Gamma)} + \|q-\Pi_h q\|_{H^{-1/2}(\Gamma)}\right)$$

and now by ~~theorem~~ 5.1 and 4.1 we deduced (5.11).

We are now getting some error estimates on $u - u_h$.

We know that

$$u(y) - u_h(y) = \frac{1}{4\pi}\int_\Gamma \frac{q(x) - q_h(x)}{|x-y|}\, d\gamma(x)$$

We can obtain estimates when y is not too closed of the surface Γ. We use a lemma

<u>lemma 3</u>

(5.12) $$\|q - q_h\|_{H^{-2}(\Gamma)} \leq c\, h^3 \,\|q\|_{H^1(\Gamma)}$$

<u>Proof.</u>

we have

$$\|q-q_h\|_{H^{-2}(\Gamma)} = \sup_{\varphi\in H^2(\Gamma)} \frac{\langle q - q_h, \varphi\rangle}{\|\varphi\|_{H^2(\Gamma)}}$$

Let us solve

$$b(g, g') = \langle \varphi, g'\rangle \qquad \forall g' \in H^{-1/2}(\Gamma)$$

we know then that $g \in H^1(\Gamma)$ and that

$\|g\|_{H^1(\Gamma)} \leq c\, \|\varphi\|_{H^2(\Gamma)}$, (as a result of the regularity of the operator associate to the bilinear form b).

So that

$$b(q-q_h, g)$$

$$= \sup_{g \in H^1(\Gamma)} \frac{b(q-q_h,\ g - \Pi_h g)}{\|g\|_{H^1(\Gamma)}}$$

$$\leq c \|q - q_h\|_{H^{-1/2}(\Gamma)} \cdot h^{3/2} \leq c\, h^3 \|q\|_{H^1(\Gamma)} .$$

∎

We deduces from that lemma the theorem

Theorem 5.3 Let u and u_h be respectively the exact solution of problem 1 and u_h the approximate one, we have

(5.13) $$|u(y) - u_h(y)| \leq \frac{c\, h^3}{(dist(y,\Gamma)) + (dist(y,\Gamma))^3} \|q\|_{H^1(\Gamma)}$$

(5.14) $$|D^\alpha u(y) - D^\alpha u_h(y)| \leq \frac{c}{(dist(y,\Gamma))^{3+|\alpha|} + (dist(y,\Gamma))^{1+|\alpha|}} h^3 \|q\|_{H^1(\Gamma)}$$

Proof

We just use the expression of $u - u_h$

$$u(y) - u_h(y) = \frac{1}{4\pi} < q - q_h ,\ \frac{1}{|x-y|} >$$

$$|u(y) - u_h(y)| \leq c \|q - q_h\|_{H^{-2}(\Gamma)} \left\| \frac{1}{|x-y|} \right\|_{H^2(\Gamma)}$$

and for $y \notin \Gamma$ we know that

$$\left\| \frac{1}{|x-y|} \right\|_{H^2(\Gamma)} \leq \frac{c}{(dist(y,\Gamma))^3} + \frac{c}{\ldots}$$

For the derivatives we have the expression

$$D^\alpha u(y) - D^\alpha u_h(y) = \langle q - q_h , D^\alpha \frac{1}{|x-y|} \rangle$$

and we know by the same argument that

$$\| D^\alpha \left(\frac{1}{|x-y|} \right) \|_{H^\ell(\Gamma)} \leq \frac{C}{(dist(y,\Gamma))^{3+|\alpha|}} + \frac{C}{(dist(y,\Gamma))^{1+|\alpha|}}$$

$\blacksquare$

Concerning global error estimates on $u - u_h$ we know already by theorem 4.1 that

$$\| u - u_h \|_{W^1(\mathbb{R}^3)} \leq C h^{3/2} \| q \|_{H^1(\Gamma)} .$$

We can prove also using a duality argument that

$$\left\| \frac{u - u_h}{(1+r^2)^{1/2}} \right\|_{L^2(\mathbb{R}^3)} \leq C h^{5/2} \| q \|_{H^1(\Gamma)} .$$

And that means that the error near Γ on the potential is not in h^3 but in h^2, the mean of the error being in $h^{5/2}$.

Remark When using polynomials of degree m for defining the finite element used in V_h, we shall obtain an error.

$$\| q - q_h \|_{H^{-1/2}(\Gamma)} \leq C h^{m+3/2} \| q \|_{H^{m+1}(\Gamma)}$$

And an error on $u - u_h$:

$$|u(y) - u_h(y)| \leq \frac{C}{\quad} h^{2m+3} \| q \|$$

The case with error on geometry.

