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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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FINITE ELEMENT METHODS FOR OPERATORS OF 4th ORDER

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0. INTRODUCTION.

We shall see in the following someone of the most interesting f.e.m. for solving the "model problem:

$$(0.0) \qquad \begin{cases} \Delta^2 w = p & \text{in } \Omega \\ w = \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \qquad \left(\Delta^2 = \nabla^4 = \Delta(\Delta) = \nabla^2(\nabla^2)\right)$$

In particular we shall see:

Lecture 1 - Classical displacement methods
" 3 - Mixed methods (I)
" 4 Mixed methods (II)
" 5 Hybrid methods

In the Lecture 2 we shall see some abstract results on approximation of saddle point that has to be used in the others lectures.

The results reported here can be founded, with appropriate references, on

G. STRANG - G. FIX "An analysis of the f.e.m." (Prentice Hall 1973)

F. BREZZI - L.D. MARINI "Hybrid f.e.m. for plate bending problems - R.A.I.R.O. (1975)

F. BREZZI - P.A. RAVIART - " Mixed f.e.m. for 4th order elliptic equations" (Proc. Conf. Numerical Analysis - DUBLIN 1976 - J.J.H. MILLER editor — to

LECTURE 1 - CLASSICAL "DISPACEMENT METHODS"

Scheme of the lecture:

1.1 The one dimensional case.
1.2 Variational formulation.
1.3 Abstract approximation.
1.4 Some examples in one dimension.
1.5 Variational formulations for the 2-dimensional case
1.6 Some examples in two dimensions.

1.1 - Consider the one dimensional formulation of problem (0.0):

$$(1.0) \quad \begin{cases} u^{(iv)}(x) = p(x) \text{ in } \Omega =]b,c[\,, \\ u = u' = 0 \text{ on } \partial\Omega,\ \text{t.i. } u(b)=u'(b)=u(c)=u'(c)=0\,. \end{cases}$$

One can think, from a physical point of view, that $u(x)$, in (1.0), represents the transversal deflection of an elastic beam, which is clamped at the endpoints and acted by a uniformely distributed load $p(x)$.

1.2. - Setting now:

$$V = H_0^2(\Omega),$$

$$a(u,v) = \int_\Omega u''(x)\,v''(x)\,dx,$$

$$(p,v) = \int_\Omega p(x)\,v(x)\,dx,$$

we get the "variational formulation" of problem (1.0):

$$(1.1)\qquad \begin{cases} \underline{\text{Find}}\ u \in V\ \underline{\text{such that}}: \\ a(u,v) = (p,v)\quad \forall v \in V \end{cases}$$

It is very easy ~~the~~ to check that, since $a(u,v)$ is V-elliptic, problem (1.1) has a unique solution, and that problems (1.0) and (1.1) are equivalent.

1.3. Let now, for each given value of a parameter h, V_h be a finite dimensional subspace of V, and consider the "approximate problem"

$$(1.2)\qquad \begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = (p, v_h)\quad \forall v_h \in V_h \end{cases}$$

Again, (1.2) has a unique solution. Moreover we have the following proposition [cfr. eg [1]]

<u>Proposition</u> 1.1. <u>Let. u and u_h be the solutions of (1.1) and (1.2) respectively; then:</u>

$$\| u - u_h \|_V \leq c \inf_{v \in V_h} \| u - v \|_V \tag{1.3}$$

<u>with c = constant independent from u and from h.</u>

~~Mo~~ It is also easy to see that (1.2) is equivalent to a system of linear algebraic equations; in fact ~~setting~~, if $w^{(1)}, \dots, w^{(N)}$ is a basis for V_h, setting

$$u_h = \sum_{i=1}^{N} \alpha_i w^{(i)}, \tag{1.4}$$

$$A_{ji} = a(w^{(i)}, w^{(j)}), \tag{1.5}$$

$$F_j = (p, w^{(j)}), \tag{1.6}$$

the unknowns $\alpha_1, \dots, \alpha_N$ of the problem are the solution of

$$(1.7) \qquad \sum_{i=1}^{N} A_{ji}\, \alpha_i = F_j \qquad j = 1, \dots, N .$$

These considerations suggest the requirement to be done on the choice of V_h :

1) V_h has to approximate V in the sense that

$$\operatorname*{Inf}_{v \in V_h} \|u - v\|_V \longrightarrow 0$$

2) V_h has to have a basis $w^{(1)}, \dots, w^{(N)}$ such that : A_{ij} and F_j $(i, j = 1, \dots, N)$ are "easy" to compute and (1.7) is "easy" to solve. In particular both the above requirements will be satisfied if V_h has a "local basis"

1.4 - Let us see now some examples of choices for V_h which satisfie requirements 1) and 2). Let, for each given $h > 0$, $\mathcal{T}_h$ be a decomposition of $\Omega =]b, c[$ into subintervals K_r of lenght $\leq h$ (see fig. 1), $r = 1, 2, \cdots, N$

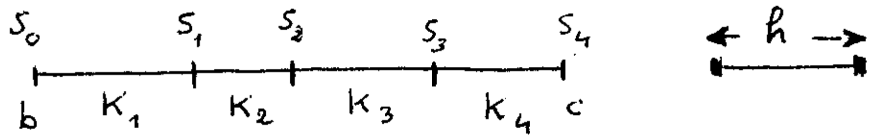


fig. 1

and let S_k be the ~~points~~ "nodes" of the decomposition ($k = 0, 1, \cdots, N$). We set

$$V_h = \{ v \mid v \in C^1(\overline{\Omega}), \; v = v' = 0 \text{ on } \partial\Omega, \; v_{|K_r} \in P_3 \;\; r = 1, \cdots, N \}$$

where P_3 is the spase of all the polynomials of degree ≤ 3. Obviously $V_h \subset V \equiv H_0^2(\Omega)$. It can also be easily proved that

$$\inf_{v \in V_h} \|u - v\|_V \leq c\, h \, \|u\|_{H^4(\Omega)} . \tag{1.8}$$

with c = constant independent of $\mathcal{T}_h$.

In order to construct a "simple" basis for V_h we do first some preliminary remarks.

