INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23 - 8

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

22 September - 26 November 1976

BASIC IDEAS IN APPLIED FUNCTIONAL ANALYSIS

(continued)

H. Lanchon
Nancy, France

In order to solve this problem, we apply the Lax Milgram theorem replacing the second member by.

$$< \tilde{f}, v > = \int_\Omega f . v \, dx + \int_\Gamma g_0 \, \gamma_0 v$$

$\tilde{f}$ defined by this way $\in (H^1(\Omega))'$ which is not a space of distributions. anyway Lax Milgram is working but the interpretation (51) is just formal.

IV.5. Mixed Homogeneous Dirichlet. Neuman problem

$$(52) \begin{cases} -\Delta u + \lambda u = f . \\ \gamma_0 u = 0 \quad \text{on } \Gamma_1 \quad \text{suppose of non zero measure} \\ \gamma_1 u = \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_2 . \end{cases}$$

Γ_1 and Γ_2 being two complementary path of Γ.

We choose.

$$V = \{ v \,/\, v \in H^1(\Omega), \ \gamma_0 v = 0 \text{ on } \Gamma_1 \} \subset H^1(\Omega) \subset L^2(\Omega)$$

Then the green formula gives $\forall \, v \in V$

$$\int_\Omega (-\Delta u + \lambda u) v \, dx = \int_\Omega \operatorname{grad} u \operatorname{grad} v \, dx - \int_\Gamma \frac{\partial u}{\partial n} v \, d\Gamma + \lambda \int_\Omega u v \, dx .$$

but $$\int_\Gamma \frac{\partial u}{\partial n} v \, d\Gamma = \int_{\Gamma_1} \frac{\partial u}{\partial n} v \, d\Gamma + \int_{\Gamma_2} \frac{\partial u}{\partial n} v \, d\Gamma = 0 .$$

because $\dfrac{\partial u}{\partial n} = 0$ on Γ_2 and $\gamma_0 v = 0$ on Γ_1.

We have therefore the same bilinear form. The Lax Milgram theory may be applied without difficulties

We can applied Lax-Milgram and we obtain for $\lambda > 0$, existence and uniqueness of $u \in H^1(\Omega)$ satisfying.

$$(49) \quad a(u,v) = (f,v) \qquad \forall\, v \in H^1(\Omega)$$

applying this relation for all $\varphi \in \mathcal{D}(\Omega) \subset H^1(\Omega)$. we obtain again.

$$(50) \qquad -\Delta u + \lambda u = f \quad \text{in } \mathcal{D}'(\Omega)$$

and in consequence $\Delta u \in L^2(\Omega)$

Let us multiply then (50) by $v \in H^1(\Omega)$ and let us integrate it over Ω. By application of the generalized green formula introduced in II. 6 in the case on $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. we obtain.

$$\int_\Omega (\operatorname{grad} u \operatorname{grad} v + \lambda u v)\, dx = \int_\Omega f \cdot v\, dx + \int_\Gamma \frac{\partial u}{\partial n} \gamma_0 v \, d\Gamma$$

$$\forall v \in H^1(\Omega).$$

By comparison with (49) we have therefore.

$$\int_\Gamma \frac{\partial u}{\partial n} \gamma_0 v \, d\Gamma = 0. \qquad \forall v \in H^1(\Omega)$$

and because γ_0 is a surjection of $H^1(\Omega)$ on $H^{1/2}(\Gamma)$, that implies

$$\int_\Gamma \frac{\partial u}{\partial n}\, g \, d\Gamma = 0 \qquad \forall\, g \in H^{1/2}(\Gamma).$$

and then $\dfrac{\partial u}{\partial n} = 0$ on γ; u is effectively the solution of. (48).

<u>Remark 16</u> : The condition $\lambda > 0$ is crucial here for the uniqueness because we are working on $H^1(\Omega)$ where the inequality of Poincaré is not available.

