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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

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RELATIONSHIP BETWEEN THEORY AND EXPERIMENT IN CONTINUUM MECHANICS

(Part II)

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3. THE EXCITATION OF SURFACE WAVES NEAR A CUT-OFF FREQUENCY†

This work consists of a theoretical & experimental investigation in a problem concerning nonlinear water waves. The reason for undertaking this work was that it provided an example in which we could employ calculational techniques that are frequently used in other applications for which it is extremely difficult to make an experimental check of their predictions. Since these procedures are purely formal ones, & since there are very few experimental checks that they actually work, it seemed worthwhile to design the theory round a particular experimental situation in order to provide a good test for the theory. The experimental situation chosen was very similar to that for the Stokes edge wave described in § 2.3, except that the 'beach' was taken to be vertical in this case. Under such circumstances there are no discrete edge modes, but only a continuous spectrum, with a cut-off frequency. In the experiment envisaged, the motions would be driven at the 'vertical beach' by making it perform small torsional oscillations about a vertical axis through the centre of

A ... L

In a similar way to that in §2.3 we let b be our length scale, & $(b/g)^{1/2}$ be our time scale, where b is the width of the channel & g is the acceleration due to gravity. Then the dimensionless velocity potential $\phi(x,y,z,t)$ & the surface displacement $\zeta(x,y,t)$ are given by

$$\nabla^2\phi = 0,$$

together with the boundary conditions:

(i) at the free surface
$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y,$$
$$\phi_t + \zeta + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0;$$

(ii) on the rigid boundaries of the channel
$$\phi_y = 0 \quad \text{on} \quad y = 0, 1 \qquad \text{and} \qquad \phi_z = 0 \quad \text{on } z = -\beta, \text{ where } \beta = d/b;$$

(iii) the condition at the wavemaker. This is discussed below;

(iv) a condition at the far end of the channel. Theoretically it is desirable to consider an open-ended channel, in which case it would be necessary to formulate a suitable radiation condition. We leave this free for the present.

3.1 The driving condition at the wavemaker

Let the angular frequency of the motion be ω & the angular displacement of the plate be θ, which we assume to be small. Then at the wavemaker we have,

$$\phi_x(0,y,z,t) = -\omega\theta(y-\tfrac{1}{2})\cos\omega t + \text{smaller},$$

which, in terms of its Fourier representation, indicates that ϕ_x can be written in the form

$$\phi_x(0,y,z,t) \approx 4\omega\theta\pi^{-2}\sum_{n=0}^{\infty}(2n+1)^{-2}\cos[(2n+1)\pi y]\cos\omega t.$$

For the linear problem the natural antisymmetric modes of the channel are of the form

$$\phi \sim \cosh \kappa(z+\beta) \cos[(2n+1)\pi y]\; e^{i\omega t}\; e^{ikx},$$

where $\quad \omega^2 = \kappa \tanh \kappa\beta \quad$ and $\quad k^2 = (2n+1)^2\pi^2 - \kappa^2$.

Thus the forcing term above would suggest that all these modes are excited, but some of the modes will be generated as exponentially decaying modes, & some as wavelike modes. If we restrict attention to frequencies γ near the lowest mode $n=0$, $k=0$, so that $\gamma^2 = \pi \tanh \pi\beta$, we see that all the other modes are generated as exponentially decaying type. Then, given the $(2n+1)^{-2}$ factor in the Fourier representation of ϕ_x, together with the rapid decay of these modes, it seems we should consider ϕ_x in the form

$$\phi_x(0,y,z,t) = 4\omega\theta\pi^{-2}\cos\pi y \cos\omega t \quad (\equiv U, \text{ say}).$$

But this forcing is independent of depth, whereas the natural mode varies as $\cosh \kappa(z+\beta)$, and this implies there must be a localized parasitic field near the wave-maker. To determine the 'effective' forcing of the wave field we use the procedure introduced by Havelock (1929). Write ϕ_x in the form

$$\phi_x(0,y,z,t) = \underbrace{U\left(1-\sigma\cosh\pi(z+\beta)\operatorname{sech}\pi\beta\right)}_{①} + \underbrace{U\sigma\cosh\pi(z+\beta)\operatorname{sech}\pi\beta}_{②}$$

and we choose σ so that the first of these terms, ①, gives rise to a purely in the form of ②, with σ thus chosen. We cannot estimate how effective this partitioning will be, & it must remain one of the unresolved questions of the theory.