We shall show, first, that each function of V_h is uniquely individuated by ~~their~~ its values and by the values of its first derivative at the nodes $S_1, \dots, S_{N-1}$ ($N=4$ in fig. 1). In fact it is well known that, for each pair of points ~~S~~ ~~[illegible]~~ R and T, with $R \neq T$ and for each set $\lambda_1, \lambda_2, \mu_1, \mu_2$ of real numbers, there exists a unique polynomial $q(x) \in P_3$ such that:

$$q(R) = \lambda_1, \quad q'(R) = \lambda_2$$

$$q(T) = \mu_1, \quad q'(T) = \mu_2.$$

We shall express this property by saying that the set of "degrees of freedom" of fig. 2 is P_3 - unisolvent

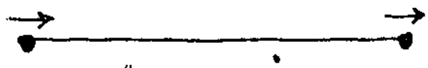


fig. 2

• value of the function
→ value of the derivative

Therefore one can conclude that each function of V_h is uniquely determined by its values and by the values of its first derivative at the "nodes" of the decomposition. In the case of fig. 1, for instance, a basis for V_h can be chosen as follows:

$$
(1.9)\quad \begin{cases}
\varphi^{(1)}(x), \dots, \varphi^{(6)}(x) \in V_h, & \\
\varphi^{(2r-1)}(s_k) = \delta_{rk} & (r = 1,2,3 \; ; \; k = 0,\dots,4) \\
\frac{d}{dx}\varphi^{(2r-1)}(s_k) = 0 & \quad " \qquad\qquad " \\
\varphi^{(2r)}(s_k) = 0 & \quad " \qquad\qquad " \\
\frac{d}{dx}\varphi^{(2r)}(s_k) = 0 & \quad " \qquad\qquad "
\end{cases}
$$

In fig. 3 someone of the functions $\varphi^{(i)}$ defined by (1.9) is represented.

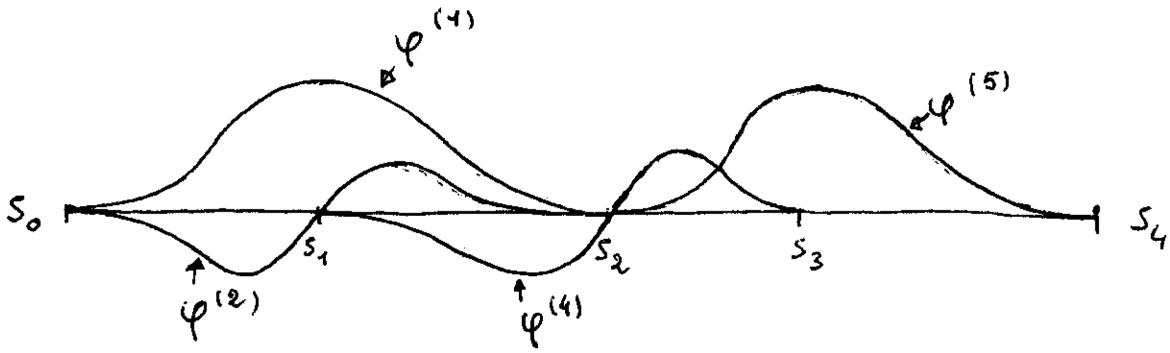


fig. 4

Onother possible choice for V_h is based on the following remark: the set of "degrees" of freedom" of fig. 5 is P_5-unisolvent:

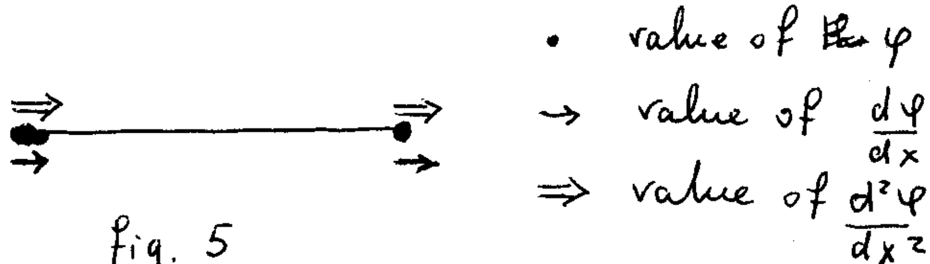


fig. 5

that is, each polynomial of degree ≤ 5 is uniquely determined by: its values, the values of its first derivative and the values of its second derivative at two distinct points.

If one considers now the space

$$V_h = \{ v \mid v \in C^2(\bar{\Omega}),\ v = v' = 0 \text{ on } \partial\Omega,\ v_{|K_r} \in P_5 \quad r = 1, \cdots, N \}$$

~~a bas~~ one can easily see that $V_h \subseteq H_0^2(\Omega)$;

a "simple" basis for V_h can be chosen as follows:

$$(1.10) \quad \begin{cases} \varphi^{(1)}, \cdots, \varphi^{(3N-3)} \in \cancel{V_h}, \psi^{(0)}, \psi^{(N)} \in V_h \\ \dfrac{d^{(3-j)} \varphi^{(3r-j+1)}}{dx^{(3-j)}}(S_k) = (\delta_{rj})(\delta_{rk}) \\ (j = 1, 2, 3;\ r = 1, \cdots, N-1, \quad k = 0, \cdots, N \\ \dfrac{d^2 \psi^{(s)}}{dx^2}(S_s) = 1 \qquad \dfrac{d^j \psi^{(s)}}{dx^j}(S_k) = 0 \quad \psi^{(s)} = \dfrac{d\psi^{(s)}}{dx} = 0 \text{ on } \partial\Omega \\ (s = 0, N;\ j = 0, 1, 2;\ k = 1, \cdots, N-1). \end{cases}$$

Moreover it can be proved that, in this case,

$$(1.11) \qquad \inf_{v \in V_h} \| u - v \|_V \le C h^3 \| u \|_{H^6(\Omega)}$$

with C constant independent of $\mathcal{T}_h$

1.5 Consider now the two dimensional case:

$$(1.12)\quad \begin{cases} \Delta^2 w(x_1,x_2) = p(x_1,x_2) \text{ in } \Omega \subset \mathbb{R}^2, \\ w = \dfrac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

One can easily check that choosing:

$$V = H_0^2(\Omega) \qquad (p,v) = \int_\Omega p v \, dx$$

$$(1.13)\quad a^{(1)}(u,v) = \int_\Omega (\Delta u)(\Delta v)\, dx$$

problem (1.12) can be formulated as follows:

$$(1.14)\quad \begin{cases} \text{Find } w \in V \text{ such that} \\ a^{(1)}(u,v) = (p,v) \ \forall v \in V. \end{cases}$$

On the other hand one ~~has~~ ~~also~~ can put

$$a^{(2)}(u,v) = \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} \right) dx$$

and have another formulation of (1.12).

$$(1.15)\quad \begin{cases} \text{Find } w \in V \text{ such that} \\ a^{(2)}(u,v) = (p,v) \ \forall v \in V \end{cases}$$

Obviously all other form $a(u,v)$ of the kind

$$a(u,v) = \sigma a^{(1)}(u,v) + (1-\sigma) a^{(2)}(u,v), \sigma \in [0,1] \tag{1.16}$$

will be usable; and other choices for $a(u,v)$ different from (1.16) will also be possible.

