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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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THE TRIPLE-DECK AND SEPARATION

(Part I)

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Introduction

The aim in these lectures is to describe the asymptotic theory of the flow of viscous fluids past a flat plate and related problems. For an incompressible fluid the mathematical problem is to determine a stream function $\psi(x,y)$ with the properties that

1) $$R^{-1}\nabla^4\psi + \frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0$$

2) $$\psi \to y \quad \text{as} \quad x^2+y^2 \to \infty$$

3) ψ and its normal derivative vanish on a closed convex curve S.

4) Outside this curve ψ is a sufficiently smooth function of x, y.

Here R is the Reynolds number, defined in physical terms as $U_\infty a/\nu$, U_∞ being the fluid velocity at infinite distances from the body, l a representative length of the body and ν the kinematic viscosity.

There are two cases to consider; $R \ll 1$ and $R \gg 1$. Each involves some subtlety of argument but the second is by far the more complicated and has only been elucidated in the last ten years. I begin with the low Reynolds number expansion.

Slow flows $R \ll 1$

The obvious thing to do here is to neglect the inertial terms when the governing equation reduces to

$$\nabla^4\psi = 0$$

but there is no solution to this equation satisfying the boundary conditions. This may be seen quite easily by considering a circular cylinder $r=1$. The solution should have the form $\psi = f(r)\sin\theta$, but on substituting this form into the equation we get

$$f(r) = Ar^3 + Br + Cr\log r + D/r.$$

In order to satisfy the boundary conditions at infinity $A=C=0$ $B=1$ whence it is impossible to satisfy the next conditions at $r=1$ $(\psi = \psi_r = 0)$. The reason for the failure is that the inviscid terms are not negligible when $r \sim R^{-1}$ no matter how small R is. Thus with the form for f above $\nabla^4\psi$ contains terms $O(r^{-3})$ $\{A=0\}$ whereas the inviscid terms are $O(r^{-2})$ when r is large. At these values of R the flow is almost uniform and so ψ may be replaced in the governing equations by y. They then reduce to

$$\nabla^4\psi - R\frac{\partial}{\partial x}(\nabla^2\psi) = 0 \qquad \text{(Oseen's equations)}$$

which is actually a uniformly valid approximation to the fundamental equation for all r in the limit as $R \to \infty$. This is because when $r \sim 1$ the term proportional to R is negligible compared with the biharmonic term.

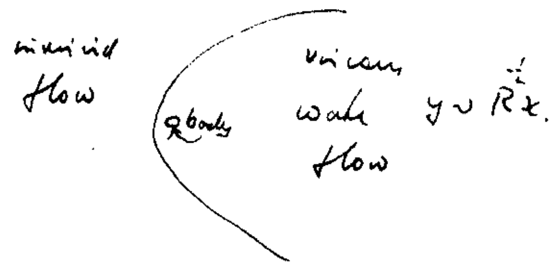
We now apply this equation to the flat plate problem. A fundamental solution takes the form

$$u = \frac{\partial\phi}{\partial x} + \frac{1}{R}\frac{\partial\chi}{\partial x} - \chi, \qquad v = \frac{\partial\phi}{\partial y} + \frac{1}{R}\frac{\partial\chi}{\partial y}$$

where $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$ are the components of velocity in the and $$\nabla^2\chi = R\frac{\partial\chi}{\partial x}.$$

The basic function for $\phi = \log r$ $\qquad r^2 = x^2+y^2$

and for χ is $e^{\frac{1}{2}Rx}K_0(\frac{1}{2}Rr)$ where K_0 is a Bessel function. Notice that when $Rr \gg 1$ χ is exponentially small except when $y = O(xR^{-1/2})$ and x is positive. This region is the wake region of flow past a body



For a flat plate the boundary conditions are that

$$u = v = 0 \quad \text{when } y \to 0 \quad |x| < 1$$

and so ϕ, χ must be smooth functions for all $|y|>0$ if $|x|<1$ and for all y if $|x|>1$. This suggests that we look for a solution in the form

$$\phi = \frac{1}{2}\int_{-1}^{+1} f(x')\log\{(x-x')^2+y^2\}\,dx' + x$$

$$\chi = R\int_{-1}^{+1} g(x')\,e^{\frac{1}{2}R(x-x')}K_0\{\tfrac{1}{2}R\sqrt{(x-x')^2+y^2}\}\,dx'$$