The calculation of σ involves a standard calculation in linear analysis, & the outcome is that we should write the forcing term at the wavemaker in the form

$$\phi_x(0, y, z, t) = \varepsilon \cos \pi y \cosh \pi(z+\beta) \operatorname{sech} \pi\beta \cos \omega t, \qquad (3.1)$$

& $\varepsilon = C(\beta, \gamma) . \theta$ is an 'effective' amplitude of the wavemaker, & C is known from the above mentioned analysis.

3.2 A theory for the nonlinear inviscid response

Since the form of the forcing (3.1) is the same as that of the natural mode we cannot expect a solution to the linearized problem when $\omega = \gamma$. We therefore look to nonlinear effects in the surface condition to limit the response near $\omega = \gamma$, & for this we need to estimate what the relevant scales are for the structure of the motions.

As the frequency $\omega \to \gamma$, the linear theory indicates that $k \to 0$ & so the horizontal scale of the motions becomes large. Therefore for the nonlinear motions we assume there is some large scale $L(\varepsilon)$ associated with variations along the channel. Let the order of the velocity potential be $\alpha(\varepsilon)$, & we have from (3.1) that $\alpha/L = O(\varepsilon)$. Another condition is needed to determine the scales, and for this we rely on heuristic arguments that will mirror the subsequent mathematics. Since L is large the x derivatives in Laplace's equation will be small, & so, locally, the waves will look like standing waves. Suppose they are of the form

$$\alpha A(L^{-1}x) \cos \pi y \; e^{i\omega t} \quad + \quad h.o.t. \; ,$$

when the higher-order terms (h.o.t.) of interest are those with the same spatial & temporal variations as the leading term. One such term arises at $O(\alpha^3)$ from the nonlinear surface condition (cf. Tadjbakhsh & Keller 1960); another comes from the neglected derivatives in Laplace's equation & is $O(\alpha L^{-2})$. Thus, for a non-trivial balance, we choose $\alpha^3 = \alpha L^{-2}$, which, together with the previous condition, indicates that we choose $\alpha = O(\varepsilon^{1/2})$ & $L = O(\varepsilon^{-1/2})$.

An estimate of the bandwidth of frequencies for the nonlinear response can be made on the expectation that the detuning term should not enter the surface condition at an earlier stage than the important nonlinear term.

These arguments suggest we look for a solution of the form

$$\phi(x,y,z,t) = \varepsilon^{1/2}\phi_0 + \varepsilon\,\phi_1 + \varepsilon^{3/2}\phi_2 + \dots\ ,$$

and $$\zeta(X,y,t) = \varepsilon^{1/2}\zeta_0 + \varepsilon\,\zeta_1 + \varepsilon^{3/2}\zeta_2 + \dots\ ,$$

where $X = \varepsilon^{1/2}x$. In addition we introduce a parameter λ (assumed to be unit order at most) such that $\omega^2 = \gamma^2 + \lambda\varepsilon$.

We now carry out a regular perturbation procedure. The first-order term ϕ_0 is given by $$\phi_0 = \tfrac{1}{2}\left\{ A_0(X)\, e^{i\gamma t} + * \right\} \cos\pi y \cosh\pi(z+\beta)\,\mathrm{sech}\,\pi\beta ,$$

where $*$ denotes the complex conjugate of the preceding term. $A_0(X)$ is a complex valued function which is arbitrary except that ϕ_0 must satisfy the boundary condition on the wavemaker, & this requires that $A_0'(0) = 1$.