In general (for the case of the <u>Dirichlet boundary conditions</u>) one has a completely free choice, among all these possibilities, for $a(u,v)$, at least from a mathematical point of view; in the practical cases, some physical consideration could suggest one of them as preferable. For the sake of simplicity we shall refer here to $a(u,v)$ of the form (1.13). In order to approximate problem (1.14) we have to construct a subspace V_h of V; since the arguments of § 1.3 were abstract, they obviously hold even in the two dimensional case; in particular we can apply proposition 1.1 and have the error bound (1.3). Again, with a given choice of the basis of V_h the approximate problem becomes a linear algebraic system like (1.7) and the requirements 1) and 2) are to be done on the choice of V_h.

1.6. Let $\mathcal{T}_h$ be a "regular" decomposition of Ω, like in fig. 6

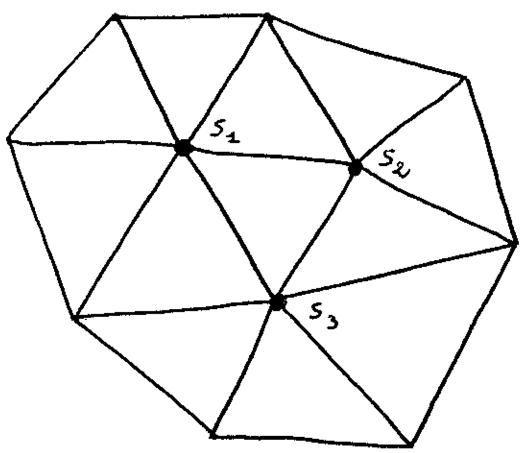


fig. 6

One can now try to construct a suitable space V_h starting from a given set of "degrees of freedom"; for instance it is easy to verify that the set of d.o.f. of figure 7 is P_3-unisolvent:

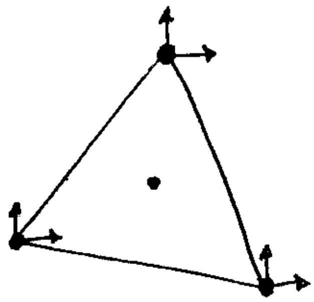


fig 7

• value of φ

→ value of $\frac{\partial \varphi}{\partial x_1}$

↑ value of $\frac{\partial \varphi}{\partial x_2}$

Nevertheless it is immediate to verify that if $\varphi^{(1)}$ and $\varphi^{(2)}$ are two polynomials of degree ≤ 3 defined on neighbouring elements K_1 and K_2 (see fig. 8)

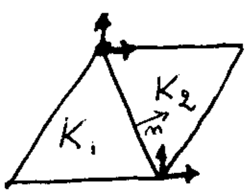


fig. 8

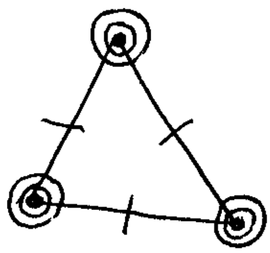
which coincide with their first derivatives at the endpoints of the common edge, then we have

$$\varphi^{(1)} \equiv \varphi^{(2)} \quad \text{on the common edge}$$

but, in general,

$$\frac{\partial \varphi^{(1)}}{\partial n} \not\equiv \frac{\partial \varphi^{(2)}}{\partial n} \quad \text{on the common edge}$$

A different choice of the degrees of freedom is therefore needed; one possible choice is the following, indicated in fig. 9



• value of φ

o values of $\varphi_{/x}$ and $\varphi_{/y}$

O " " $\varphi_{/xx}$, $\varphi_{/xy}$, $\varphi_{/yy}$

+ value of $\varphi_{/n}$

fig. 9

which gives a P_5-unisolvent set of d.o.f. It is now easy to see that if $\varphi^{(1)}$ and $\varphi^{(2)}$ are polynomials of degree ≤ 5 defined on neighbouring elements K_1 and K_2, such that $\varphi^{(1)}$ and $\varphi^{(2)}$ coincides at the endpoints of the common edge with their derivatives up to the order 2 and $\varphi^{(1)}_{/n}$ and $\varphi^{(2)}_{/n}$ coincide at the midpoint of the common edge, then

$$\left.\begin{aligned} \varphi^{(1)} &\equiv \varphi^{(2)} \\ \varphi^{(1)}_{/n} &\equiv \varphi^{(2)}_{/n} \end{aligned}\right\} \text{ on the common edge.}$$

If one choose now:

$$V_h = \Big\{ v \,\Big|\, v \in C^1(\bar{\Omega}),\ v_{|K} \in P_5 \ \forall K \in \mathcal{T}_h,\ v_{/xx}, v_{/xy},$$

$$v_{/yy} \ \underline{\text{continuous at the vertices of}}\ \mathcal{T}_h,\ v = v_{/n} = 0 \ \underline{\text{on}}\ \partial\Omega \Big\}$$

it is easy to see that a "simple" basis can be constructed for V_h in a way similar to (1.10). Moreover one has (cfr. e.g. [1])

$$\operatorname*{Inf}_{v \in V_h} \| u - v \|_V \le c\, h^3 \| u \|_{H^6(\Omega)}$$

with c independent of $\mathcal{T}_h$.

