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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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RELATIONSHIP BETWEEN THEORY AND EXPERIMENT TO MECHANICS

(Part III)

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4. WITHDRAWAL FROM A RESERVOIR OF STRATIFIED FLUID †

The work I would like to describe now is a first step towards understanding what appears to be a very subtle problem. Many of the arguments presented are very tentative & should be seen as attempts to identify the relevant physical processes & thereby formulate an appropriate mathematical problem. To this end, a rather 'simple' set of experiments provided crucial initial guidance & the theoretical considerations then pointed to the need for further experiments to help clarify our ideas. But I think it would have been virtually impossible to have guessed the answer to this problem without the help of the experimental evidence. As I have said, this is only a first step towards understanding the problem, & accordingly it opens up many new research areas, but also I think it is a very salutary exercise about the importance of ~~[illegible]~~ linking mechanics closely to experiment & of being careful in interpreting one's experimental observations in a consistent way.

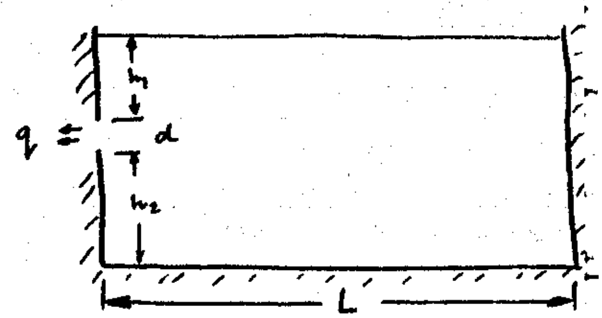
4.1 Outline of the problem

It turns out that fresh-water reservoirs are often stratified, & almost but results in density variations in the water. These density variations can have a profound effect on the dynamics involving fluid motions in the reservoir.

The reason for particular interest in this problem is that the engineers involved in managing such resources are concerned about controlling the 'quality' of the water supply, especially with regard to the oxygen content of the liquid. But suppose the situation arises in which the water becomes contaminated in some way, or some poison may find its way into the reservoir, the question arises of how to remove this contaminant. To answer such a question we need to know exactly what fluid is removed when we draw off some liquid from the reservoir. This suggests we consider a situation of the kind shown in the sketch.

outlet orifice, with coordinate system defined.

flow rate [q (cm²/s)] per unit width of tank.



Assume that the initial density distribution in the tank is $\rho_0(1-\beta y)$, β = const. The fluid is forced out through the orifice by the hydrostatic head of the liquid in the tank, & the basic question we would like answered is -

'what is the source, in the reservoir, of the fluid withdrawn at a given time after opening the outlet?'.

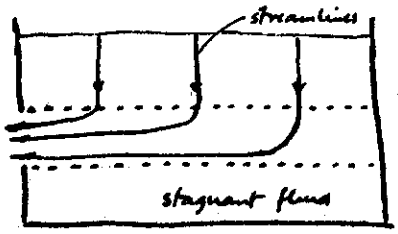
4.2 Previous investigations

At the time of starting this work there had been a number of previous investigations made into this problem, of both a theoretical and an empirical kind. All of the theoretical studies were based on solutions for the domain $x > 0$ & the domain in y either bounded or unbounded. However, they all had the property that the mass flux through a vertical plane was independent of the distance from the outlet. Clearly this can not hold in a 'real' reservoir, & because we would expect the kinematic contraints at the free surface to keep the water level there very nearly horizontal throughout the withdrawal, it would appear that this mass flux should decrease linearly to zero at the ~~[illegible]~~ end of the reservoir opposite the outlet.