An equation for A_0 is obtained from the $O(\varepsilon^{3/2})$ terms in the free-surface boundary conditions. Fortunately, the very messy calculation of the 'detuning term' has been carried out by Tadjbakhsh & Keller (1960), & using this together with the usual orthogonality condition to avoid secularities, we find that A_0 satisfies the consistency condition

$$a A_0'' + (\lambda + D A_0 A_0^*) A = 0\ ,$$

where a & D are (known) constants determined by the value of β. This equation, together with the boundary condition $A_0'(0) = 1$ & an appropriate boundary condition at infinity provide the means for determining the spatial variation of the wavefield along the channel.

3.3 A theory for dissipative effects

Here we take account of the influence the boundary layers on the side walls & on the bottom of the channel have on the wave motion at frequencies near γ, for small amplitude waves. For such waves the boundary-layer equations are approximately linear & it is possible to calculate a displacement-thickness correction without knowing the details of the potential flow.

For example, consider the component u of the flow velocity near the wall $y=0$. It satisfies the equation

$$u_t = \mu^2 u_{yy} ,$$

where $\mu^2 = \nu (b^3 g)^{-1/2}$ is the reciprocal of the Reynolds number. When the outer flow is sinusoidal, with period ω, it follows that, near $y=0$,

$$u = u_\infty e^{i\omega t} \left\{ 1 - \exp\left[-(i\omega)^{1/2} \mu^{-1} y \right] \right\} ,$$

where u_∞ is the velocity 'outside' the boundary layer & $i^{1/2}$ has been used these two results we can deduce from the continuity equation the velocity component normal to the wall. It accordingly follows that the appropriate boundary condition for the outer (potential) flow is

$$\phi_y(x, 0, z, t) = [\mu/(i\gamma)^{1/2}](\phi_{xx} + \phi_{zz}),$$

in place of the original condition that $\phi_y = 0$ on $y = 0$. These boundary conditions, together with similar conditions on the other rigid boundaries, & the condition $\phi_z = \omega^2\phi$ on $z = 0$ (the linear free surface condition) provide an eigenvalue problem for $\nabla^2\phi = 0$. For $\omega - \gamma$ small, it is found that the wavenumber k, in the x direction, can be written in the form

$$k^2 = C_1 + i C_2 + a^{-1}(\omega^2 - \gamma^2) + O((\omega^2 - \gamma^2)^2),$$

where C_1 & C_2 are constants. The idea is that we should determine C_1 & C_2 from empirical observations on waves with wavenumber $k = 0$, because theoretical estimates of wave damping usually turn out to be very unreliable in predicting the empirical observations.

A wave number k of the above form implies an equation for A_0 of the form

If we now assume that the nonlinear correction & the dissipative correction are linearly additive (which is tantamount to an application of Taylor's theorem) we have an equation, for the variation of wave amplitude along the channel, of the form

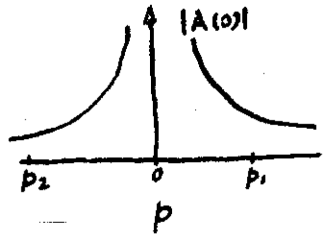
$$A'' + (p + iq + rAA^*)A = 0 \quad , \qquad (3.2)$$

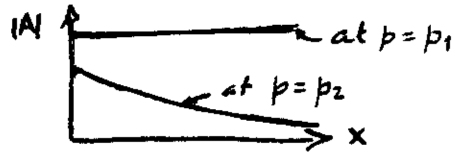
with the obvious redefinitions. Note (i) that p is a measure of the frequency difference $\omega - \gamma$, (ii) q is a measure of the dissipative effects, & (iii) r may be of either sign, depending on the water depth (for a given width).

3.4 <u>The response near the cut off frequency</u>

a) <u>Inviscid linear theory</u>. The solution of (3.2) for $q=0$, $r=0$, satisfying the boundary conditions is

$$A(x) = \begin{cases} (-p)^{-1/2} \exp[-(-p)^{1/2} x + i\pi] & , \ p<0 \\ p^{1/2} \exp[-i(p^{1/2} x - \pi/2] & , \ p>0 \end{cases} .$$