In this case the first thing is to define what is the mapping Ψ from Γ_h to Γ that lead us to the mappings

$$q_h \to \tilde{q}_h = q_h \circ \Psi^{-1}$$

The first idea is to define the mapping Ψ using on each triangle the mapping $\phi_{ih}|_T$ of T on Γ_h. So that

$$\Psi = \phi_j \circ F_T^{-1} = \phi_i \circ \phi_{ih}^{-1} \quad (\,F_T \text{ is the restriction of } \phi_{ih} \text{ to } T\,)$$

It happens that this mapping lead you to a non-optimal error analysis. So we are lead to choose an other mapping. We defined it that way:

Let $x \in \Gamma$ and $n(x)$ the normal to Γ at that point. That normal cut the surface Γ_h in one point that we call $\Psi^{-1}(x)$. This is our mapping. It turns out that when $h \to 0$ you are sure that Ψ is a bijection from Γ_h onto Γ. So Ψ is a kind of projection on Γ along the normal.

We shall restrict us to the case when Γ_h is a union of flat triangles and the space V_h is the space of functions constant on each triangles. The proof in other cases is similar but with more technical results.

We give the final result and we shall prove it.

<u>Theorem 5.4</u> Let Ψ be defined as above (using the normal). We have

(5.15) $$\|q - q_h \circ \Psi^{-1}\|_{H^{-1/2}(\Gamma)} \leq c \left(\|u_0 - u_{0h} \circ \Psi^{-1}\|_{H^{1/2}(\Gamma)} + h^{3/2} \|q\|_{H^{1}(\Gamma)} \right)$$

▨

We shall used a long serie of lemmas to get our result. The first one concern the difference between $\Psi \circ F_T$ and F_T

<u>lemma 4.</u> If all the triangles satisfies the condition of angles $\theta \geq \theta_0 > 0$.

(5.16) $$\sup_{x \in T} | \Psi \circ F_T(x) - F_T(x) | \leq c\, h^2 \sup_{x \in T} |D^2 \phi_i(x)| \ ;$$

(5.17) $$\sup_{x \in T} |D(\Psi \circ F_T)(x) - DF_T(x)| \leq c\, h \sup_{x \in T} |D^2 \phi_i(x)| \ .$$

<u>Proof</u> we know already that F_T is the interpolate of ϕ_i by a polynomial of degree 1. So that using the Taylor expansion of ϕ_i, you can prove that when $T \in D_i$

$$\sup_{x \in T} | \phi_i(x) - F_T(x) | \leq c\, h^2 \sup_{x \in T} |D^2 \phi_i(x)|$$

$$\sup_{x \in T} |D \phi_i(x) - DF_T(x)| \leq c\, h \sup_{x \in T} |D^2 \phi_i(x)|$$

We know also that the mapping Ψ is regular and has bounded derivatives.

We write

$$\phi_i^{-1} \circ \psi \circ F_T(x) - x = \phi_i^{-1} \circ \psi \circ F_T(x) - \phi_i^{-1} \circ \psi \circ \phi_i(x)$$

and by differentiation

$$D\phi_i^{-1}\, D(\psi \circ F_T) - I = D(\phi_i^{-1} \circ \psi)(F_T(x)) \circ DF_T(x) - D(\phi_i^{-1} \circ \psi)(\phi_i(x)) \circ D\phi_i(x)$$

So that using the continuity of $\phi_i^{-1} \circ \psi$ and its derivatives

$$\sup_{x\in T} |\phi_i^{-1} \circ \psi \circ F_T(x) - x| \leq C \sup_{x\in T} |\phi_i(x) - F_T(x)|$$

and then "applying ϕ_i" leads you to (5.16).

We have also

$$\sup_{x\in T} \|(D\phi_i)^{-1} \circ D(\psi \circ F_T) - I\| \leq C \sup_{x\in T} |F_T(x) - \phi_i(x)|\,|DF_T(x)|$$
$$+ C \sup_{x\in T} |DF_T(x) - D\phi_i(x)| \leq Ch \sup_{x\in T} |D^2 \phi_i(x)|$$

By applying $D\phi_i$ we obtain

$$\sup_{x\in T} |D(\psi \circ F_T) - D\phi_i| \leq Ch \sup_{x\in T} |D^2 \phi_i(x)|$$

and then using the inequality between $D\phi_i$ and DF_T we obtain (5.17)

■

The next lemma concern the difference of the area elements on Γ_h and Γ at point corresponding through ψ. Let us recall that the area element is defined by (on Γ_h and Γ respectively)

$$J(F_T) = \left| \frac{\partial F_T}{\partial \xi_1} \wedge \frac{\partial F_T}{\partial \xi_2} \right| \quad ; \quad J(\psi \circ F_T) = \left| \frac{\partial(\psi \circ F_T)}{\partial \xi_1} \wedge \frac{\partial(\psi \circ F_T)}{\partial \xi_2} \right|$$

We notice that the difference between $J(F_T)$ and $J(\phi_i)$ is $O(h)$. Here we obtain a difference in $O(h^2)$.