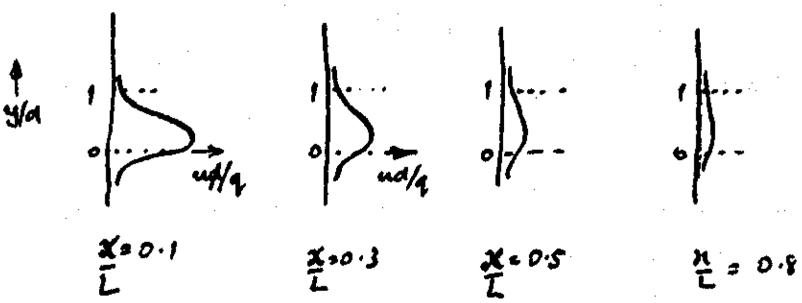
Thus we made an experiment to see if this was in fact realized, & the results confirmed our ideas that the mass flux decreased linearly with distance from the outlet. Thus, it appeared that a new set of ~~[illegible]~~ experiments should be made.

4.3 The initial experiments

It was anticipated that a crude model for the flow might be as shown in the sketch. Therefore, let us scale the vertical coordinate by the size of the orifice & the horizontal velocities by the mean flow speed at the outlet, viz q/d.



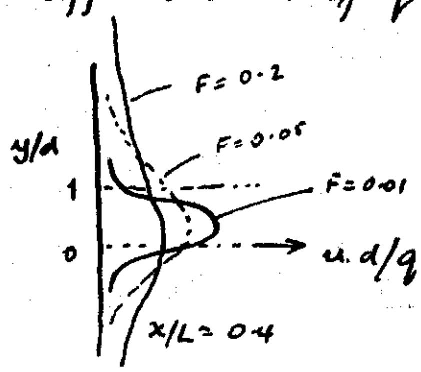
a) The horizontal velocities. Measurements of the horizontal velocities $u(y)$ at various stations along the tank gave results as shown.



Expt. for a given flow rate.

These experiments indicated that the width of the layer in which there were significant horizontal motions was nearly uniform, & that the scale of the horizontal velocity field decreased with x/L.

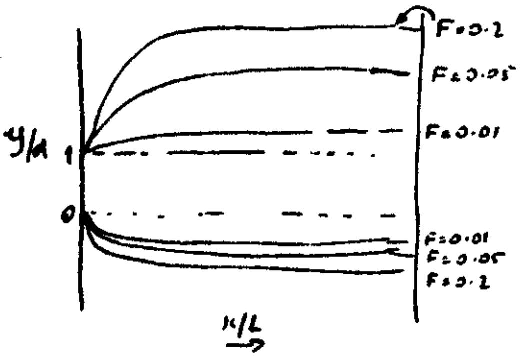
Measurements of $u(y).d/q$, made at different values of q, for a given value of x were as shown.

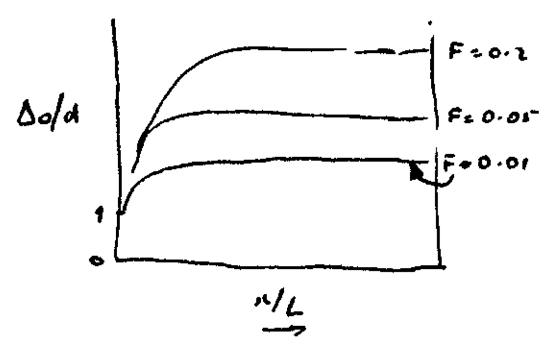


Here the parameter F, called the Froude number, is given by $F = \dfrac{q}{d^2 (\beta g)^{1/2}}$.

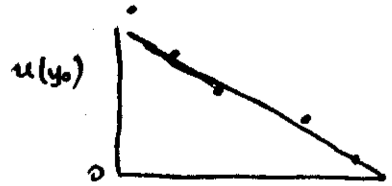
b) <u>The steadiness of the flow field.</u> Although the experiment is basically an unsteady one, as the free surface is falling throughout the experiment, we found experimentally that the velocity field was very nearly steady after a short period, in which the motions were being established. This test for the steadiness of the velocity field was made rather carefully.

c) <u>The width of the withdrawal layer.</u> The measurements in part (a) suggested that the width Δ_0 of the withdrawal layer was nearly independent of distance from the outlet. Suppose that we define Δ_0 to be given by the vertical distance between the levels at which the velocity $u(y) = 0.1\, u_m$, where u_m is the maximum velocity observed at the station under consideration. The experiments indicate that Δ_0 is as shown.





d) These results suggest that $u(y_0)$, y_0 fixed, should depend linearly on x/L, & the experimental evidence seems to support this. This was checked for a number of different values of y_0 & for each of the experiments, & it seemed to be a good description of the results, except for x near the outlet end of the tank.