On applying the boundary condition on v it follows that

$$f(x) = g(x)$$

and on applying the condition $u=0$ when $y\to 0$ $|x|<1$ we get

$$1 + \int_{-1}^{+1}\left\{\frac{f(x')}{x-x'} - \left(\frac{R}{2}e^{\frac{1}{2}R(x-x')}K_0\{\ \} - \tfrac{1}{2}R\,e^{\frac{1}{2}R(x-x')}K_1\{\ \}\right)f(x')\right\}dx'$$

Expanding in powers of R and neglecting R^2, this reduces to

$$1 + \frac{R}{2}\int_{-1}^{+1} f(x')\left\{\log\tfrac{1}{4}R(x-x') + \gamma - 1\right\}dx' \qquad \gamma = .577\ldots$$

with unique, physically acceptable solution

$$f(x) = \frac{\bar{A}}{\sqrt{1-x^2}}, \qquad \bar{A} = -\frac{2}{\pi R\{\log R/8 + \gamma - 1\}}.$$

Notice that only the leading term of this solution is strictly correct for the basic Navier-Stokes equation i.e.

$$f(x) = -\frac{2}{\pi R \log R\sqrt{1-x^2}} + O(R^{-1})$$

The main features of this solution also apply to the solution for any value of R.

Thus 1) at large distances

$$\phi \simeq \log r \int_{-1}^{+1} f(x)dx + x = -\frac{2}{R\log R}\log r + x$$

so that the plate acts like a source at large distances with fluid having a broad flux away from it.

2) at large distances

$$\chi \simeq R e^{\frac{1}{2}Rx} K_0(\tfrac{1}{2}Rr) \int_{-1}^{+1} g(x)\,dx$$

and the fluid has wake like properties with an inflow towards the plate in this region compensating for the outflow under (1).

3). The vorticity at the plate $\left(= -\frac{\partial\chi}{\partial y}\text{ in Oseen flow}\right)$ has a singularity at both leading & trailing edges $\left(\frac{\partial\chi}{\partial y} = -R\pi g(x)\text{ when } y=0\right)$ which is also preserved at all values of R

Thus if we expand the solution of the basic equations about, for example, the leading edge, by writing $x+1 = s\cos\theta$, $y = s\sin\theta$ and $s \ll 1$ and assuming that $\psi \sim s^{3/2}$ near $s=0$ the form of the solution near $s=0$ is $\psi \sim Bs^{3/2}(\cos\theta/2 - \cos 3\theta/2)$ $\left(\text{so that } \psi = \psi_\theta = 0 \text{ when } \theta = 0, 2\pi\right)$

and the correction due to inviscid terms is $O(s^2\log s)$ and turns out to be the third term in the expansion of ψ in powers of s about $s=0$.

We may expect therefore that the singularity at $x=\pm 1$ revealed by our study of the low Reynolds number flow is preserved for all R & so it seems to prove. [Notice however we are not proving anything in this lecture series – and except for $R \ll 1$ everything we shall do is conjectural – in the best mathematical sense. For $R \ll 1$ all our remarks can & have been substantiated]

4. Numerical studies have been carried out by Dennis (unpublished) for values of R up to 400 and they confirm the persistence of the singularities at $x = \pm 1$. They also show that the graph of the vorticity on the plate ($\partial u/\partial y$) becomes asymmetric generally approaching a limit $\frac{0.332}{2^{1/2}\sqrt{1+x}} R^{1/2}$ (the Blasius limit) but still with a singularity at $x = 1$.

Let us now look at the asymptotic expansion of the solution for $R \gg 1$. This gives rise to the classical boundary layer theory originated by Prandtl but is much more as I shall show you. The heuristic method of approach which we adopt is hierarchical i.e. by constructing a sequence of solutions non of which is complete but the errors induced are systematically of higher order in R^{-1}. Thus the solution has the capability of being an asymptotic expansion in the strict mathematical sense but it has not yet been proved to be one.

