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REMARKS ON NONLINEAR ELLIPTIC PROBLEMS

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## REMARKS ON NONLINEAR ELLIPTIC PROBLEMS

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#### O. INTRODUCTION

Our main purpose here is to consider existence, nonexistence and multeplicity results for boundary value problems of the type

(1) 
$$\Delta u + p(u) = g(x)$$
 on  $\Omega$ ,  $u = 0$  on  $\partial \Omega$ ,

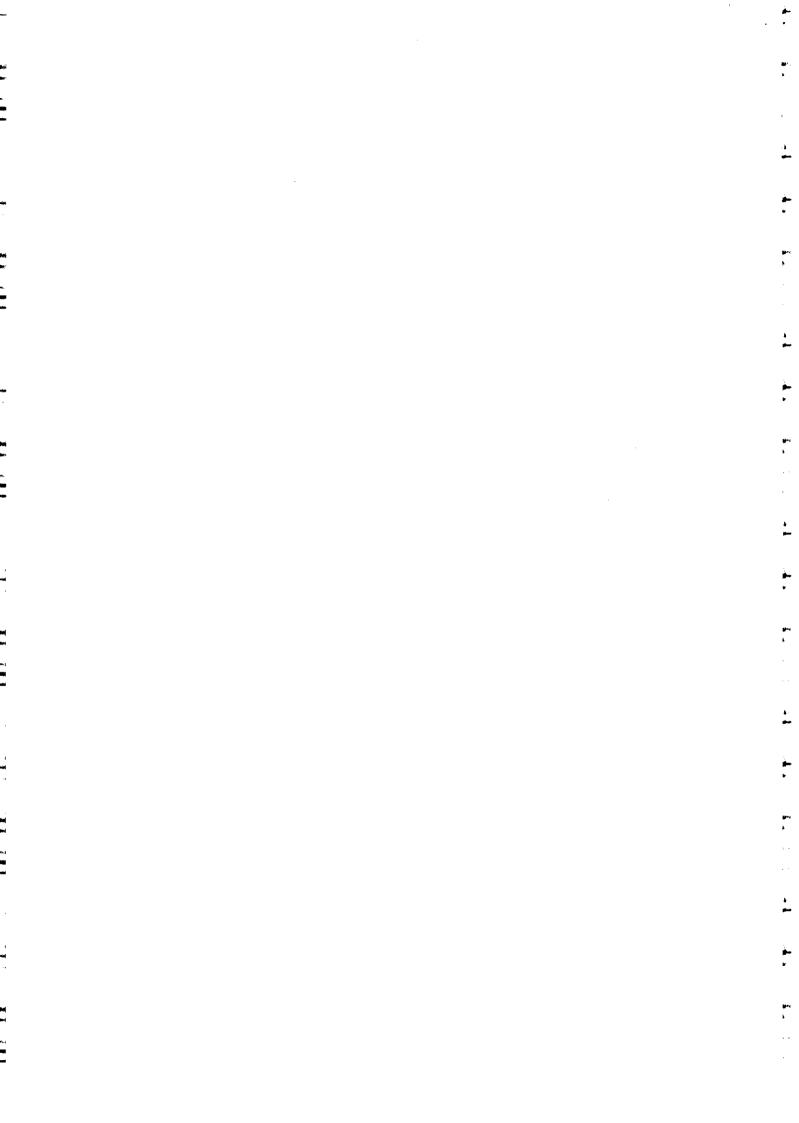
where  $\Omega \subset \mathbb{R}^N$  is open, bounded and  $p \in C(\mathbb{R},\mathbb{R})$ . Clearly we consider for semplicity the Laplace operator  $\Delta$  instead of more general elliptic operator, as well as we could consider p depending on x, etc. Obviously the method to study (1) will depend on the behaviour of p. We shall consider in what follows 3 tipical cases.

## 1. A VARIATIONAL APPROACH

The first is a variational approach and applies, for ex., when g=0 and p(s) is superlinear at infinity. We will sketch briefly the framework of this method and refer for more details to [1]. The solutions of (1) are the critical points of the functional

$$J(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} P(u(x)) dx, \quad P(u) = \int_{0}^{u} p(s) ds$$

in E =  $W_o^{1,2}(\Omega)$  where  $||u||^2 = \int |\nabla u|^2$  is taken as norm . These critical points are investigated by means of the Lusternik-Schni-



relman theory. The basic tool of such a teory is the definition of a topological inveriant: we will define here the genus.