[1] : G. Strang – G. Fix "An analysis of the f. e. m." Prentice Hall

2. APPROXIMATION OF SADDLE POINTS

2.0 – We shall be interested, in the following to the approximation, by means of finite dimensional problems, of saddle point problems of the kind:

$$(2.0)\quad \begin{cases} \underline{\text{find}}\ (u,\psi)\ \underline{\text{in}}\ V\times W\ \underline{\text{such that}}: \\ a(u,v)+b(v,\psi)=(f,v)\quad \forall v\in V, \\ b(u,\varphi)=(g,\varphi)\quad \forall \varphi\in W. \end{cases}$$

In general we shall suppose that V is a real Hilbert space, W a real Banach space, $a(u,v)$ and $b(v,\varphi)$ are bilinear continuous form on $V\times V$ and $V\times W$ respectively and $f\in V'$, $g\in W'$ are given functionals. More precise assumption will be done in each particular case. However we can set, up to now:

$$\|a\| = \sup_{u,v\in V} a(u,v)\,\|u\|_V^{-1}\,\|v\|_V^{-1}$$

$$\|b\| = \sup_{v\in V,\,\varphi\in W} b(v,\varphi)\,\|v\|_V^{-1}\,\|\varphi\|_W^{-1}$$

A finite dimensional approximation of (2.0) will always be of the following kind: we choose two finite dimensional spaces $V_h \subset V$ and $W_h \subset W$ and we consider the "discrete problem":

$$(2.1)\quad \begin{cases} \underline{\text{find}}\ (u_h, \psi_h)\ \underline{\text{in}}\ V_h \times W_h\ \underline{\text{such that}}: \\ a(u_h, v) + b(v, \psi_h) = (f, v)\ \forall v \in V_h, \\ b(u_h, \varphi) = (g, \varphi)\ \forall \varphi \in W_h . \end{cases}$$

We remark immediately that (2.1) is equivalent to a system of linear algebraic equations; in fact if $v^{(1)}, \dots, v^{(N)}$ is a basis for V_h and $\mu^{(1)}, \dots, \mu^{(M)}$ a basis for W_h, setting

$$u_h = \sum_{i=1}^{N} U_i\, v^{(i)} \qquad \psi_h = \sum_{\tau=1}^{M} \mu^{(\tau)} \Psi_\tau^T$$

$$A^{ij} = a(v^{(j)}, v^{(i)}) \qquad B^{i\tau} = b(v^{(i)}, \mu^{(\tau)})$$

$$F_{\xi}^{i} = (f, v^{(i)}) \ , \quad G^{\tau} = (g, \mu^{(\tau)})$$

problem (2.1) becomes the following linear system

$$(2.2)\quad \left(\begin{array}{c|c} A & B \\ \hline B^T & 0 \end{array}\right) \cdot \left(\begin{array}{c} U \\ \hline \Psi \end{array}\right) = \left(\begin{array}{c} F \\ \hline G \end{array}\right)$$

We shall not be interested, in general in the theory of existence and uniqueness of the solution of (2.0). We just suppose that (2.0) has a unique solution; however the following theorem (cfr. [ ]) will provide, in some cases, the existence and uniqueness for the solution of (2.0)

Theorem 1. If W is an Hilbert space and if the following properties are satisfied:

(2.3) $\exists\, \alpha > 0 \quad ; \quad \forall v \in V \quad a(v,v) \geq \alpha \|v\|_V^2$,

(2.4) $\exists\, \beta > 0 \quad ; \quad \forall \varphi \in W \quad \sup_v\, b(v,\varphi)\|v\|_V^{-1} \geq \beta \|\varphi\|_W$

Then (2.0) has a unique solution.

We shall give in the following three different choices of assumptions in order that:

a) the discrete problem (2.1) has a unique solution

b) an "error bound" can be found for the distance between (u,ψ) (exact solution) and (u_h, ψ_h) (approximate solution).

2.1

Suppose that V_h and W_h be such that the following "discrete analogue" of (2.4) is satisfied:

(2.5) $\exists \bar{\beta} > 0 \; ; \forall \varphi \in W_h \; , \; \sup_{v \in V_h} b(v,\varphi)\|v\|_V^{-1} \geq \bar{\beta}\|\varphi\|_W$

Theorem 2. If (2.3) and (2.5) are satisfied, then (2.1) has a unique solution; moreover we have:

(2.6) $\|u-u_h\|_V + \|\psi-\psi_h\|_W \leq c\left(\inf_{v \in V_h}\|u-v\|_V + \inf_{\varphi \in W_h}\|\psi-\varphi\|_W\right).$

Proof. The existence and uniqueness of the solution of (2.1) is an exercise; from (2.0) and (2.1) we get, for all v_h in V_h and φ_h in W_h that

$$\alpha\|u-u_h\|_V^2 \leq a(u-u_h,u-u_h) \leq \|a\|\,\|u-u_h\|\,\|u-v_h\| + a(u-u_h,v_h-u_h) =$$

$$= \|a\|\,\|u-u_h\|\,\|u-v_h\| + b(\psi-\psi_h, u_h-v_h) \leq$$

$$\leq \|a\|\,\|u-u_h\|\,\|u-v_h\| + \|b\|\,\|\psi-\varphi_h\|\,\|u_h-v_h\| + b(\varphi_h-\psi_h,u_h-v_h) =$$

$$= \|a\|\,\|u-u_h\|\,\|u-v_h\| + \|b\|\,\|\psi-\varphi_h\|\,\|u_h-v_h\| + b(\varphi_h-\psi_h,u-v_h) \leq$$

$$\leq \|a\|\,\|u-u_h\|\,\|u-v_h\| + \|b\|\,\|\psi-\varphi_h\|\,\|u_h-v_h\| + \|b\|\,\|\varphi_h-\psi_h\|\,\|u-v_h\|.$$

$$\bar{\beta}\|\varphi_h-\psi_h\|_W \leq \sup_{v_h} b(v_h,\varphi_h-\psi_h)\|v_h\|^{-1} \leq \|b\|\,\|\psi-\varphi_h\| +$$

$$+ \sup_{v_h} b(v_h,\psi-\psi_h)\|v_h\|^{-1} \leq$$

$$\leq \|b\|\,\|\psi-\varphi_h\| + \|a\|\,\|u-u_h\|$$

2.2.

Suppose now that a pair of Banach spaces H, M can be found in such a way that:

(2.7) $\quad V \subseteq H \qquad \|v\|_H \leq \|v\|_V \quad \forall v \in V, \ \dfrac{a(w,v)}{\|w\|_H \|v\|_H} \leq \xi$

(2.8) $\quad W \subseteq M \qquad \|\varphi\|_M \leq \|\varphi\|_W \quad \forall \varphi \in W,$

(2.9) $\quad \exists\, \alpha^* > 0 \quad \forall v \in V \quad a(v,v) \geq \alpha^* \|v\|_H^2,$

(2.10) $\quad \exists\, \beta^* > 0 \quad \forall \varphi \in W_h \quad \sup_{v \in V_h} b(v,\varphi)\|v\|_V^{-1} \geq \|\varphi\|_M \cdot \beta^*.$

Then the following theorem holds.

