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APPLICATIONS OF NONLINEAR FUNCTIONAL ANALYSIS TO DIFFERENTIAL EQUATIONS

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REMARKS ON NONLINEAR ELLIPTIC PROBLEMS

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0. INTRODUCTION

Our main purpose here is to consider existence, nonexistence and multeplicity results for boundary value problems of the type

$$\Delta u + p(u) = g(x) \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset R^N$ is open, bounded and $p \in C(R,R)$. ~~Clearly~~ we consider for semplicity the Laplace operator Δ instead of a more general elliptic operator, as well as we could consider p depending on x, etc. Obviously the method to study (1) will depend on the behaviour of p. We shall consider in what follows 3 tipical cases.

1. A VARIATIONAL APPROACH

The first is a variational approach and applies, for ex., when g=0 and p(s) is superlinear at infinity. We will sketch briefly the framework of this method and refer for more details to [1].
The solutions of (1) are the critical points of the functional

$$J(u) = \frac{1}{2}\|u\|^2 - \int_\Omega P(u(x))\,dx, \qquad P(u) = \int_0^u p(s)ds$$

in $E = W_0^{1,2}(\Omega)$ where $\|u\|^2 = \int|\nabla u|^2$ is taken as norm. These critical points are investigated by means of the Lusternik-Schni-

relman theory. The basic tool of such a teory is the definition of a topological invariant: we will define here the genus.

Definition. Let $X \subset E-\{0\}$, X symmetric. The genus $\gamma(X)$ of X is defined as the least integer n such that $\exists\ \phi \in C(X,R^n-\{0\})$, ϕ odd.

One of the main property of the genus is that if $h \in C(E,E)$ is odd, then $\gamma(h(X)) \geq \gamma(X)$. The genus permits us to define some minimax level which, under suitable conditions on J, are critical, i.e. at such level there is at least one critical point of J.

More precisely let us assume:

A1) $|p(s)| \leq a + b|s|^{\alpha}$, $1 \leq \alpha < \frac{N+2}{N-2}$ for $N>2$, and any α for $N \leq 2$;

A2) $s^{-1}p(s) \to \infty$ as $|s| \to \infty$; $p(s) = o(|s|)$ as $s \to 0$;

A3) $\exists a$: for $|s| \geq a$ $P(s) \leq \theta p(s)s$ with $\theta \in]0,\frac{1}{2}[$;

A4) p(s) is odd.

Using the genus we define a class of sets G_n with the propertiy: for every $X \in G_n$ $h(X) \in G_n$ whenever $h \in C(E,E)$, h odd and $J(h(u)) \leq J(u)$ $\forall u \in E$. By A2 it follows G_n is not empty and thus we can define

$$c_n = \inf_{X \in G_n} \sup_{u \in X} J(u)$$

A2 implies $0<c_n$ and thus c_n will be not the trivial level. Moreover A1 and A3 imply J satisfies a suitable compactness condition (the P-S condition); therefore if c_n is not a critical level it is possible to define a deseendent flow h of J in such a way that $\exists\ \varepsilon$

$$(2) \qquad \sup_X J(u) < c_n + \varepsilon \quad \Rightarrow \quad \sup_{h(X)} J(u) < c_n - \varepsilon;$$

From A4 it follows that J is even and thus h can be taken odd. But in this case $X \in G_n$ implies $h(X) \in G_n$ and thus (2) contradicts the definition of c_n. Finally it is possible to show A2 implies $c_n \uparrow \infty$.

Then we can state:

Theorem 1. Let the assumptions A1-A4 be satisfied. Then the boundary value problem (1) has infinitely many solutions.

We end this sections with some remarks: the condition A4 is essential in the arguments above. When this parity conditions is dropped, then we known only weak results. For example for the problem

$$\Delta u + u^3 = g \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{3}$$

it is only known [2] that $\forall n\ \exists \varepsilon$ such that (3) has at least n solutions (provided $|g| < \varepsilon$). Another abstract result in the study of critical points relaxing the ~~evenness~~ parity condition can be found in [3].

2. A NONVARIATIONAL METHOD

If p(s) is asymptotically linear at infinity the approach above cannot be applied. We will treat this case in this section; the results we expose are contained in a forthcoming joint paper with G. MANCINI [4]. Consider

$$\Delta u + \lambda_k u + f(u) = g \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{4}$$

where λ_k is an eigenvalue of Δ and f is C^1 and bounded. Problem (4) has been studied by LANDESMAN & LAZER [5], assuming λ_k is simple, $f(s) \to f(\mp\infty)$ as $s \to \mp\infty$, and

$$f(-\infty) < f(s) < f(+\infty) \tag{5}$$

They are able to prove the following interesting result:

Theorem 2. Under the conditions above, (4) has solutions if and only if (denoted by v_k the eigenfunction associated to λ_k)

$$f(-\infty)\int_{v_k>0} v_k + f(+\infty)\int_{v_k<0} v_k < \int_{\Omega} g v_k < f(+\infty)\int_{v_k>0} v_k + f(-\infty)\int_{v_k<0} v_k$$

For a simple proof, see HESS [6].