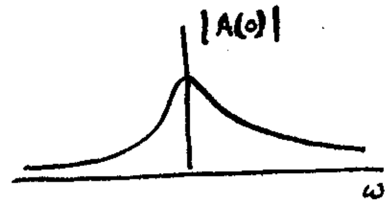




b) <u>Viscous linear theory</u>. The solution of (3.2) for $r=0$, $q \neq 0$,

$$A = h \exp\{-(K_1 + iK_2)X\} \quad ,$$

where $h = (-K_1 + iK_2)/(K_1^2 + K_2^2)$; $K_1 = \{\frac{1}{2}[-p + \sqrt{(p^2+q^2)}]\}^{1/2}$; $K_2 = -q\{2[-p + \sqrt{(p^2+q^2)}]\}^{-1/2}$.



c) Nonlinear inviscid theory. This has $q = 0$ in (3.2), & for the ensuing discussion I shall assume $r > 0$. The arguments can fairly easily be modified to the case $r < 0$.

Introduce real-valued functions R, ψ, representing the amplitude & the phase of the waves, by $A(x) = R(x)\, e^{i\psi(x)}$. Then (3.2) can be written as

$$R'' - R\psi'^2 + pR + rR^3 = 0 \, ,$$

and

$$(R^2\psi')' = 0 \, .$$

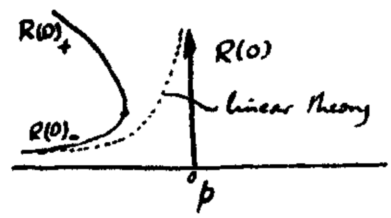
The second of these equations indicates that $R^2\psi' = M$, where M is a constant. Using this in the first equation, multiplying by R' & integrating, we have that

$$R'^2 + M^2R^{-2} + pR^2 + \tfrac{1}{2}rR^4 = N \, , \qquad (3.3)$$

where N is a constant. Thus (3.3) is an equation for the wave the boundary condition at the wavemaker & the nature of the far field from which N & M are to be evaluated. Some of the possibilities are:

(a) Solutions that have the same character as the linear solutions for $p \ll 0$. These decay monotonically, to zero, & this determines both N & M as being zero. The boundary condition at $x=0$ then implies that $R'(0) = -1$ & $\psi(0) = \pi$.

Thus we find that $R(0) = \frac{1}{r}\left\{-p \pm \sqrt{(p^2-2r)}\right\}$.

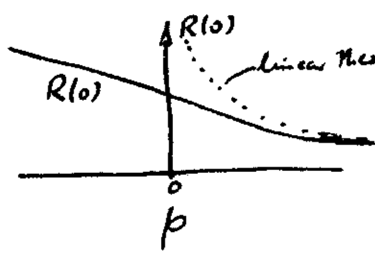


Then $R(x)$ can be found by a further integration.

Note that these solutions can apply only when $p \leq -\sqrt{(2r)}$.

(b) Solutions that have the same character as the wave-like solutions for $p \gg 0$. These solutions are of the form $A = R(0)\, e^{i(cx+\chi)}$, where $R(0)$, c & χ are all real constants. It follows from the basic equation & the boundary condition at $x=0$, that $\chi = -(\pi/2)\,\mathrm{sgn}\, c$ & (for $r>0$), that

$$R(0) = \left\{\left[-p + \sqrt{(p^2+4r)}\right] / 2r\right\}^{1/2}.$$



This solution is possible for all $p \in \mathbb{R}$.

Care should be exercised in interpreting this solution, especially for $p<0$, because the waves satisfy a sort of generalized linear questionable whether or not this mode could be realized in practice.

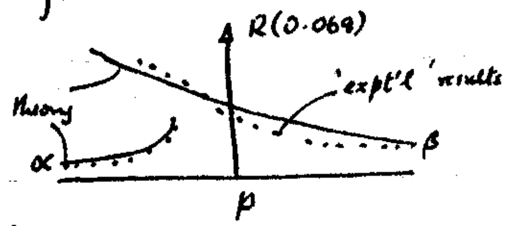
(c) There is a two-parameter family of spatially periodic solutions depending on the nature of the far field. In fact, it follows from (3.3) that

$$R' = \pm R^{-1}\left\{-\tfrac{1}{2}rR^6 - pR^4 + NR^2 - M^2\right\}^{1/2}, \qquad (3.4)$$

& for R to be well defined for each X this implies that R must lie in the interval $[R_L, R_U]$, where R_L^2 & R_U^2 are given by the two positive roots of the polynomial $\left\{-\tfrac{1}{2}rY^3 - pY^2 + NY - M^2\right\}$.