Step 1. Neglect viscous terms

$$\Longrightarrow \quad \nabla^2\psi = f(\psi) \qquad f \text{ arbitrary}$$

since at infinite distances upstream flow is (presumably – but check) undisturbed $f \equiv 0$ for unseparated flows

$$\therefore \quad \psi = y$$

This solution actually satisfies full equation but is not uniformly valid because it does not satisfy no-slip condition at the plate ($\psi_y \neq 0$ at $y = 0$ $|x| < 1$)

Step 2 Assume solution of Step 1 is almost uniformly valid i.e. breakdown is confined to neighbourhood of the plate. Then to make viscous terms significant $\frac{\partial}{\partial y} \gg 1$. It is also clear

The balance between viscous & inviscid terms requires that in fact

$$\frac{\partial}{\partial y} \sim R^{1/2} \quad \text{so let} \quad y R^{1/2} = Y$$

and $\psi \sim R^{-1/2}$ i.e. $v = R^{-1/2} V(x, Y)$ in this region (where $\frac{\partial}{\partial y} \gg 1$)

Take the limit $R \to \infty$ & get

$$u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2} \quad : \quad \frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0 \quad : \quad \frac{\partial p}{\partial Y} = 0$$

This is Prandtl's boundary layer equation set.

Outside boundary layer we should return to solution of step 1 i.e. $u = 1$, $v = 0$ (to order 1) Hence $p \to 0$ as $Y \to \infty$ & $\therefore p \equiv 0$

$u = V = 0$ on the plate $\overset{Y=0}{L}$, $|x| \le 1$: $V = \frac{\partial u}{\partial Y} = 0$ on $Y = 0$, $|x| > 1$ since flow is smooth outside the plate & symmetrical about $Y = 0$.

This is a parabolic equation and a consistent solution can be found as follows

2a. $x < -1$ $\quad Y \sim 1$

$$u \equiv 1 \quad , V = 0.$$

2b. Boundary Layer

$$\psi = R^{-1/2} (x+1)^{1/2} F\left(\frac{Y}{\sqrt{x+1}}\right) \qquad |x| < 1$$

where $\quad F''' + \frac{1}{2} F F'' = 0 \quad$ and $\quad F(0) = F'(0) = 0, \quad F'(\infty) = 1.$

This solution satisfies the conditions at $Y = 0$ $|x| < 1$ matches with the solution at $x = -1$ for $Y > 0$ and to leading order approaches inviscid solution as $Y \to \infty$

Notice however that there is a singularity at $x = -1$ and $V \not\to 0$ as $Y \to \infty$ $\quad \left(V = \frac{1}{2} R^{-1/2} (x+1)^{-1/2} \left(\frac{Y}{\sqrt{x+1}} F' - F \right) \right.$ & $\to$ non zero limit as $Y \to \infty$ $\left. \right)$

At $x = +1$, let

$$u = U_0(Y).$$

2c. At $x = +1$ there is a change in the boundary condition at $Y = 0$ and this gives rise to a double-structure in the solution of the boundary layer equations in $x > 1$. This is necessary because $\partial u/\partial Y$ drops discontinuously from a value $U_0'(0) = a_1$ at $x = 1-0$ to zero at $x = 1+$. Define $x - 1 = \bar{x}$ and consider the

Inner region, where $Y \sim \bar{x}^{1/3}$:

Specifically write $\eta = \dfrac{Y}{3(2\bar{x})^{1/3}}$

$$\text{and} \quad \psi = 2^{-1/3}\,\bar{x}^{2/3} \sum_{n=0}^{\infty} (\tfrac{1}{4}\bar{x})^n f_n(\eta).$$

Then $f_n(0) = f_n''(0) = 0$ and

$$f_n \sim \frac{3a_{n+1}}{3n+2}\, 6^{3n+1} \eta^{3n+2} \quad \text{as } \eta \to \infty$$

where $$U_0(Y) = \sum_{n=0}^{\infty} a_{n+1} Y^{3n+1}$$

By direct substitution into the boundary layer equations we get a set of ordinary differential equations for the functions f_n all of which are linear after the first

$$f_0''' + 2f_0 f_0'' - f_0'^2 = 0.$$

Numerical integration of this equation gives

$$f_0'(\eta) - 18a_0\eta \to 6{\cdot}134\, a_0^{2/3} = 3\delta a_0 \quad \text{as } \eta \to \infty$$

This means that when $\eta \gg 1$, i.e. $\bar{x}$ small and Y not too small

$$\psi \simeq \tfrac{1}{2} a_0 \left[Y^2 + 2Y\delta(2x)^{1/3} + \delta^2 (2x)^{2/3} \right] + O(Y^5, x)$$

The solutions for f_n in fact appear to be unique.