IV. 4 <u>non Homogeneous Neuman problem</u>

$$(51) \begin{cases} -\Delta u + \lambda u = f & \text{on } \Omega \qquad f \text{ given in } L^2(\Omega) \end{cases}$$

with $\tilde{f} = f + \Delta w_0 + \lambda w_0 \in H^{-1}(\Omega)$
in fact f and $\lambda w_0 \in L^2(\Omega) \subset H^{-1}(\Omega)$
and $\ell(v) = \int_\Omega \Delta w_0 . v \, dx = - \int_\Omega \operatorname{grad} w_0 \operatorname{grad} v \, dx . \quad \forall v \in H_0^1(\Omega)$

$$|\ell(v)| \leq \|w_0\|_{H^1(\Omega)} \|\|v\|\|_{H_0^1(\Omega)}$$

ℓ (and by identification Δw_0) is therefore a continuous linear form on $H_0^1(\Omega)$. Whence $\Delta W_0 \in H^{-1}(\Omega)$

We obtain existence and uniqueness of v by preceding results for each w_0 and therefore existence u satisfying (47)

What about uniqueness of u?
Suppose there exists two solutions u_1 and u_2; we can introduce $w = u_1 - u_2$; w verifies

$$\begin{cases} -\Delta w = 0 & \text{on } \Omega \\ \gamma_0 w = 0 & \text{on } \Gamma \end{cases}$$

and we know that $w = 0$ is the only possible solution of this problem (Existence and Uniqueness of solution for Homogeneous Dirichlet problem.)

IV. 3. Homogeneous Neumann problem.

$$(48) \quad \begin{cases} -\Delta u + \lambda u = f \text{ on } \Omega ; \quad \lambda > 0 . \\ \gamma_1 u = 0 \quad \left(\text{that is to say } \dfrac{\partial u}{\partial n} = 0\right) \text{ on } \Gamma . \end{cases}$$

Ω : bounded open subset of R^n; f given in $L^2(\Omega)$.

We can again to choose a frame work corresponding to the situation of inclusion of 2 Hilbert spaces described in I. 2;
$H = L^2(\Omega)$, $V = H^1(\Omega)$ these spaces being supplied with their usual Hilbert structure.

The injection of V in H is continuous: $|u| \leq \|u\| \quad \forall u \in V$ and V which contains $\mathcal{D}(\Omega)$ is dense in H.

The bilinear form a is the same as precedently. So the change of space V constitutes the only difference.

Which the inequality announced in the theorem but for $\varphi \in \mathcal{D}(\Omega)$
we have just now to extend this result for $u \in H_0^1(\Omega)$
and that is trivial by continuity because the density of
$\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$

Remark 14: We have just used the hypothesis that
Ω is included in the set of $x \in R^n$ such that $-A < x_1 < A$.
The result (16) is therefore still true for any Ω of
finite thikness in one direction.

26

IV Applications of the Lax Milgram Theorem to several problems.

IV.1 Homogeneous Dirichlet problem for $\lambda = 0$.

We want now to resolve the following problem.
Ω given bounded open subset of R^n with a boundary Γ
sufficiently regular
f given in $L^2(\Omega)$
we look for u such that.

$$(44) \quad \begin{cases} -\Delta u = f & \text{on } \Omega \\ \gamma_0 u = 0 & \text{on } \Gamma \end{cases}$$

If we adopt the same framework as precedingly we
know that we shall have a difficulty about the coercivity.
because

$$a(u,v) = \int_\Omega \operatorname{grad} u \operatorname{grad} v \, dx \quad \text{implies} \quad a(u,u) = \int\!\!\int_\Omega |\operatorname{grad} u|^2 dx$$

and
$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\operatorname{grad} u|^2 dx + \int_\Omega |u|^2 dx.$$

so we have a priori no possibility of inequality such as.

$$a(u,u) \geq \alpha \|u\|^2 \quad \text{except with } \alpha = 0.$$

However, because we are working on $H_0^1(\Omega)$ with a bounded
Ω, we are going to prove that there exists a more convenient