<u>Theorem</u> 3 <u>Under the hypotheses</u> (2.7) ... (2.10) <u>problem</u> (2.1) <u>has a unique solution and moreover</u>:

(2.11) $$\|u-u_h\|_H + \|\psi-\psi_h\|_M \leq C\, S_1(h)\, S_2(h) \left(\inf_{v \in V_h} \|u-v\|_V + \inf_{\varphi \in W_h} \|\psi-\varphi\|_W \right),$$

<u>with</u>

(2.12) $$S_1(h) = \sup_{v \in V_h} \frac{\|v\|_V}{\|v\|_H}, \qquad S_2(h) = \sup_{\varphi \in W_h} \frac{\|\varphi\|_W}{\|\varphi\|_M}.$$

<u>Proof</u> - Identical to the previous one, just being more carefull with the different norms.

2.3

Suppose again that hypotheses (2.7) ⋯ (2.10) are satified and define, for each $\tilde{g} \in W'$,

$$B(\tilde{g}) = \{ v \mid v \in V,\ b(v, \varphi) = (\tilde{g}, \varphi)\ \forall \varphi \in W \}$$

$$B_h(\tilde{g}) = \{ v \mid v \in V_h,\ b(v, \varphi) = (\tilde{g}, \varphi)\ \forall \varphi \in W_h \}$$

and suppose that the following condition is satisfied

$$(2.13) \qquad B_h(0) \subseteq B(0).$$

<u>Theorem</u> 4. If hypotheses (2.7) ⋯ (2.10) and (2.13) are satified, then we have

$$(2.14) \quad \| u - u_h \|_H \le c \inf_{v \in B_h(g)} \| u - v \|_H$$

$$(2.15) \quad \| \psi - \psi_h \|_M \le c \left(\inf_{\varphi \in W_h} \| \psi - \varphi \|_W + \| u - u_h \|_H \right)$$

Proof. Let us prove first (2.14); for all $v_h \in B_h(g)$ one has

$$\alpha^* \| u - u_h \|_H^2 \le a(u - u_h, u - u_h) \le \bar{\alpha} \| u - u_h \|_H \| u - v_h \|_H + \cancel{b(\psi - \psi_h, u_h - v_h)}.$$

and (2.14) follows. Let us prove (2.15); for all φ_h in W_h one has

$$\| \psi - \psi_h \|_M \le \| \psi - \varphi_h \|_M + \| \varphi_h - \psi_h \|_M$$

$$\beta^* \| \varphi_h - \psi_h \|_M \le \sup_{v \in V_h} b(v, \varphi_h - \psi_h) \| v \|_V^{-1} \le \| b \| \, \| \psi - \varphi_h \|_W + \bar{\alpha} \| u - u_h \|_H \underbrace{\frac{\| v \|_H}{\| v \|_V}}_{\le 1}$$

In the applications, the verification of (2.10) is sometimes the main difficulty. Therefore, the following proposition will often be interesting.

<u>Proposition 2.1.</u> <u>Suppose that the following assumptions are satisfied</u>:

(2.16) $\exists \quad \Pi_h \in \mathcal{L}(V, V_h)$ s.t. :

$$\text{(i)} \qquad b(v - \Pi_h v, \varphi) = 0 \quad \forall \varphi \in W_h \quad \forall v \in \mathcal{V},$$

with $\mathcal{V}$ = subspace of V,

$$\text{(ii)} \qquad \|\Pi_h v\|_V \leq \gamma \|v\|_V \quad \forall v \in \mathcal{V}, \ \gamma \text{ indep. of } h,$$

(2.17) $\exists \ \tilde{\beta} > 0 \ ; \quad \sup\limits_{v \in \mathcal{V}} b(v, \varphi) \|v\|_V^{-1} \geq \tilde{\beta} \|\varphi\|_M \quad \forall \varphi \in W;$

<u>Therefore</u> (2.10) <u>is satisfied</u>.

<u>Proof</u>. We have for all $\varphi_h \in W_h \subseteq W$:

$$\tilde{\beta} \|\varphi_h\|_M \leq \sup_{v \in \mathcal{V}} \frac{b(v, \varphi_h)}{\|v\|_V} = \sup_{v \in \mathcal{V}} \frac{b(\Pi_h v, \varphi_h)}{\|v\|_V} \leq$$

$$\leq \gamma^{4} \sup_{v \in \mathcal{V}} \frac{b(\Pi_h v, \varphi_h)}{\|\Pi_h v\|_V} \leq \gamma \sup_{v_h \in V_h} \frac{b(v_h, \varphi_h)}{\|v_h\|_V}$$

that is (2.10) holds with $\beta^* = \tilde{\beta}/\gamma$.

3. Mixed methods (I)

3.0 – Let us consider the model problem in the case of one dimension.

$$(3.0)\qquad \begin{cases} w^{(iv)}(x) = p(x) & \text{in } \Omega =]S, E[\\ w = w' = 0 & \text{on } \partial\Omega \equiv \{S\} \cup \{E\} \end{cases}$$

We split-up the problem introducin a pair of new variables

$$\psi = w \qquad u = w''$$

The ~~problem~~ pair (u, ψ) is obviously a solution of the problem:

$$(3.1)\qquad \begin{cases} \text{find } (u,\psi) \text{ in } L^2(\Omega) \times H_0^2(\Omega) \text{ s.t.} \\ \displaystyle\int_\Omega u v \, dx = \int_\Omega v \psi'' \, dx \quad \forall v \in L^2(\Omega) \\ \displaystyle\int_\Omega u \varphi'' \, dx = \int_\Omega p \varphi \, dx \quad \forall \varphi \in H_0^2(\Omega) \end{cases}$$

Since (3.1) ~~sati~~ is of the kind (2.0) and satisfies (2.3), (2.4), the solution of (3.1) is unique and (3.0) is equivalent to (3.1).

If the solution (u,ψ) of (3.1) is such that $u \in H^1(\Omega)$, then (u,ψ) will be also a solution of the following problem:

$$
(3.2) \quad \begin{cases} \text{find } (u,\psi) \text{ in } H^1(\Omega) \times H^1_0(\Omega) \text{ s.t.} \\ \int_\Omega u v \, dx = - \int_\Omega v' \psi' \, dx \quad \forall v \in H^1(\Omega) \\ - \int_\Omega u' \varphi' \, dx = \int_\Omega p \varphi \, dx \quad \forall \varphi \in H^1_0(\Omega) \end{cases}
$$

Problem (3.2) is again of the kind (2.0), but now (2.7) will be satisfied instead of (2.3).