Definition. Let  $X \subset E - \{0\}$ , X symmetric. The genus  $\gamma(X)$  of X is defined as the least integer n such that  $\exists \phi \in C(X, \mathbb{R}^n - \{0\})$ ,  $\phi$  odd.

One of the main property of the genus is that if  $h \in C(E,E)$  is odd, then  $\gamma(h(X)) > \gamma(X)$ . The genus permits us to define some minimax level which, under suitable conditions on J, are critical, i.e. at such level there is at least one critical point of J. More precisely let us assume:

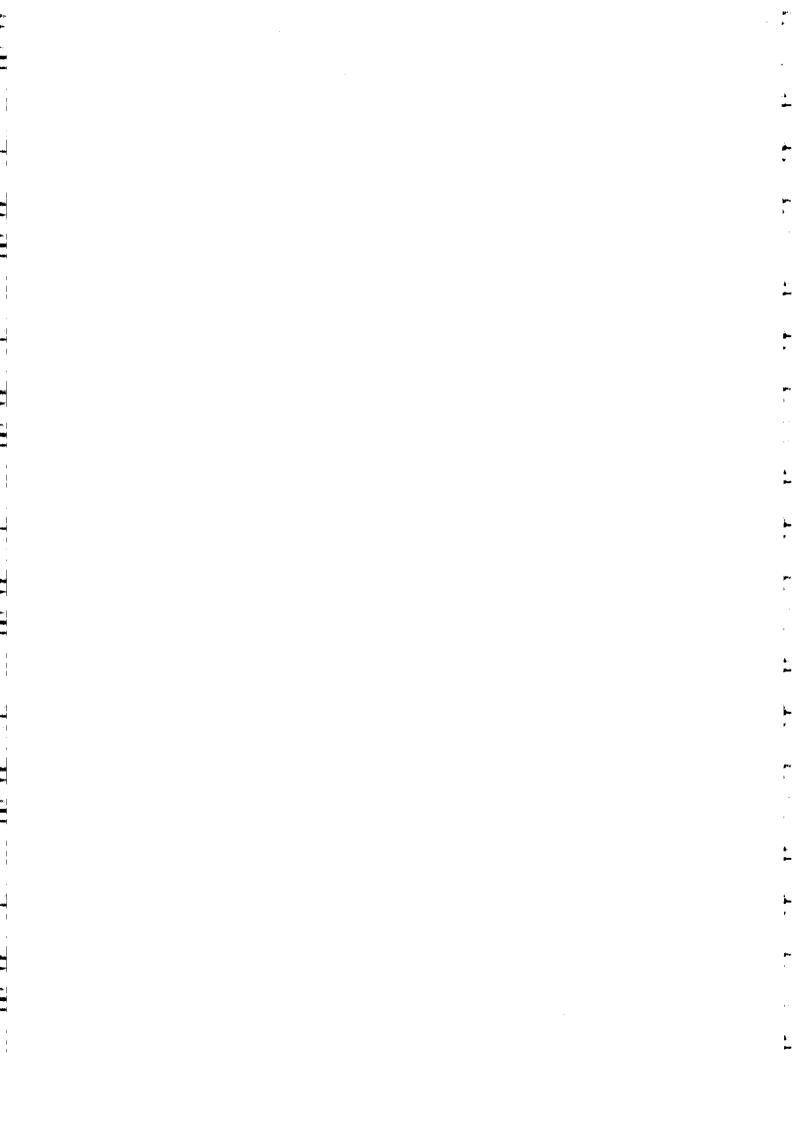
- A1)  $|p(s)| \le a + b|s|^{\alpha}$ ,  $1 \le \alpha < \frac{N+2}{N-2}$  for N>2, and any  $\alpha$  for  $N \le 2$ ;
- A2)  $s^{-1}p(s) \rightarrow \infty$  as  $|s| \rightarrow \infty$ ; p(s) = o(|s|) as  $s \rightarrow 0$ ;
- A3)  $\exists a: \text{ for } |s| \ge a \quad P(s) \le \theta p(s) s \text{ with } \theta \in [0, \frac{1}{2}[];$
- A4) p(s) is odd.