<u>lemma 5</u> We have the following inequalities:

(5.18) $$\sup |J(F_T)(x) - J(\psi \circ F_T)(x)| \leq Ch^2 \sup |D^2 \phi_i(x)|$$

$\forall\, T, T' \in \mathcal{G}_h \;,\; x \in T \;,\; y \in T'$

(5.19) $$c \; |F_T(x) - F_{T'}(y)| \leq |\Psi_0 F_T(x) - \Psi_0 F_{T'}(y)| \leq C \; |F_T(x) - F_{T'}(y)|$$

(5.20) $$\Big|\, |F_T(x) - F_{T'}(y)|^2 - |\Psi_0 F_T(x) - \Psi_0 F_{T'}(y)|^2 \Big| \leq C h^2 \; |F_T(x) - F_{T'}(y)|^2 .$$

Proof We do not get it here because it is rather technical. We refer to Nedelec [ ] for that proof. The main idea is that the fact of using the normal to Γ for defining Γ give us an extra h in the error for every tangential vector.

lemma 6

Let $\tilde{V}_h = \{ \hat{q}_h \;,\; \hat{q}_h = q_h \circ \Psi^{-1} \;,\; q_h \in V_h \}$

We have

(5.21) $$\| \hat{q}_h \|_{L^2(\Gamma)} \leq \frac{C}{\sqrt{h}} \; \| \hat{q}_h \|_{H^{-1/2}(\Gamma)} \quad ; \quad \forall\, \tilde{q}_h \in V_h$$

Let s_h be the projector in the L^2 norm on the space $\tilde{V}_h$ of $L^2(\Gamma)$.

We have

(5.22) $$\| q - s_h q \|_{L^2(\Gamma)} \leq C \, h \; \| q \|_{H^1(\Gamma)}$$

(5.23) $$\| q - s_h q \|_{H^{-1/2}(\Gamma)} \leq C \, h^{3/2} \; \| q \|_{H^1(\Gamma)} \quad ; \quad \forall\, q \in H^1(\Gamma) .$$

Proof It is the same proof as for the theorem 5.1 plus the fact that the mapping Ψ^{-1} is bounded and its derivative also.

Proof of theorem 5.4. We use the theorem 4.2

We have already the first term by lemma 6

$$\inf_{q'_h \in V_h} \| q - q'_h \|_{H^{-1/2}(\Gamma)} \leq C \, h^{3/2} \; \| q \|_{H^1(\Gamma)}$$

Look at the term

(5.24) $$b(\hat{q}'_h, \hat{w}_h) - b_h(q'_h, w_h)$$

We transform integrals on Γ_h into integrals on Γ by using our mapping Ψ. We obtain

$$b_h(q'_h, w_h) = \frac{1}{4\pi} \int_\Gamma\int_\Gamma \frac{\hat{q}'_h(x)\hat{w}_h(y)}{|\Psi^{-1}(x) - \Psi^{-1}(y)|} J(\Psi^{-1}(x))\, J(\Psi^{-1}(y))\, d\gamma_x\, d\gamma_y$$

So ~~the~~ the difference (5.24) gives us two terms of errors. One is due to the difference of area element, and leads a bound by:

$$c \int_\Gamma\int_\Gamma \frac{|\hat{q}'_h(x)|\,|\hat{w}_h(y)|}{|\Psi^{-1}(x) - \Psi^{-1}(y)|}\; h^2\, d\gamma_x\, d\gamma_y$$

The other ones is due to the error on the kernel which is

$$\left| \frac{1}{|\Psi^{-1}(x) - \Psi^{-1}(y)|} - \frac{1}{|x-y|} \right| = \frac{\big|\, |x-y|^2 - |\Psi^{-1}(x) - \Psi^{-1}(y)|^2 \,\big|}{|x-y|\, |\Psi^{-1}(x) - \Psi^{-1}(y)|}$$

We use now for those two terms the lemma 5, such that we obtain

$$|b(\hat{q}'_h, \hat{w}_h) - b_h(q'_h, w_h)| \le C h^2 \int_\Gamma\int_\Gamma \frac{|\hat{q}'_h(x)|\,|\hat{w}_h(y)|}{|x-y|}\, d\gamma(x)\, d\gamma(y)$$

$$\le C h^2\, \|\hat{q}_h\|_{L^2(\Gamma)}\, \|\hat{w}_h\|_{L^2(\Gamma)}$$

Using then the inverse inequality (5.21) we obtain

$$\frac{|b(\hat{q}'_h, \hat{w}_h) - b_h(q'_h, w_h)|}{\|\hat{w}_h\|_{H^{-1/2}(\Gamma)}} \le C h^{3/2}\, \|\hat{q}_h\|_{L^2(\Gamma)}$$