<u>Exercise</u> 3.1 Prove that (2.4) still holds

<u>Exercise</u> 3.2 Prove uniqueness (not the existence) of the solution of (3.2).

<u>Remark</u> 3.0 From the practical point of view, the method $\left((3.0) \Longrightarrow (3.2)\right)$ has two advantages:

1) One can use trial functions which are in $C^0(\bar{\Omega})$ but not in $C^1(\bar{\Omega})$
2) One has $u = w''$ as an independent variable: in practice this means (think to the deflection of a beam) direct computation of the "moment" and more handable situation in nonlinear case

3.1. The discretisation of (3.2) is now very easy; for instance let r be an integer ≥ 2 and set, for each decomposition $\mathcal{T}_h$ of Ω into subintervals K,

$$(3.3) \quad V_h = \{ v \mid v \in C^0(\bar{\Omega}),\ v_{|K} \in P_r \quad \forall K \in \mathcal{T}_h \} \subseteq V = H^1$$

$$(3.4) \quad W_h = \overset{\circ}{V}_h = \{ v \mid v \in V_h \quad v = 0 \text{ on } \partial\Omega \} \subseteq W = M = H^1_0$$

Since we have, in this case, $W_h \subseteq V_h$, the verification of (2.10) is an easy <u>exercise</u>. It can be proved moreover that if the ratio between the maximum and the minimum leght of the elements K is bounded independently of h, then

$$(3.5) \qquad S_1(h) = \sup_{v \in V_h} \|v\|_{H^1} \cdot \|v\|^{-1}_{L^2} \leq c\, h^{-1}$$

with c constant independent of h. Moreover $S_2(h) = 1$ in our case since we chose $M = W = H^1_0$.

By theorem 3 we get therefore:

$$(3.6) \qquad \|u - u_h\| + \|\psi - \psi\| = O(h^{r-1})$$

3.2. Let us consider now the two-dimensional case:

$$(3.7)\qquad \begin{cases} \Delta^2 w = p \text{ in } \Omega \subseteq \mathbb{R}^2, \\ w = w_{/n} = 0 \text{ on } \partial\Omega, \end{cases}$$

and look for a suitable splitting up. It is easy to see that the operator Δ^2 can be splitted-up in the form $\Delta^2 = T^* T$ in a number of different ways. Two of them are particularly important for the applications

$$(3.8)\qquad \psi = w \qquad u = \Delta w$$

$$(3.9)\qquad \psi = w \qquad \underline{u} = (u_{ij}), \; u_{ij} = w_{/ij}.$$

Since the approach, for both cases, is the same (and essentially coincide with the one-dimensional case), we shall deal only ~~fo~~ with the choice (3.8). ~~&~~ We shall also spend few words about (3.9) in the next lecture.

The first saddle point formulation is now:

$$(3.10)\quad \begin{cases} \text{find } (u,\psi) \in L^2(\Omega) \times H^2_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega u v \, dx = \int_\Omega v \Delta\psi \, dx \quad \forall v \in L^2(\Omega) \\ \displaystyle\int_\Omega u \Delta\varphi \, dx = \int_\Omega p \varphi \, dx \quad \forall \varphi \in H^2_0(\Omega) \end{cases}$$

Since (3.10) is of the kind (2.0) and satisfies (2.3), (2.4) we immediately get the equivalence between (3.7) and (3.10).

Suppose now that $u \in H^1(\Omega)$; then (u,ψ) will be also a solution of the "second saddle point formulation":

$$(3.11)\quad \begin{cases} \text{find } (u,\psi) \in H^1 \times H^1_0 \text{ such that} \\ \displaystyle\int_\Omega u v \, dx = -\int_\Omega \nabla v \cdot \nabla\psi \, dx \quad \forall v \in H^1 \\ \displaystyle -\int_\Omega \nabla u \cdot \nabla\varphi \, dx = \int_\Omega p \varphi \, dx \quad \forall \varphi \in H^1_0 \end{cases}$$

Problem (3.11) is still of the kind (2.0) but now (2.7) (with $H = L^2(\Omega)$) will be satisfied instead of (2.3). Moreover, as in the one dimensional case, (2.4) and uniqueness still hold and the considerations of remark 3.0 can be extended. We also remark that different splitting-up will be preferable, in practice, following the different kinds of physical problem to be treated. For instance (3 9) will be a good choice mainly in the fluid dinamics problems (ψ = streamfunction, u = = vorticity).

Let now $\mathcal{T}_h$ be a decomposition of Ω into triangular subdomains K and r be an integer ≥ 2. We set

$$V_h = \{ v \mid v \in C^0(\bar{\Omega}) \quad v_{|K} \in P_r \quad \forall K \in \mathcal{T}_h \} \subseteq V = H^1$$

$$W_h = \overset{\circ}{V}_h = \{ v \mid v \in V_h \, , \, v_{|\partial\Omega} = 0 \} \subseteq W = M = H^1_0$$

With similar arguments to the one dimensional case, the following error bound can be deduced from theorem 3:

$$(3.12) \quad \|u - u_h\|_{L^2} + \|\psi - \psi_h\|_{H^1} \le c\,|h|^{r-1}$$

with c constant independent of h.

4. MIXED METHODS (II)

Let us go back to the procedure of lecture 3 in the case of the decomposition

$$(4.0)\qquad \psi = w \qquad \underline{u} = (u_{ij}) \quad u_{ij} = w/_{ij}$$

The first saddle point formulation is now:

$$(4.1)\quad \begin{cases} \text{find } (\underline{u}, \psi) \in (L^2)^4_s \times H^2_0 \text{ such that} \\ \displaystyle\int_\Omega u_{ij} v_{ij}\, dx = \int_\Omega v_{ij}\, \psi/_{ij}\, dx \quad \forall \underline{v} \in (L^2)^4_s \\ \displaystyle\int_\Omega u_{ij}\, \varphi/_{ij}\, dx = \int_\Omega p\, \varphi\, dx \quad \forall \varphi \in H^2_0 \end{cases}$$

with $(L^2)^4_s = \{\underline{v} \mid v_{ij} \in L^2 \ (i,j = 1,2) \ v_{12} = v_{21}\}$ and where, here and in the following, we use the convention of summation of repeated indices. F

Following the same idea of the previous lecture we should take $(H^1)^4_s$ instead of $(L^2)^4_s$ and $-\int_\Omega v_{ij/i}\, \varphi/_j$ instead of $\int_\Omega v_{ij}\, \varphi/_{ij}$; this is obviously possible, but would lead us to the same "lost" as in (3.6) and (3.12): that is, for instance, with piecewise quadratic moments we would have an $O(h)$ rate of convergence for the L^2-error on the moment field, which is not very satisfactory.