In summary we found from these experiments that the width Δ_0 appeared to depend on the value of the flow rate. This indicates that a strictly linear theory cannot be used to describe the results, since such a model would imply a width independent of q. (After complet[ing] these experiments there was a paper published by Imberger & Fandry (1975) using a linear viscous, diffusive model with kinematic constraints of the kind observed in these experiments. The fact that Δ_0 depended on q mean[s] that their theory cannot describe the experimental results.) Also the results suggest that for $x/L \gtrsim 0.3$ the width Δ_0 was approximately constan[t] for any of the experiments. These experiments covered a range of Froude numbers between $F \sim 0.01$ & $F \sim 1.5$.

4.4 A Universal Curve?

Here we ask the question, is there a scaling for the width of the withdrawal layer? Let us introduce a time scale $(\beta g)^{-1/2}$ & a length scale $\ell = q^{1/2}/(\beta g)^{1/4}$. Let the density at the point (x,y) be given by $\rho_0\,(1+\beta\ell\,(-y+\rho))$, & define the streamfunction to be $q\psi$ & the vorticity to be $(\beta g)^{1/2}\,\omega$. Then (in dimensionless form) the equations of motion for the two-dimensional flow of a Boussinesq fluid are

$$\frac{\partial \omega}{\partial t} + \frac{\partial(\omega, \psi)}{\partial(x,y)} = \rho_x + \frac{\nu}{q} \nabla^2 \omega , \tag{4.1}$$

$$\nabla^2 \psi = \omega$$

and
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho, \psi)}{\partial(x,y)} + \psi_x = \frac{\kappa}{q} \nabla^2 \rho , \tag{4.2}$$

where ν is the kinematic viscosity & κ the diffusivity of the fluid. To fix these equations we need (i) Reynolds number q/ν ; (ii) the Peclet number q/κ ; (iii) the length of the tank $\varepsilon^{-1} = L/\ell$; (iv) the Froude number $F = q/a^2(\beta g)^{1/2}$; (v) h_1/ℓ ; (vi) h_2/ℓ , (see definition sketch).

We do not believe that h_1/ℓ or h_2/ℓ will be very important, unless the outlet is very near the free surface or the bottom of the tank. Thus it would appear that the width Δ, however it be defined, should be determined in the form $\Delta/\ell = \mathscr{D}_0(q/\nu, q/\kappa, \varepsilon, F)$. Since Δ/ℓ was about 1/100 for the experiments we can expect the sizes of the terms in the vorticity equation to be, for steady flows, roughly

$$\mathscr{D}_0^{-3}\varepsilon \quad , \quad \mathscr{D}_0 \varepsilon \quad , \quad \nu/q \, \mathscr{D}_0^{-4} .$$

So the 'effective' Reynolds number is $(q/\nu)\varepsilon\mathscr{D}_0$ & this was about 1/15 for the experiments, and it would appear that inertial effects are unlikely to play a dominant role in determining the structure of the flow. Thus one might expect a balance to be struck such that $\mathcal{D}_0 \varepsilon = \frac{\nu}{q} \mathcal{D}_0^{-4}$. This suggests that, by extricating a factor $(\nu/\varepsilon q)^{1/5}$ from $\mathcal{D}_0$ we might remove all sensitive dependence of Δ on $\mathcal{D}_0$, for our experiments. It appea therefore, that we should introduce a new vertical length scale

$$\mathcal{L} = (\nu q L/\beta g)^{1/5},$$

& a velocity scale, given by (q/Δ), $\mathcal{U} = (q^4 \beta g/\nu L)^{1/5}$.