2d. The outer region when $\bar{x}$ is small ?? has a solution in the form

$$\psi = \Psi_0(Y) + (2\bar{x})^{1/3}\Psi_1(Y) + \dots$$

and again by direct substitution into the equation and using $\Psi_0'(Y) = U_0(Y)$ we obtain $\Psi_1 = A_1 U_0^*(Y)$ $\quad A_1$ constant

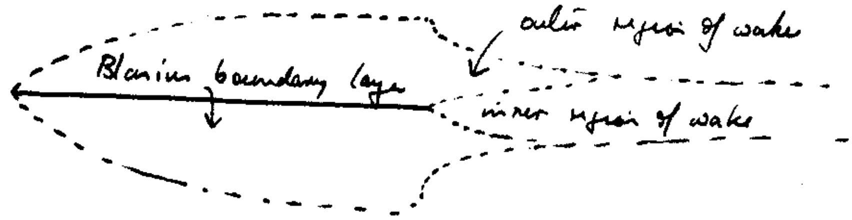
On matching with the inner solution

$$A_1 = \delta$$

etc.

When $\bar{x} \sim 1$ it is best to proceed by numerical methods and when $\bar{x} \gg 1$

$$u = 1 - \frac{a_1\sqrt{2}}{\sqrt{\pi \bar{x}}} e^{-Y^2/4\bar{x}} + \dots$$



3 Further improvements needed are

1) To remove singularities at $x = \pm 1$ $y = 0$

2) To match with inviscid flow.

3a. From 2. we have that as $Y \to \infty$

$$V \not\to 0$$

$-1 < x < 1$ $\quad V \to -\frac{1}{2\sqrt{x+1}} \underset{\theta \to \infty}{Lt} \left(F(\theta) - \theta F'(\theta)\right) = \frac{1.7321}{2\sqrt{x+1}}$

if $x > 1$ $\quad V \to -\frac{1}{3}\left(\frac{2}{\bar{x}^2}\right)^{1/3} \Phi_1(\infty) - \dots$ as $Y \to \infty$ when $\bar{x}$ is small

$V \propto \bar{x}^{-3/2}$ when $Y \to \infty$ & $\bar{x}$ is large.

Let $V(\infty) = \delta'(x)$ when $x > -1$
$\qquad = 0 \qquad x < -1$

δ is a known function

and write

$$u = 1 + R^{-1/2} u_1(x,y)$$
$$v = R^{-1/2} v_1(x,y) \operatorname{sgn} y$$

when $y \sim 1$ $\quad x \sim 1$.

Then $(u_1, v_1) = \operatorname{grad} \phi$ when $\phi = \frac{1}{2\pi}\int_{-1}^{\infty} \delta'(x_1) \log\left((x-x_1)^2 + y^2\right) dx_1 \quad y > 0$

Notice that ϕ acts like a source

36. Singularities at the leading edge $x=-1$, $y=0$

as $x \to -1$ $\quad v \to \infty$

$$\frac{\partial}{\partial x} \sim \frac{1}{x+1} \qquad \frac{\partial}{\partial y} \sim R^{1/2} / \sqrt{x+1}$$

& assumption of boundary layer theory – that $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$ – breaks down when $1+x \sim R^{-1}$, $y \sim R^{-1}$.

Now all terms are equally significant and solution can be completed by using parabolic coordinates

$$x+1 = R^{-1}(\xi^2 - \eta^2) \qquad y = R^{-1} \cdot 2\xi\eta$$

$x > -1 \quad y = 0 \iff \eta = 0, \xi > 0$ & here $\psi = \partial\psi/\partial\eta = 0$

$x < -1 \quad y = 0 \iff \eta > 0, \quad \xi = 0$ & $\quad \psi = \psi_{\xi\eta} = 0$