We decide therefore that a different choice of the "second saddle point formulation" would be preferable. In particular, instead require the "continuity" of the moment field we shall only require that, for a given decomposition $\mathcal{T}_h$ of Ω into triangular elements K, the "normal bending moment"

$$M_n(\underline{v}) = v_{ij}\, n_i n_j$$

has to be continuous across the interelement boundaries. More precisely we set

(4.2) $V = \{\underline{v} \mid \underline{v}_{|K} \in (H^1(K))^4_s \ \forall K \in \mathcal{T}_h,\ M_n(\underline{v})$ <u>continuous</u>

We look now for a bilinear form $b(\underline{v},\varphi)$ such that

$$(4.3)\qquad b(\underline{v},\varphi)=\int_\Omega v_{ij}\,\varphi_{/ij}\quad \forall\,\underline{v}\in V,\ \forall\varphi\in H_0^2;$$

in order to see what (4.3) actually means, we need the following "Green formula":

$$\int_K v_{ij}\,\varphi_{/ij}\,dx=-\int_K v_{ij/i}\,\varphi_{/j}\,dx+\int_{\partial K}\left(M_n(\underline{v})\varphi_{/n}+M_{nt}(\underline{v})\varphi_{/t}\right)d\sigma$$

where $\vec{n}$ = unit outward normal to ∂K, $\vec{t}$ = unit tangent, $M_n(\underline{v})=v_{ij}n_in_j$, $M_{nt}(\underline{v})=v_{ij}n_it_j$

Summation over $K\in\mathcal{T}_h$ yields, for each $\underline{v}\in V$ (defined by (4.2)) and $\varphi\in H_0^2$:

$$(4.4)\qquad \int_\Omega v_{ij}\,\varphi_{/ij}\,dx=\sum_K-\int_K v_{ij/i}\,\varphi_{/j}\,dx+\sum_K\int_{\partial K}M_{nt}(\underline{v})\varphi_{/t}\,d\sigma+\sum_K\int_{\partial K}M_n(\underline{v})\varphi_{/n}\,d\sigma$$

and the last term in (4.4) vanishes due to the "continuity requirements. We put then

$$(4.5)\qquad b(\underline{v},\varphi)=\sum_{K\in\mathcal{T}_h}\left\{-\int_K v_{ij/i}\,\varphi_{/j}\,dx+\int_{\partial K}M_{nt}(\underline{v})\varphi_{/t}\,d\sigma\right\}$$

Note that $b(\underline{v}, \varphi)$, defined by (4.5) takes sense, if $\underline{v} \in V$, for each $\varphi \in W_0^{1,p}(\Omega)$ (for all $p > 2$). We put therefore

$$(4.6) \qquad W = W_0^{1,p}(\Omega) \qquad p > 2$$

and consider as "second saddle point formulation" the following one:

$$(4.7) \begin{cases} \text{find } (\underline{u}, \psi) \text{ in } V \times W \text{ such that:} \\ \int_\Omega u_{ij} v_{ij} \, dx = b(\underline{v}, \psi) \quad \forall v \in V, \\ b(\underline{u}, \varphi) = \int_\Omega p \varphi \, dx \quad \forall \varphi \in W. \end{cases}$$

It is easy to prove that, if one takes $H = (L^2)^4_s$, $M = H_0^1(\Omega)$, hypotheses (2.7) ... (2.9) are satisfied; Moreover (4.7) has a unique solution that coincide with the solution of (4.1). Take now for each integer number $k \geq 0$

$$V_h = \{\underline{v} \mid \underline{v} \in V, \; v_{ij|K} \in P_k \;\; \forall K \in \mathcal{T}_h\}$$

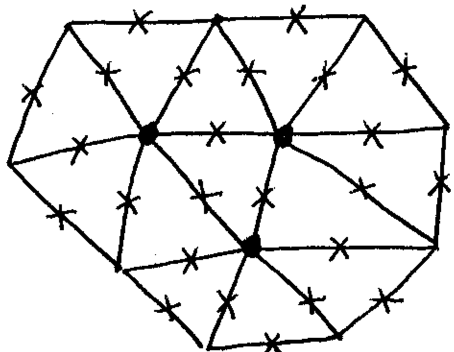
$$W_h = \{\varphi \mid \varphi \in W, \; \varphi_{|K} \in P_{k+1} \;\; \forall K \in \mathcal{T}_h\}$$

In fig. 10 a suitable choice of the degrees of freedom for V_h and W_h is represented, in the case of $k = 0$

d.o.f. for W_h

• value of φ

$\dim(W_h) = 3$



d.o.f. for V_h

x value of $M_n(\underline{v})$

$\dim(V_h) = 24$

fig. 10

It is now possible to prove that the present choice of $V, W, b(\underline{v},\varphi), H, M, V_h, W_h$ is such that (2.13) is also satisfied, and (2.16), (2.17) are satisfied for instance with $\mathcal{V} = V$. Therefore (2.10), by means of proposition 2.1, is satisfied and we can apply theorem 4. A correct evaluation of the "interpolation error" yieds then:

$$\|\underline{u} - \underline{u}_h\|_{(L^2)^4_s} + \|\psi - \psi_h\|_{H^1_0} \leq C\,|h|^{k+1}$$

with c = constant independent of h.

5 - HYBRID METHODS.