These scalings appear to correlate the data very well. For a given experiment we can form the mean values of Δ_0 & U_m for $X > 0.3$ (where $u(X,y) = (1-X)\,U(y)$ & $X = \tilde{x}/L$, with $\tilde{x}$ being the physical distance from the outlet end); call these mean values $\bar{\Delta}_0$ & $\bar{U}_M$. Then, comparing the values of $\bar{\Delta}_0$ & $\bar{U}_M$ for each of the experiments we found they were nearly constant if scaled by $\mathcal{L}$ & $\mathcal{U}$. In fact, the results gave

$$\bar{\Delta}_0/\mathcal{L} = 4.54 \quad (\text{std. dev. of } 0.15 \text{ or } 3.3\%),$$
$$\bar{U}_m/\mathcal{U} = 0.422 \quad (\text{std. dev of } 0.012 \text{ or } 2.8\%).$$

Thus, the experimental results suggest that, scaling in this way, the (velocity) data for $X \gtrsim 0.3$, lies very near to a universal curve.

4.5 A similarity structure for the observed flow?

The experimental observations suggest we should look for a solution of the form $\psi = (1-X)\,\Psi(y)$, where $X = \varepsilon x$. Let us consider the implications of such a choice, assuming steady velocities have been achieved. It follows from (4.1) that the density gradient can be written as

$$\rho_X = -2(1-X)\,R(y),$$

where R is some function defined explicitly in terms of Ψ. This implies that we can write the density as

$$\rho = \rho_1(y,t) + (1-X)^2 R(y),$$

where ρ_1 is a function to be determined. Using this expression for ρ in (4.2) we find that the equation can be partitioned into two parts, namely

$$2R\Psi' - R'\Psi + \frac{\kappa}{q\varepsilon} R'' = 0, \qquad (4.3)$$

and

$$\frac{\partial \rho_1}{\partial t} + \varepsilon\Psi \frac{\partial \rho_1}{\partial y} - \varepsilon\Psi - \frac{\kappa}{q}\left(2\delta^2 R + \frac{\partial^2 \rho_1}{\partial y^2}\right) = 0. \qquad (4.4)$$

The vorticity equation reduces to

$$\Psi'''\Psi - \Psi''\Psi' + 2R - \frac{\nu}{q\varepsilon}\Psi^{iv} = 0. \qquad (4.5).$$

The boundary conditions for these equations are

$$\left.\begin{array}{ll} R(-\infty) = 0, & R(\infty) = 0 \\ \Psi(-\infty) = 0, & \Psi(\infty) = -1 \\ \Psi'(-\infty) = 0, & \Psi'(\infty) = 0 \end{array}\right\} \qquad (4.6)$$

These conditions together with (4·3) & (4·5) might be expected to determine Φ and R, & then (4·4) could be used to determine ρ_1. This decoupling would probably explains why steady velocity fields were observed experimentally.

However, it turns out that the problem posed by (4·3), (4·5) & (4·6)' does not have a uniquely determined solution with properties consistent with the experimental observations. For, consider solutions Φ, S of the equations

$$\Phi'''\Phi - \Phi''\Phi' + 2S - \Phi^{iv} = 0,$$
$$2S\Phi' - S'\Phi + \tfrac{\kappa}{\nu} S'' = 0,$$

subject to the boundary conditions (4·6). It follows there is a 1-1 correspondence

$$\Phi(y) = \Phi\left(\tfrac{\varepsilon q}{\nu} y\right),$$
$$R(y) = \left(\tfrac{q\varepsilon}{\nu}\right)^3 S\left(\tfrac{q\varepsilon}{\nu} y\right),$$

between the two sets of differential equations. Thus, corresponding to any discrete solution of the rescaled problem, there would be a solution Φ whose dimensionless width would vary as $\nu/\varepsilon q$. This was contradictory to the experimental observations.