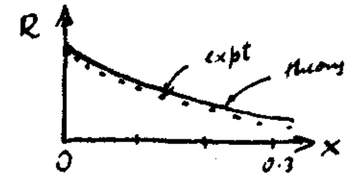
3.5 Experiments

A set of experiments was made in which the amplitude $R(X)$ was measured. The responce near the wavemaker ~~R(0)~~ gave values in fairly good agreement with the theoretical predictions from the inviscid nonlinear solutions described above in § 3.4. The

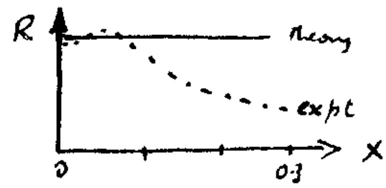


empirical evidence was clear that the linear viscous model could not describe the experimental observations, except for very small values of the wavemaker amplitude ε.

The variation of R with X was also found to be in fairly good agreement with the theory.



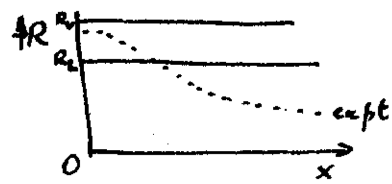
Waves corresponding to branch α



Waves corresponding to branch β

The agreement between the theory & experiment for waves corresponding to the solution branch labelled α in the above sketches was rather close, but for branch β the experiment indicated a variation along the channel that was not predicted theoretically. However the channel was rather short & it looked as though the beach, which began at $X \sim 0.32$, was probably influencing the structure of the motions, & that would probably account for the observed variation with X. At this stage a draft manuscript was prepared, & in the process of doing this we found, in the course of checking that the experimental results agreed with the linear theory at large values of $|p|$, that the phase of the waves at $p \ll 0$ did not agree with the linear theory; the error was about $\pi/4$. The resolution of this difficulty was a very small amount of 'flexing' of the wavemaker, & on stiffening the paddle this discrepancy was removed. But this meant we should repeat all the experiments.

Thus, a new set of experiments was carried out & the measurements of R were very much the same as those described above. At about this time we realized how to take account of the presence of the beach, in an empirical way, & check whether or not the inviscid model could describe the experimental situation. The idea was that the constants N & M in (3·4) should be determined empirically say by specifying the values of $R(0)$ & $\psi(0)$. Then the range of the solution set $R(x)$ could be determined in the way indicated in §3.4. The values of R_L & R_U thus found, were as sketched.



Since R_L & R_U were very insensitive to errors in the measurement of $R(0)$ & $\psi(0)$ it would accordingly appear that the inviscid model could not possibly describe the empirical results.

However, the bandwidth of the nonlinear response is quite small. The stability of the frequency control for the wavemaker was very good (about 1 part in 10000 variability over a period of hours), but the actual value of γ is determined by the width & the depth of the channel along the channel might 'force' the solution into one of the decaying type (for branch α) & measurements of the width & depth suggested that this was very likely to be the case.

Therefore a new, longer channel was built, with these errors kept as small as possible, & a new set of measurements made. The results were essentially as described above, with the same kinds of conclusion. Thus, if the model is to make any sense at all (in terms of the experimental situation) it would appear that the damping effects must be important. To understand the influence of the damping we resorted to numerical methods

3.6 Nonlinear dissipative solutions

Fortunately it is possible to arrange the numerical calculations so that they can proceed as solutions to an initial-value, rather than to a boundary-value, problem. This happens because, with damping present, all the solutions of the differential equation tend to zero as $x \to \infty$. Thus, for sufficiently large x, the solutions are necessarily dominated by the linear equation with dissipation & must be of the two equations similar to those given in § 3.4 (c). But, for the numerical computations it is convenient to transform these equations, & we have used two different versions.