Several people extended the Landesman & Lazer result to the case of multiple eigenvalue, but assuming ~~anyway~~ always (5). Among others we recall WILLIAMS [7] and FUČIK, KUČERA & NEČAS [8]. On the contrary we know only few and partial results about the solvability of (4) without the hypothesis (5). Our goal here is to study such a problem. Precisely we are mainly interested not only to prove existence and nonexistence theorems, ~~and~~ but if possible, to find multiple solutions of (4).

The approach we use is a global version of the classical Lyapunov-Schmidt method. Such a technique has been used by PRODI and the author in [9] for an hyperbolic equation, and by BERGER & PODOLAK [10] and FUČIK [11] in a problem studied in [12] by means of a global inversion theorem.

In what follows we shall assume:

H1) λ_k is a simple eigenvalue of Δ with eigenfunction v_k;

H2) λ_{k-1} <const. $\leq \lambda_k + f'(s) \leq$ const. $< \lambda_{k+1}$;

H3) $f(s)$ is bounded.

(Here v_k indicate the eigenfunction associated to λ_k) Let V_k be denote the kernel of $\Delta + \lambda_k$ and let $V_k^{\perp}$ be his ortogonal complement; denoted by P_k and Q_k the projections on V_k and $V_k^{\perp}$ resp., we set $u = tv_k + w$, with $w \in V_k^{\perp}$ and project (4) by means of P_k and Q_k. We obtain the equivalent system:

(6') $$\Delta w + \lambda_k w + Q_k f(tv_k + w) = Q_k g$$

(6") $$P_k f(tv_k + w) = P_k g$$

Define the mapping $\Phi(w) = \Delta w + \lambda_k w + Q_k f(tv_k+w)$ from $V_k^\perp$ in itself. Using the variational carachterization of λ_k, H2 implies Φ is locally invertible in every w. By H3 it follows Φ is prOper and therefore by global inversion theorem we can conclude Φ is a diffeomorphism from $V_k^\perp$ onto $V_k^\perp$. This enables us to state:

<u>Lemma 1. Assume H1-H3. Then for every $g \in L^2$ and $t \in R$, (6') has a unique solution $w(g,t)=w_g(t)$. Moreover $w_g(t)$ is a continuous function of t and $\| w_g(t) \| \le c$, with c constant not depending on t</u>.

Fixed $g \in L^2$, we put $w_g(t)$ in (6") and obtain

$$\Gamma_g(t) = \int_\Omega f(tv_k+w_g(t))\, v_k = \int_\Omega g v_k \tag{7}$$

We study now the function $\Gamma_g(t)$. Let us introduce the following symbols:

$$\underline{f}(\mp\infty) = \min\lim_{s\to\mp\infty} f(s), \qquad \overline{f}(\mp\infty) = \max\lim_{s\to\mp\infty} f(s)$$

$$\underline{b}^{\mp} = \underline{f}(\mp\infty)\int_{v_k>0} v_k + \underline{f}(\pm\infty)\int_{v_k<0} v_k$$

$$\overline{b}^{\mp} = \overline{f}(\mp\infty)\int_{v_k>0} v_k + \overline{f}(\pm\infty)\int_{v_k<0} v_k$$

<u>Lemma 2. Assume H1-H3. Then for every $g \in L^2$ it results</u>:

i) $\min\lim_{t\to+\infty} \Gamma_g(t) \ge \underline{b}^+$

ii) $\min\lim_{t\to-\infty} \Gamma_g(t) \ge \underline{b}^-$

iii) $\max\lim_{t\to+\infty} \Gamma_g(t) \le \overline{b}^+$

iv) $\max\lim_{t\to-\infty} \Gamma_g(t) \le \overline{b}^-$

<u>Proof</u>. Let $t_n \to \infty$. By Lemma 1 $\| w_g(t_n)\| \le c$ and then $w_n = w_g(t_n)$ contains a converging subsequence (which we label w_n too) in L^2. Then if $x \in \Omega$: $v_k > 0$ we have $t_n v_k + w_n \to \infty$, while if $x \in \Omega$: $v_k < 0$ then $t_n v_k + w_n \to -\infty$. An application of Fatou's Lemma gives i. The same

arguments prove ii, iii and iv.