Using the genus we define a class of sets  $G_n$  with the propertiy: for every  $X \in G_n$   $h(X) \in G_n$  whenever  $h \in C(E,E)$ , h odd and  $J(h(u)) \nleq J(u) \forall u \in E$ . By A2 it follows  $G_n$  is not empty and thus we can define

$$c_n = \inf_{X \in \mathcal{L}_n} \sup_{u \in X} J(u)$$

A2 implies  $0 < c_n$  and thus  $c_n$  will be not the trivial level. Moreover A1 and A3 imply J satisfies a suitable compactness condition (the P-S condition); therefore if  $c_n$  is not a critical level it is possible to define a descendent flow h of J in such a way that  $\exists$   $\epsilon$ 

From A4 it follows that J is even and thus h can be taken odd. But in this case  $X \in G_n$  implies  $h(X) \in G_n$  and thus (2) contradicts the definition of  $c_n$ . Finally it is possible to show A2 implies  $c_n \uparrow \infty$ .



Then we can state:

Theorem 1. Let the assumptions A1-A4 be satisfied. Then the boundary value problem (1) has infinitely many solutions.

We end this sections with some remarks: the condition A4 is essential in the arguments above. When this parity conditions is dropped, then we known only weak results. For example for the problem

(3) 
$$\Delta u + u^3 = g \quad \text{on } \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega$$

it is only known [2] that  $\forall$ n  $\exists \varepsilon$  such that (3) has at least n solutions. Another abstract result in the study of critical points relaxing the eveness condition can be found in [3].

#### 2. A NONVARIATIONAL METHOD

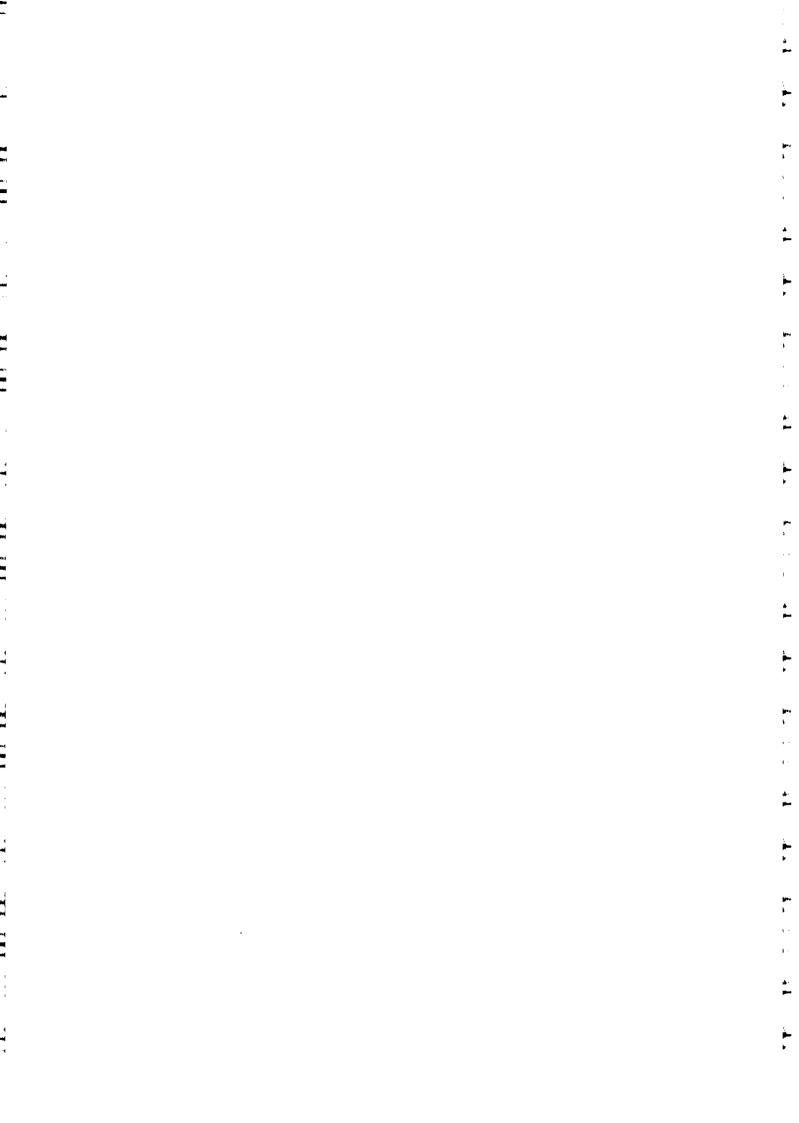
If p(s) is asymptotically linear at infinity the approach above cannot be applied. We will treat this case in this section; the results we expose are contained in a joint paper with G. MANCINI [4]. Consider