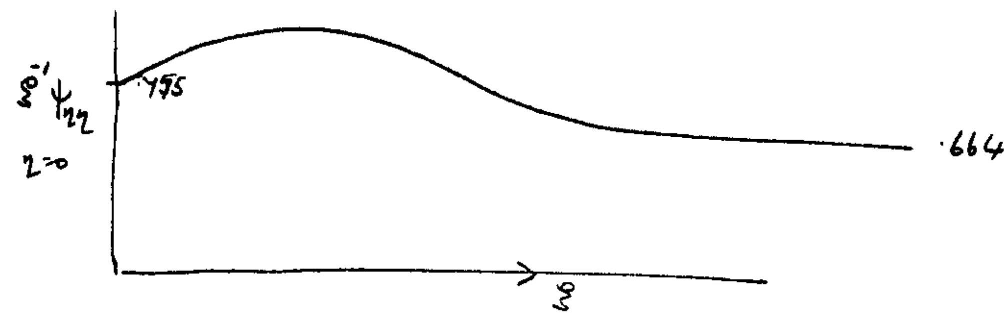
as $\eta \to \infty$ we have uniform flow – $\psi_\eta \to 2\xi$

as $\xi \to \infty$ we hope to get Blasius flow

so $\psi \to \xi F_0(2\eta)$

Governing equation $\quad \nabla^2\left(\frac{1}{\xi^2+\eta^2}\nabla^2\psi\right) = \dfrac{\partial\left(\psi, \frac{1}{\eta^2+\xi^2}\nabla^2\psi\right)}{\partial(\eta, \xi)}$

solution by Van de Vooren & Dijkstra.



Skin friction varies from $\dfrac{\cdot 378}{\sqrt{x+1}} R^{-1/2}$ when $(x+1)R \ll 1$

to $\dfrac{\cdot 332}{\sqrt{x+1}} R^{-1/2}$ when $(x+1)R \gg 1$.

3c. The problems of the trailing edge.

1. as with the ~~trailing~~ leading edge there is a breakdown near $x=1$, $y=0$ because $v \sim R^{-1/2}(x-1)^{-1/3}$ (Refer to $\mathcal{L}_0$)

and $u \sim R^{1/2} y$ (Blasius)

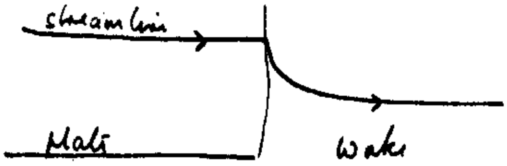
then $\frac{\partial}{\partial x}$ & $\frac{\partial}{\partial y}$ are comparable

when $x-1 \sim y$ i.e. $(x-1) \sim y \sim R^{-3/4}$ $(v \sim u)$

and this region could be troublesome.

2. More seriously the hierarchical principle breaks down when $x=1$ $y \neq 0$

because δ' is not bounded



The induced pressure must satisfy

$$\frac{\partial p}{\partial y} = -\delta''(x) \text{ at } y=0 \quad \left(\sim (x-1)^{-5/3} \text{ if } x>1.\right.$$

and we find in fact that

$$p \sim -p_0(|x-1|)^{-2/3} \quad \text{if } x \to 1- \qquad p_0 \text{ constant.}$$

$$p \sim \tfrac{1}{2}p_0(x-1)^{-2/3} \quad \text{if } x \to 1+$$

This singular pressure breaks the hierarchical ~~to~~ chain.

4. The trailing edge Triple-Deck.

The purpose of this structure is to accommodate the singularity in p as $x \to 1\pm$. and as the name implies consists of three superposed layers.

Define $X = R^{3/8}(x-1)$ $\quad Y = R^{1/2}y$ $\quad \epsilon = R^{-1/8}$

4.1 Main deck Physically, this occupies most of boundary layer near trailing edge

Write $u = U_0(Y) + \epsilon U_1(X,Y) + \cdots$ expansion in powers of ϵ

$v = \epsilon^2 V_1(X,Y) + \cdots$

$p = \epsilon^2 P_2(X,Y) + \cdots$

Substitute these forms into the basic equation and equate successive powers of ϵ to zero. We get

$$V_1 = -A'(X)U_0(Y) \quad , \quad U_1 = A(X)\,U_0'(Y)$$

where A is an arbitrary function of X

and P_2 is independent of Y

As $Y \to \infty$ i.e. leaving main deck upwards

$$u \to 1 + O(\epsilon^2) \quad , \quad v \to -\epsilon^2 A'(X), \quad p = \epsilon^2 P_2(X)$$