Thus, the only form of solution to the rescaled problem, which will not lead to a contradiction of this kind, is one for which there is a family of solutions $(\Phi(y,\delta), S(y,\delta))$, where δ is a parameter specifying, say, the vertical scale of the solution, for all real values δ in an appropriate interval.

Even so, if there is such a continuous branch of solutions, we would be unable to select one which would serve to predict the thickness of the withdrawal layer.

Moreover, an analysis of the equations obtained by linearizing about the asymptotic state $\Phi = 1$, for large y, suggests that it might not be possible to find any acceptable solution to the boundary value problem posed by (4·3), (4·5) & (4·6).

4.6 A structure for the withdrawal layer

In spite of these considerations, the experimental results strongly point to a similarity form of structure of the kind assumed in §4.5, & it is plausible to conjecture that the similarity structure breaks down in small-scale localized regions at the top & bottom of the withdrawal layer. Thus, restrict the use of the similarity form to the withdrawal layer itself.

Let Δ be the observed (dimensional) thickness of the layer, & define a new vertical coordinate y by the height $y\Delta$ above the streamline $\psi = 0$ located near the upper edge of the recirculating flow (just below the withdrawal layer). Then, it turns out that a fairly good approximation to the resulting equations is given by $R=0$ & $\Psi^{iv}=0$. It follows that the horizontal velocity in the withdrawal layer is given by

$$U(y) = 6y(1-y),$$

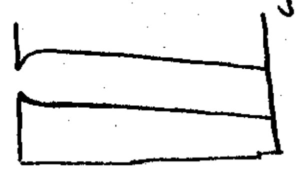
&, by choosing Δ to comply with the results, this gave fairly good agreement with the experimental observations.

However, this kind of solution implies that the viscous stresses completely dominate all other terms in the vorticity equation in the steady state, which is rather embarasing in view of our having obtained the scalings that led to the Universal curve on the basis of a balance between the terms f_x & $\nu\nabla^2\omega$. We return to this point shortly.

Let us first consider some of the features of the above model.

(i) There is a pressure gradient needed to drive the flow. A straightforward calculation shows that this corresponds to a change in head of less than 10^{-3} mm of water, which is very small.

(ii) A consideration of all the possibilities suggests that this head is most probably derived from a tilt which might develop on the withdrawal layer.



A slope of no more than 8×10^{-4} radians is needed to drive the flow in the withdrawal layer, & this would have been too small to observe experimentally, by the methods employed for our measurements.

(iii) A description is needed for the flow in the upper 'edge' layer at the top of the withdrawal layer. At the lower edge of the withdrawal layer the structure there probably has some similarities to the 'Stewartson layer' in rotating flows.

(iv) The vertical location of the layer is not determined in the above model. This is not unreasonable, since the similarity solution does not recognize the existence of the orifice & assumes that it can accept the mass flux supplied by the withdrawal layer. Nevertheless, the general level of the withdrawal layer must be determined near the outlet & it seems likely that a detailed calculation of the flow in the region near a specific outlet orifice, using the similarity solution as the far field, would locate the layer.

(v) The width Δ of the withdrawal layer cannot be determined by any of the above considerations.

4.7 <u>What determines the width of the withdrawal layer</u>?

We consider here some exploratory ideas which, nonetheless, led us to some novel results, & may therefore provide a setting for a more thorough investigat.

As indicated above, in the steady flow, the viscous terms dominate all other terms in the momentum equation. Yet the extremely good correlation of the experimental results was obtained with a scaling deduced on the basis of a balance between the viscous terms & the horizontal density gradients. Thus, if this balance is to be achieved, it would have to be in the initial unsteady part of the motion, & in turn this would suggest that the width might be determined by the way in which the steady state is established & therefore it might not be uniquely determined. For, consider two flows with the same value of q & different values of Δ. The vorticity levels in each flow would change as q/Δ^2. From the vorticity equation we have that

$$\omega(x,y) = \int_0^\infty \left[\nu \nabla^2 \omega + \rho_x - \frac{\partial(\omega,\psi)}{\partial(x,y)} \right] dt .$$