(4) 
$$\Delta u + \lambda_k u + f(u) = g \quad on \Omega, \quad u = 0 \quad on \ \partial \Omega$$

where  $\lambda_k$  is an eigenvalue of  $\Delta$  and f is  $C^1$  and bounded. Problem (4) has been studied by LANDESMAN & LAZER [5], assuming  $\lambda_k$  is simple,  $f(s) \to f(\tau)$  as  $s \to \tau$ , and

(5) 
$$f(-\infty) < f(s) < f(+\infty)$$

They are able to prove the following interesting result:



Theorem 2. Under the conditions above, (4) has solutions if and only if (denoted by  $v_k$  the eigenfunction associated to  $\lambda_k$ )

For a simple proof, see HESS [6].

Several people extended the Landesman & Lazer result to the case of ways multiple eigenvalue, but assuming anyway (5). Among others we recall WILLIAMS [7] and FUCIK, KUCERA & NECAS [8]. On the contrary we know only few and partial results about solvability of (4) without the hypothesis (5). Our goal here is to study such a problem. Precisely we are mainly interested to prove existence and nonexistence theorems, and, possible, to find multiple solutions of (4).

The approach we use is a global version of the classical Lyapunov-Schmidt method. Such a technique has been used by PRODI and the author in [9] for an hyperbolic equation, and by BERGER & PODOLAK [10] and FUCIK [11] in a problem studied in [12] by means of a global inversion theorem.

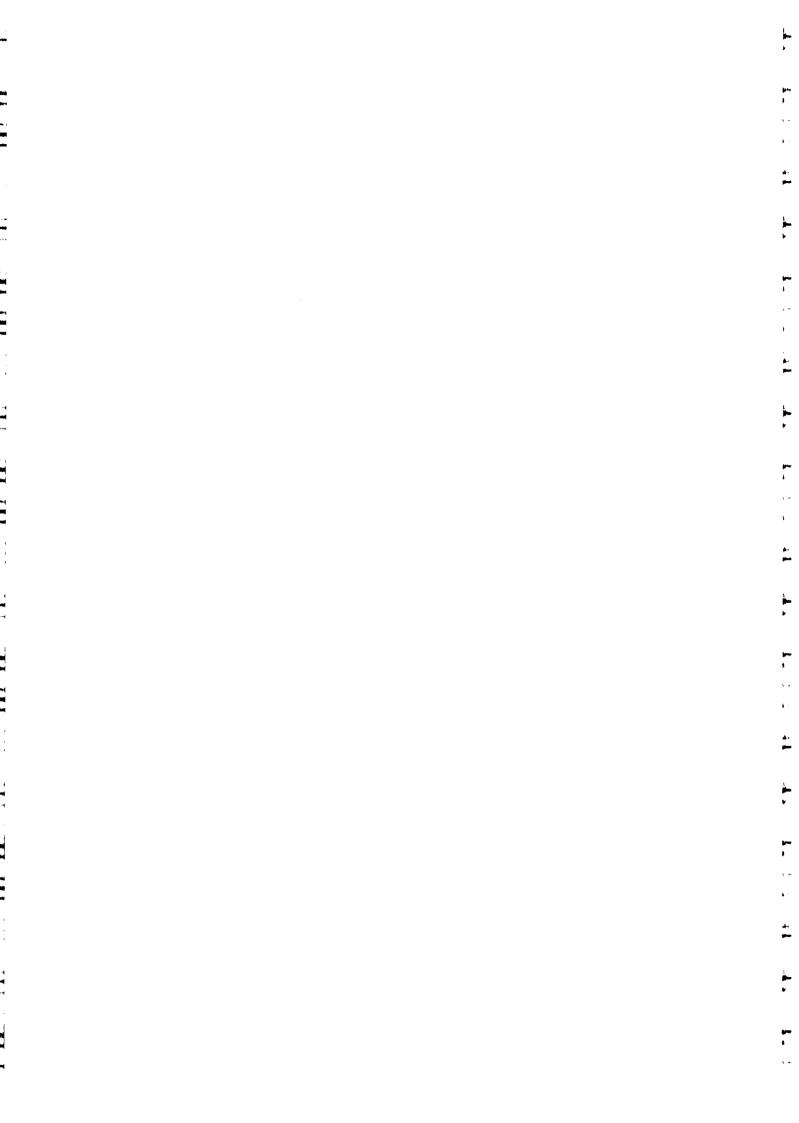
In what follows we shall assume:

- H1)  $\lambda_k$  is a simple eigenvalue of  $\Lambda$  with eigenfunction  $v_k$ ;
- H2)  $\lambda_{k-1}$  <const.  $\leq \lambda_k + f'(s) \leq const. < \lambda_{k+1}$ ;
- H3) f(s) is bounded.

(Here  $v_k$  indicate the eigenfunction associated to  $\lambda_k$ ) Let  $V_k$  be denote the kernel of  $\Delta + \lambda_k$  and let  $v_k^\perp$  be his ortogonal complement; denoted by  $P_k$  and  $Q_k$  the projections on  $V_k$  and  $V_k^\perp$  resp., we set  $u=tv_k^\perp+w$ , with  $w\in V_k^\perp$  and project (4) by means of  $P_k$  and  $Q_k$ . We obtain the equivalent system:

$$\Delta w + \lambda_k w + Q_k f(tv_k + w) = Q_k g$$

(6") 
$$P_{k}f(tv_{k}+w) = P_{k}g$$



Define the mapping  $\phi(w) = \Delta w + \lambda_k w + Q_k f(tv_k + w)$  from  $V_k^{\perp}$  in itself. Using the variational carachterization of  $\lambda_k$ , H2 implies  $\phi$  is locally invertible in every w. By H3 it follows  $\phi$  is proper and therefore by global inversion theorem we can conclude  $\phi$  is a diffeomorphism from  $V_k^{\perp}$  onto  $V_k^{\perp}$ . This enables us to state:

Lemma 1. Assume H1-H3. Then for every  $g \in L^2$  and  $t \in R$ , (6') has a unique solution  $w(g,t)=w_g(t)$ . Moreover  $w_g(t)$  is a continuous function of t and  $\|w_g(t)\| \le c$ , with c constant not depending on t.

Fixed  $g \in L^2$ , we put  $w_g(t)$  in (6") and obtain

(7) 
$$r_{g}(t) = \int_{\Omega} f(tv_{k} + w_{g}(t)) v_{k} = \int_{\Omega} gv_{k}$$

We study now the function  $\Gamma_g(t)$ . Let us introduce the following symbols:

$$\frac{f}{f}(\overline{+}\infty) = \min \lim_{s \to \pm \infty} f(s), \quad \overline{f}(\overline{+}\infty) = \max \lim_{s \to \pm \infty} f(s)$$

$$\underline{b}^{\overline{+}} = \underline{f}(\overline{+}\infty) \int v_k + \underline{f}(\underline{+}\infty) \int v_k$$