If we streghten H3, lemma 2 can be improved:

Lemma 3. Assume H1, H2 and

H4) $$\lim_{s \to \mp\infty} f(s) = f(\mp\infty)$$

Then:

i) $\lim_{t \to \infty} \Gamma_g(t) = \underline{b}^+ = \overline{b}^+ = b^+$

ii) $\lim_{t \to -\infty} \Gamma_g(t) = \underline{b}^- = \overline{b}^- = b^-$.

Now we state our main theorems:

Theorem 3. Assume H1, H2 and H3. Then for every $g \in L^2$ there exist $\underline{a}_g \leq \overline{a}_g$ with $\underline{a}_g \leq \min(\overline{b}^+, \overline{b}^-)$, $\overline{a}_g \geq \max(\underline{b}^+, \underline{b}^-)$ such that: if $P_k g \in]\underline{a}_g, \overline{a}_g[$ then (4) has at least one solution, while if $P_k g \notin [\underline{a}_g, \overline{a}_g]$ then (4) has no solutions.

Theorem 4. Assume H1, H2 and H4. Then for every $g \in L^2$ there exist $\underline{a}_g \leq \overline{a}_g$ with $\underline{a}_g \leq \min(b^+, b^-)$, $\overline{a}_g \geq \max(b^+, b^-)$ such that the same results of Theorem 3 obtain.

The proofs are easy consequence of Lemmas 2 and 3 above, taking $\underline{a}_g$ as the minimum of Γ_g and $\overline{a}_g$ the maximum.

Corollary If we assume also (5) then the necessary and sufficient condition of Theorem 2 obtain.

Remark. If it results $\underline{a}_g < \min(b^+, b^-)$, namely if (4) has solution for same g, with $P_k g < \min(b^+, b^-)$, then for such a g (4) possesses ~~another~~ a more solution. We shall see these kinds of multeplicity results in examples below.

The method above applies to a large variety of situations, and in

In many cases, under additional hypotheses on f, the theorems 3 and 4 can be considerably streghtened. Among others, we expose here tree examples:

Example 1. The problem

$$\Delta u + \lambda_1 u + \frac{u}{\sqrt{1+u^2}} = g \quad \text{on } \Omega \ , \quad u = 0 \quad \text{on } \partial\Omega$$

has exactly one solution if and only if $|\int g v_1| < \int v_1$.

Example 2. Consider the problem

$$\text{(8)} \qquad \Delta u + \lambda_1 u + \frac{u}{1+u^2} = g \quad \text{on } \Omega \ , \quad u = 0 \text{ on } \partial\Omega$$

Then for every $g \in L^2$ there exist $\underline{a}_g < 0 < \overline{a}_g$ such that:

i) (8) has solutions if and only if $\int g v_1 \in [\underline{a}_g, \overline{a}_g]$;

ii) if $\int g v_1 \ ]\underline{a}_g, 0[\cup]0, \overline{a}_g[$ then (8) has at least two distinct solutions.

Example 3. Consider the problem

$$\text{(9)} \qquad \Delta u + \lambda_1 u + \frac{1}{1+u^2} = g \quad \text{on } \Omega \ , \quad u = 0 \quad \text{on } \partial\Omega$$

Then for every $g \in L^2$ there exist $\overline{a}_g > 0$ such that:

i) (9) has solutions if and only if $\int g v_1 \in]0, \overline{a}_g]$;

ii) if $\int g v_1 \in]0, \overline{a}_g[$ then (9) has at least two distinct solutions.

Remarks. Example 1 is considered in [13]. Problem (8) is studied in a recent paper of KASDAN & WARNER [14]. Our results improves their statement. In general many of the results of [14] can be precised using the method above, which seems to be simpler too.

3. THE SUBLINEAR CASE: AN APPLICATION TO NONLINEAR ELASTICITY

The global Lyapunov-Schmidt method can be applied also when either $f(s) \longrightarrow \infty$ as $s \longrightarrow -\infty$ or $f(s) \longrightarrow -\infty$ as $s \longrightarrow \infty$ or both. In this case theorems as Theorem 3 or 4 can be proved but either $\underline{a}_g$ or $\overline{a}_g$ or both allow to be infinite. It can be useful to recall that if $s^{-1} f(s) \longrightarrow -\infty$ for $|s| \longrightarrow \infty$ then it is possible to use also the variational approach. Besides weakening the assumption at infinity, here we can study the existence of multiple solutions without any parity condition. To illustrate the sort of results which can be proved, we shall consider an application to nonlinear elasticity: the Van Karman equations for a clamped plate (we deal for semplicity with a plate, but the same arguments apply in the case of a shell sufficiently shallow)