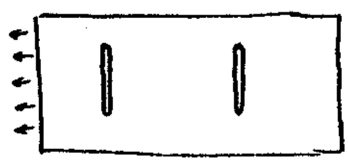
During the initial stages the isopycnics undergo considerable deflections from the horizontal, thereby generating vorticity. The inertial terms serve to redistribute this vorticity; however the establishment times for the steady flow are fairly short so that the total movement of fluid elements is not large, suggesting that the ultimate level of vorticity might be determined mainly by a balance between vorticity generation & vorticity diffusion. But this would suggest that, if the ultimate vorticity level in the layer were somehow changed then a new width would ensue.

(Note that similar ideas can be developed via a momentum argument.)

Thus, some new experiments were made in which the initiation procedure was varied, & the vorticity levels were influenced geometrically.

(i) A flow was established & the measurements indicated that it lay near the previous 'universal' curve. Then the flow rate was suddenly increased by a factor of three. When the new steady state was achieved it was found that the physical width of the layer was essentially the same as before the increase in q; yet to give results in agreement with the 'Universal' curve it would have had to increase by a 23%. This experiment shows that hysteresis effects are possible, & that the 'Universal' curve obtained earlier no longer applies, & so it was somewhat fortuitous.

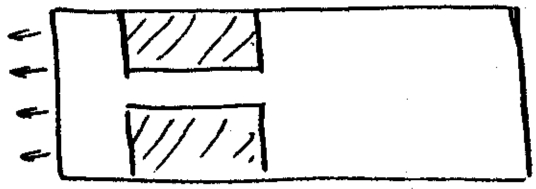
(ii) Some obstructions were placed in the tank, as shown in the sketch. A flow



plan view

was establised, & it was found that these had no effect on the width of the withdrawal layer. During this experiment the flow rate decreased steadily & yet the ~~or~~ flow closely resembled that given by the 'Universal curve', if the local value of q was used for the scaling.

(iii) A narrow constriction was placed in the tank, as shown in the sketch. Again a withdrawal layer, of uniform thickness along the tank, was established,



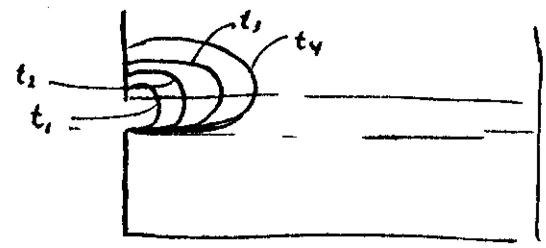
plan view

but the thickness of the layer was much larger than that expected on the basis of the 'Universal' curve. This could be accounted for by a contraction of the vortex lines (on entering the constricted region) leading to increased diffusion of vorticity & thereby establishing a different balance from that obtained with no constriction present. (Note that, in experiment (ii), the obstructions were so short there would not have been time for the vorticity to diffuse much while a fluid element passed through the narrow region.

Thus, in summary, these experiments suggest that the width of the withdrawal layer is not uniquely determined, & that, by starting the flow in different ways the width can be varied. So there is no such thing as a 'Universal curve' as suggested previously.

4.8 <u>What is the source of the fluid taken from the reservoir</u>?

For a given value of Δ, & a similarity structure of the kind indicated above, the velocity components are known. Thus, at a given time T, it is possible to calculate the initial position in the reservoir of those particles which are just about to pass out of the tank. Such a calculation indicates that the fluid is taken from regions, as shown in the sketch, at times $t_1 < t_2 < t_3 < t_4$.



The fluid elements on these curves at $t=0$, occupy the positions at the outlet orifice at times t_1, t_2, t_3, t_4 respectively.

<u>4.9</u> I think this example shows just how fruitful the interplay between experiment & theory (albeit very vague in places) has helped in achieving a much better understanding of the problem. Without the experiments I think it would have been very unlikely for someone to have guessed the outcome of ~~such~~ a situation such as this.