The technique for the term containing u_0 and u_{0h} is exactly the

Theorem 5.5. Let u be the potential given by

$$u(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x)}{|x-y|} \, d\gamma_x$$

and u_h the approximate potential given by

$$u_h(y) = \frac{1}{4\pi} \int_{\Gamma_h} \frac{q_h(x)}{|x-y|} \, d\gamma_x$$

then we have

$$(5.25) \qquad |u(y) - u_h(y)| \leq \frac{C}{\mathrm{dist}(y,\Gamma) + \mathrm{dist}^2(y,\Gamma)} \left(h^2 \|q\|_{H^1(\Gamma)} + \|u_0 - \hat{u}_{0h}\|_{L^2(\Gamma)} \right)$$

Proof. The idea is the same as in the case without error on the boundary Γ. We transform the expression of u_h using Ψ

$$u_h(y) = \frac{1}{4\pi} \int_\Gamma \frac{\tilde{q}_h(x)\, J(\Psi^{-1})(x)}{|y - \Psi^{-1}(x)|} \, d\gamma_x$$

So that we have

$$u(y) - u_h(y) = \frac{1}{4\pi} \int_\Gamma \frac{q(x) - \hat{q}_h(x)\, J(\Psi^{-1})}{|y-x|} \, d\gamma_x$$

$$+ \frac{1}{4\pi} \int_\Gamma \hat{q}_h \, J(\Psi^{-1}) \left(\frac{1}{|y-x|} - \frac{1}{|y - \Psi^{-1}(x)|} \right) d\gamma_x$$

The second term will give an error in h^2 by using the lemma 5 and inequality **(5.20)**.

For treating the first term we use a lemma

Lemma 7. $\ldots \leq c \left(h^2 \|q\| + \|u_0 - \tilde{u}_{0h}\|_{L^2(\Gamma)} \right).$

Proof

We use again the same trick

$$\| q - \hat{q}_h J(\psi^{-1}) \|_{H^{-1}(\Gamma)} = \sup_{\varphi \in H^1(\Gamma)} \frac{< q - \hat{q}_h J(\psi^{-1}), \varphi >}{\|\varphi\|_{H^1(\Gamma)}}$$

and we solve

$$b(g, g') = < \varphi, g' > \qquad \forall g' \in H^{-1/2}(\Gamma)$$

We obtain a unique g such that

$$\|g\|_{L^2(\Gamma)} \leq C \|\varphi\|_{H^1(\Gamma)}$$

$$\| q - \hat{q}_h J(\psi^{-1}) \|_{H^{-1}(\Gamma)} = \sup_{g \in L^2(\Gamma)} \frac{b(q - \hat{q}_h J(\psi^{-1}), g)}{\|g\|_{L^2(\Gamma)}}$$

We compute then

$$a(q - \hat{q}_h J(\psi^{-1}), g) = b(q - \hat{q}_h J(\psi^{-1}), g - s_h g) + b(q - \hat{q}_h J(\psi^{-1}), s_h g)$$

The first of these terms give an error bound (using lemma 6)

$$C\sqrt{h} \| q - \hat{q}_h J(\psi^{-1}) \|_{H^{-1/2}(\Gamma)}$$

The second of these terms can be written as

$$(u_0, s_h g)_{L^2(\Gamma)} - \frac{1}{4\pi} \int_{\Gamma_h} \int_{\Gamma_h} \frac{q_h(x)\, s_h g(y)}{|\psi(x) - \psi(y)|} \, d\gamma_x \, d\gamma_y$$

$$= (u_0, s_h g)_{L^2(\Gamma)} - (u_{0h}, s_h g)_{L^2(\Gamma_h)}$$

$$- \frac{1}{4\pi} \int_{\Gamma_h} \int_{\Gamma_h} q_h(x)\, s_h g(y) \left(\frac{1}{|\psi(x) - \psi(y)|} - \frac{1}{|x-y|} \right) d\gamma_x \, d\gamma_y$$

And we obtain the final result by using again lemma 6.

■

Now for $y \notin \Gamma$ we can bound the remaining term in $u - u_h$ by :

$$c \; \| q(x) - \hat{q}_h(x) \, J(\Psi^{-1}) \|_{H^{-1}(\Gamma)} \; \left\| \frac{1}{|\cdot - y|} \right\|_{H^{1}(\Gamma)}$$

and that finish the proof .

∎

<u>remark</u>

We cannot obtain a global estimate between u and u_h
The reason why , is that the normal derivative of u has a discontinuity when crossing Γ . and the normal derivative of u_h has a discontinuity when crossing Γ_h .
So that u and u_h are quite different at a point between Γ and Γ_h .

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