5.0. Consider again the model problem

$$(5.0)\qquad \begin{cases} \Delta^2 w = p & \text{in } \Omega \subset \mathbb{R}^2 \\ w = w_{/n} = 0 & \text{on } \partial\Omega \end{cases}$$

and, refering to the case of a plate bending problem, consider the moment field associated to w:

$$(5.1)\qquad \underline{M} = (M_{ij}) \quad M_{ij} = w_{/ij}.$$

For a given decomposition $\mathcal{T}_h$ of Ω into subdomains K, and for each K consider the complementary energy associated with K

$$(5.2)\qquad CE(K) = \frac{1}{2}\int_K M_{ij} M_{ij}\, dx + \int_{\partial K} (M_n w_{/n} + M_{nt} w_{/t} - Q_n v)\, d\sigma$$

where $\vec{n}$ = unit outward ~~direction~~ normal to ∂K, $\vec{t}$ = unit tangent, and

$$(5.2)\qquad M_n = M_{ij} n_i n_j \,;\; M_{nt} = M_{ij} n_i t_j \,;\; Q_n = M_{ij/i} n_j$$

(normal bending moment, twisting moment and normal shear stress). By the principle of the minimum of complementary energy, $CE(K)$

must be minimum among all the moment fields which equilibrates the load p inside K; that is, among all the moment fields $\tilde{M}$ such that

$$\tilde{M}_{ij/ij} = p \quad \text{inside } K. \tag{5.3}$$

Summation over K yields

$$CE = \sum_{K \in \mathcal{T}_h} CE(K) = \int_\Omega M_{ij} M_{ij} + b(\underline{M}, w) \tag{5.4}$$

with

$$b(\underline{M}, w) = \sum_K \int_{\partial K} \left(M_n w_{/n} + M_{nt} w_{/t} - Q_n w \right) d\sigma. \tag{5.5}$$

Again, if $\tilde{M}$ is another moment field ~~such that~~ which satisfies (5.3) for each K, we must have

$$CE(\underline{M}, w) = \int_\Omega M_{ij} M_{ij} + b(\underline{M}, w) \leq CE(\underline{\tilde{M}}, w) \tag{5.6}$$

Moreover the <u>continuity</u> of the actual moment field across the interelement boundaries yields

$$b(\underline{M}, \varphi) = 0 \quad \forall \varphi \in H_0^2(\Omega). \tag{5.7}$$

we can conclude that the functional $C(\tilde{M}, \varphi)$ has a stationary point at (M, w) if $\tilde{M}$ varies in

$$\mathcal{H}_p = \left\{ \underline{\tilde{M}} \;\middle|\; \tilde{M}_{ij} \in L^2 ,\ \tilde{M}_{12} = \tilde{M}_{21} ,\ \tilde{M}_{ij/ij} = p \text{ inside each } K \right\}$$

and φ varies in $H^2_0(\Omega)$. It will be easier, instead of treating $\mathcal{H}_p$ directly, to choose an element $\underline{f}$ in $\mathcal{H}_p$ and consider

$$\mathcal{H}_0 = \mathcal{H}_p - \underline{f}$$

which is a linear manyfold. The actual moment field $\underline{M}$, given by (5.1) will also be written as

(5.8) $$\underline{M} = \underline{u} + \underline{f} \quad , \underline{u} \in \mathcal{H}_0$$

The stationariety conditions ~~yield~~ are now:

(5.9) $$\int_\Omega u_{ij} v_{ij} + \int_\Omega f_{ij} v_{ij} + b(\underline{v}, w) = 0 \qquad \forall \underline{v} \in V = \mathcal{H}_0$$

(5.10) $$b(\underline{u} + \underline{f}, \varphi) = 0 \qquad \forall \varphi \in H^2_0(\Omega)$$

We remark that since $b(\underline{v}, w)$ depends only on the values of w and $w_{/i}$ $(i=1,2)$ on $\Sigma = \bigcup_K \partial K$, the equations (5.9), (5.10) can not have a unique solution. We introduce therefore

$$(5.11) \quad W = \{ \varphi \,/\, \varphi \in H^2_0(\Omega),\ \Delta^2 \varphi = 0 \text{ in each } K \}$$

and let

(5.12) ψ be the unique element of W which coinides with w (with $\psi_{/i} = w_{/i}$) on Σ.

It is easy to see now that tha following problem:

$$(5.13) \quad \begin{cases} \text{find } (\underline{u}, \psi) \in V \times W \text{ such that} \\ \int_\Omega u_{ij} v_{ij} + b(\underline{v}, \psi) = -\int_\Omega f_{ij} v_{ij} \quad \forall \underline{v} \in V \\ b(\underline{u}, \varphi) = -b(\underline{f}, \varphi) \quad \forall \varphi \in W \end{cases}$$

has a unique solution (use theorem 1) which is given by (5.1), (5.8), (5.12).

Let now consider, for each triplet of integer numbers (r, s, m) the following finite dimensional spaces:

$$W_h = \{\varphi \mid \varphi \in W,\ \varphi_{|\partial K} \in P_r,\ \frac{\partial \varphi}{\partial n}_{|\partial K} \in \tilde{P}_s(\partial K) \forall K\}$$

(where $\tilde{P}_s(\partial K)$ is the set of the functions which are polynomials of degree $\leq s$ on each edge of ∂K),

$$V_h = \{\underline{v} \mid \underline{v} \in V,\ v_{ij|K} \in P_m \quad \forall K\}$$

It can be proved that, for each pair (r, s) with $r \geq 3$, $s \geq 1$, if m is such that

$$m \geq r-2 \quad \text{if } r-1 > s$$

$$m \geq r-2 \quad \text{if } r-1 = s,\ s \text{ odd}$$

$$m \geq r-1 \quad \text{if } r-1 = s,\ s \text{ even}$$

$$m \geq s-1 \quad \text{if } r-1 < s$$

then V_h and W_h satisfie the condition (2.5)

If this is the case, by means of theorem ? and a correct evaluation of the interpolation error, one obtains

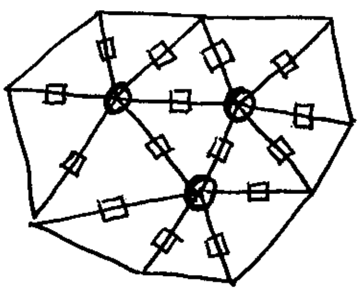
$$\|\underline{u}-\underline{u}_h\|_{L^2}+\|\psi-\psi_h\|_{H^2}\leq c|h|^q$$

$$q=\min(m+1,s,r).$$

The lowest "optimal case" is therefore:

$$r=4\ ,\ s=3\ ,\ m=2\ .$$

Fig11 shows a possible choice for the degrees of freedom in W_h for $r=4$ and $s=3$



○ = values of φ, $\varphi_{/x}$, $\varphi_{/y}$

□ = values of φ, $\varphi_{/m}$, $\varphi_{/t}$

The choice of the d.o.f. of V_h can be performed in (almost) any reasonable manner since the elements of V_h are independently assumed in each element. It is also important to remark that the final matrix of the discretised problem has again the form (2.2). However, in the ~~ca~~ present case, ~~mat~~ the submatrix A is block-diagonal and therefore the computation of A^{-1} is easy to perform. The final system, therefore, will be in the W_h-unknowns only, with matrix $B^T A^{-1} B$.