<u>Upper deck</u> Here the scaling for x and y must be the same

$$y = R^{-3/8}\hat{y} \quad , \quad \hat{y} = R^{-1/8} Y, \quad x-1 = R^{-3/8}\hat{x} \qquad \hat{x} = X$$

$$u = 1 + \epsilon^2 \hat{u}_2(\hat{x}, \hat{y}) + \cdots$$

$$v = \epsilon^2 \hat{v}_2(\hat{x}, \hat{y}) + \cdots$$

$$p = \epsilon^2 \hat{p}_2(\hat{x}, \hat{y}) + \cdots$$

In order to effect a match with the main deck

$$\hat{p}_2(\hat{x}, 0) = P_2(X) \quad : \quad \hat{v}_2(\hat{x}, 0) = -A_0'(\hat{x})$$

On substituting these forms into the full equations we find $\hat{u}_2$, $\hat{v}_2$, $\hat{p}_2$ satisfy the linearised inviscid equations and hence since we need $\hat{u}_2$ $\hat{v}_2 \to 0$ as $\hat{x}^2 + \hat{y}^2 \to \infty$ it follows that

$$P_2(\hat{x}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A'(\hat{x}_1)\, d\hat{x}_1}{\hat{x} - \hat{x}_1} \qquad (*)$$

<u>Lower deck</u> as $Y \to 0$ i.e. leaving main deck downwards

$$u \to a_1(Y + \epsilon A(X)) \quad , \quad v \to -a_1 Y \epsilon^2 A'(X) \qquad p \to \epsilon^2 P_2(X)$$

Lower deck scaling is

$$u = \epsilon \tilde{u} \quad , \quad v = \epsilon^3 \tilde{v} \quad , \quad p = \epsilon^2 P_2(X) \quad , \quad x-1 = R^{-3/8}\tilde{x}, \quad \tilde{x} = X$$

$$\tilde{y} = R^{+5/8} y \quad , \quad \tilde{y} = R^{1/8} Y.$$

$\tilde{u}$, $\tilde{v}$ are functions of $\tilde{x}$, $\tilde{y}$.

For leading terms in the expansion of the solution of the basic

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \;, \quad \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{dP_2}{d\tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}$$

with boundary conditions

$\tilde{u} = \tilde{v} = 0 \qquad \tilde{y} = 0 \qquad \tilde{x} < 0$ No-slip.

$\tilde{v} = \frac{\partial \tilde{u}}{\partial \tilde{y}} = 0 \qquad \tilde{y} = 0 \qquad \tilde{x} > 0$ Wake symmetry

XX

$\tilde{u} \to \tilde{y} a_1 \quad \tilde{x} \to -\infty \quad \tilde{y} > 0$: to match with Blasius

$\tilde{u} \to a_1(\tilde{y} + A^*(\tilde{x}))$ as $\tilde{y} \to \infty$ to match with main deck.

These two equations are sufficient to enable us to determine lower deck solution & hence ~~the~~ the triple-deck properties. The numerical procedure is to guess a form for $A(\tilde{x})$ solve (xx) to obtain a form for p_2 and then solve (x) to obtain a better form for A^*, after integration with respect to $\tilde{x}$. This procedure is rapidly convergent.

The most striking result of this theory is the estimate for the drag on one side of a flat plate of length l in a fluid of kinematic viscosity ν and undisturbed velocity U_∞.

We note that this depends on

$$\int_{-1}^{+1}\left(\frac{\partial u}{\partial y}\right)_{y=0} dx \qquad (0)$$

and that $\left.\frac{\partial u}{\partial y}\right|_{y=0} = \left.\frac{\partial \tilde{u}}{\partial \tilde{y}}\right|_{\tilde{y}=0}$

Hence the triple deck adds a term $R^{-3/8}\int_{-\infty}^{0}\left(\frac{\partial \tilde{u}}{\partial \tilde{y}} - 1\right)d\tilde{y}$ to (0)

and the drag coefficient $C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 l} = \frac{1.328}{R^{1/2}} + \frac{2.67}{R^{7/8}} + O(R^{-1})$

the number 2.67 having been computed from the triple-deck ~~the~~ equations by Melnik & Chow. A comparison between the asymptotic formula & Dennis' numerical results is set out below

R	C_D (Dennis)	Blasius	Asymptotic
1	3.79	1.33	4.1
10	[illegible]	0.421	0.777