(i) Let R be the independent variable & define $V = R'$ & $W = \psi'$. Then we have that

$$\frac{d}{dR}\left(\tfrac{1}{2}V^2\right) = \left(W^2 - p - rR^2\right)R$$

and

$$\frac{dW}{dR} = -q/V - 2W/R.$$

For given initial values for V & W (which are determined from the linear viscous solution) we can integrate these equations. The integration can be continued until the point is reached at which V vanishes, but to retain numerical accuracy it is necessary to curtail the integration before this happens, & change to new variables. We then use $S = R^2W$ (which is a strictly monotonic function of x) & let the dependent variables be V & R. Thus we have that

$$dV/ds = \left(rR^6 + pR^4 - S^2\right)/\left(qR^5\right),$$

and

$$dR/ds = -V/qR^2.$$

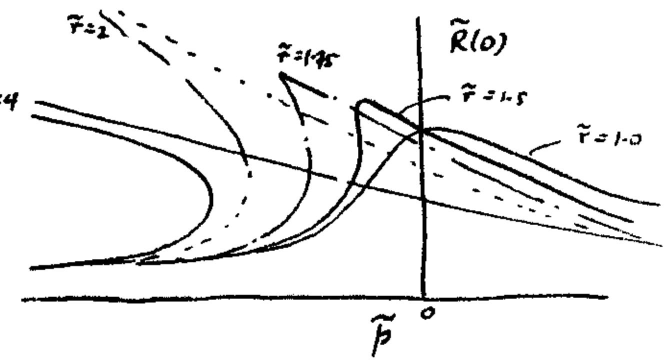
This integration can be continued as long as desired.

Thus we have converted the computation into one of solving by the point at which the boundary condition is satisfied, & the distance from the origin is obtained by an additional quadrature. Thus, for a given far field there may be more than one point of the solution set at which the boundary condition is satisfied, & each point represents a possible origin for a solution to the forced motion.

(Note that the boundary condition may be written as
$J \equiv V^2 + R^2W^2 - 1 = 0$ (case 1), or $J \equiv V^2 + S^2R^{-2} - 1 = 0$ (case 2).)

In carrying out the numerical computations we have made $q = 1$, so effectively we are examining a scaled problem. The solutions for $|R(0)|$ are of the form shown, where the tilde refers to scaled variables.

The results of these computations showed a number of interesting features.



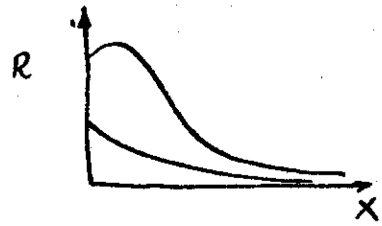
i) For $\tilde{r} \lesssim 1.43$ there appeared to be only one solution for any given $\tilde{p}$.

ii) For $1.43 \lesssim \tilde{r} < 2.0$ there was a range of $\tilde{p}$ for which there were three solutions for $\tilde{R}(0)$ & outside this bounded interval there was only one solution.

iii) For $\tilde{r} \geq 2.0$ the interval of $\tilde{p}$ for which three solutions were found appeared not to be bounded below.

iv) The results of these computations suggest that the dissipative theory gives radically different answers from the inviscid theory.

computations was as shown in the sketch,



for values of the parameters corresponding to the experiments described above. Thus, the dissipative effects have made a large change in the variation of $R(x)$ compared with the nonlinear theory. In this sketch I have shown the two solutions that are observed experimentally, at an appropriate value of p & for a suitably large ε (larger ε corresponds to larger values of $\tilde{F}$). These results were in fairly good agreement with the experimental observations.

In summary it would appear that the weakly nonlinear, dissipative theory gives a good description of the general features of the experiments, which is all one could expect from an approximate theory of this kind.

I think this work illustrates just how difficult it can be to make an experiment in which one has sufficient confidence to claim that a theory has been invalidated; & even then one can't be sure. Thus, just as one should have a healthy respect for both the power & the abuse of mathematics, the same criteria should also be applied to experiment.