We consider a thin plate whose shape is $\Omega \subset R^2$. The equilibrium states of the plate subjected to forces along $\partial\Omega$ and to external force q are the solutions of the following system

$$(10) \qquad \begin{aligned} \Delta^2 \psi &= -\frac{1}{2}[u,u]^{(')} \\ \Delta^2 u &= [\psi, u] + q \end{aligned} \qquad \text{on } \Omega$$

$$u = u_x = u_y = 0, \qquad \psi = \lambda \psi_1, \quad \frac{\partial \psi}{\partial n} = \lambda \psi_2 \qquad \text{on } \partial\Omega$$

In the equations above u is the deflection of the plate from the initial state, ψ is the Airy stress function, ψ_1 and ψ_2 are the edge stresses and λ is their magnitude. Finally the boundary conditions on u signify the plate is clamped and n is the outer normal at $\partial\Omega$ which is supposed smooth. Let Ψ_0 be the solution of the Dirichlet problem $\Delta^2 \Psi_0 = 0$ on Ω, $\Psi_0 = \psi_1$, $\frac{\partial \Psi_0}{\partial n} = \psi_2$ on $\partial\Omega$ (we assume ψ_1 ψ_2 smooth). We pose $\psi = \lambda \Psi_0 + F$ and substitute in (10). It results

$$\begin{aligned} \Delta^2 F &= -\frac{1}{2}[u,u] \\ \Delta^2 u &= \lambda[\psi_0,u] + [F,u] + q \end{aligned} \qquad \text{on } \Omega \tag{11}$$

$$u = \frac{\partial u}{\partial n} = 0, \qquad F = \frac{\partial F}{\partial n} = 0 \qquad \text{on } \partial\Omega$$

We look for solutions of (11) for $\mu_1 < \lambda < \mu_2$, where μ_i are the eigenvalues of $\Delta^2 u = \mu[\psi_0,u]$ with Dirichlet boundary conditions.
Using arguments similar to those of section 2 in connection with a global inversion theorem in presence of singularities (see [12]) it is possible to prove the following result:

<u>Theorem</u> 5. <u>Assume</u> $\mu_1 < \lambda < \mu_2$. <u>Then (11) has at least one solution</u> (u,F). <u>Moreover there exists a set</u> $A \subset L^2$ <u>such that if</u> $q \notin A$ <u>then</u> <u>(11) has at least tree distinct solutions</u>.

For other results on the Van Karman equation we refer, among others, to BERGER & FIFE [15], BERGER [16], BERGER [17], RABINOWITZ [18], [19] NAUMANN [20], and [21].

REFERENCES

1. A.AMBROSETTI & P.H.RABINOWITZ. J. Funct. Anal. 1973
2. A.AMBROSETTI. M.R.C. Techn. Summ. Rep. N.1442, 1974
3. ------------ A note on the Lusternik-Schnierelman theory for functionals which are not even. To appear on Ric. di Mat.
4. A.AMBROSETTI & G.MANCINI. Existence and multeplicity results for nonlinear elliptic boundary value problems. To appear.
5. E.M.LANDESMAN & A.C.LAZER. J. Math. Mech. 1970
6. P.HESS. Ind. Univ. Math. J. 1974
7. S.A.WILLIAMS. J. Diff. Eq. 1969
8. S.FUČIK, M.KUČERA & J.NEČAS. J. Diff. Eq. 1975
9. G.PRODI & A.AMBROSETTI. Analisi non lineare, Quad. I. Scuola Norm. Sup. Pisa 1975
10. M.S.BERGER & E.PODOLAK. Ind. Univ. Math. J. 1975
11. S.FUČIK. Boll. U.M.I. 1975
12. A.AMBROSETTI & G.PRODI. Ann. Mat. Pura Appl. 1973
13. H.BREZIS. Proc. Scheveningen Meet. on P.D.E., North Holland 1976
14. J.L.KASDAN & F.W.WARNER. Comm. Pure Appl. Math. 1975
15. MS.BERGER & P.FIFE. Bull. A.M.S. 1966
16. M.S.BERGER. Comm. Pure Appl. Math. 1967
17. ---------- Ind. Univ. Math. J. 1971
18. P.H.RABINOWITZ. Ind.UNiv. Math J. 1974
19. -------------- A note on topological degree for potential operators. To Appear
20. J.NAUMANN. J. de Mech. 1974
21. A.AMBROSETTI. Ric. di Mat. 1974