$$\overline{b}_{k>0} \quad v_{k<0}$$

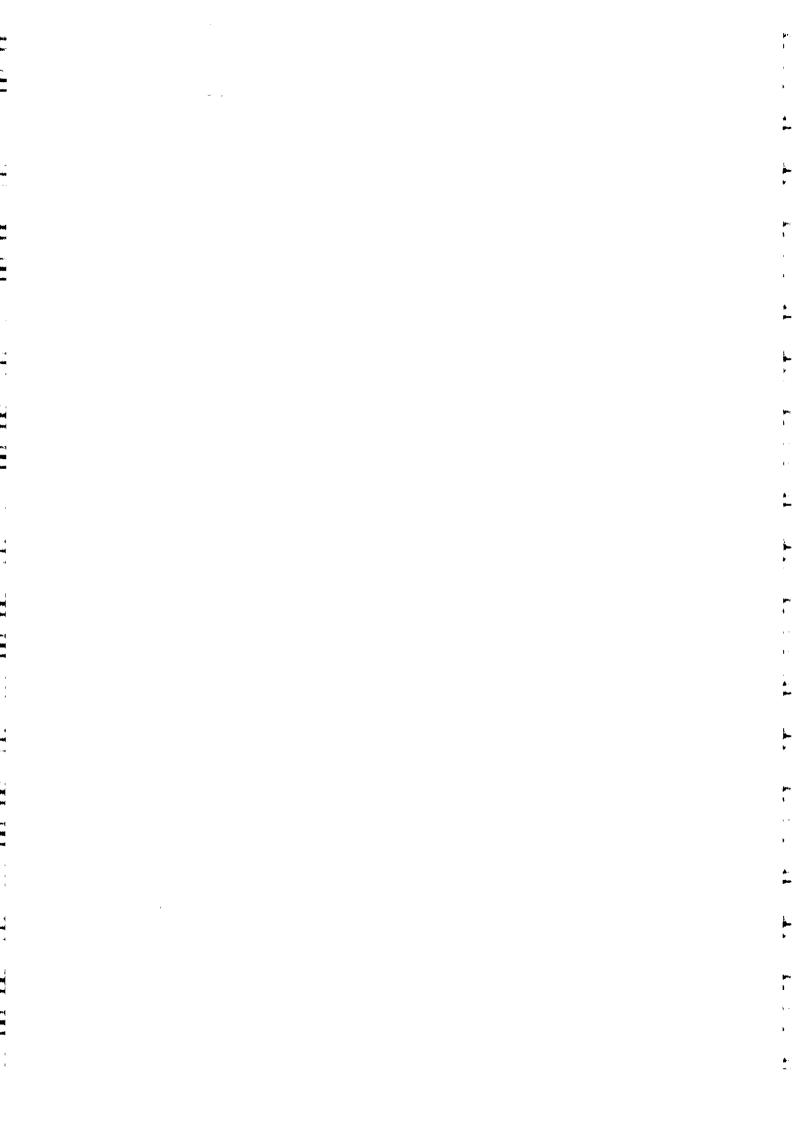
$$\overline{b}_{k} = \overline{f}(x) \left( v_k + \overline{f}(\underline{+}\infty) \right) \left( v_k + \overline{f}(\underline{+}\infty$$

$$\overline{b}^{\mp} = \overline{f}(F^{\infty}) \int_{V_{k}>0} V_{k} + \overline{f}(\tilde{T}^{\infty}) \int_{V_{k}<0} V_{k}$$

Lemma 2. Assume H1-H3. Then for every  $g \in L^2$  it results:

- i) min lim  $\Gamma_g(t) \ge \underline{b}^{\dagger}$
- ii) min lim  $\Gamma_g(t) \ge \underline{b}$
- iii)  $\max_{t \to +\infty} \lim_{g} (t) \leq b^+$
- iv)  $\max_{x} \lim_{x \to \infty} \Gamma_{g}(t) \leq b$

<u>Proof.</u> Let  $t_n \to \infty$ . By Lemma 1  $\|\mathbf{w}_g(t_n)\| \le c$  and then  $\mathbf{w}_n = \mathbf{w}_g(t_n)$  contains a converging subsequence (which we label  $\mathbf{w}_n$  too) in L<sup>2</sup>. Then if  $\mathbf{x} \in \Omega$ :  $\mathbf{v}_k > 0$  we have  $t_n \mathbf{v}_k + \mathbf{w}_n \to \infty$ , while if  $\mathbf{x} \in \Omega$ :  $\mathbf{v}_k < 0$  then  $t_n \mathbf{v}_k + \mathbf{w}_n \to -\infty$ . An application of Fatou's Lemma gives i. The same



arguments prove ii, iii and iv.

If we streghten H3, lemma 2 can be improved:

#### Lemma 3. Assume H1, H2 and

H4) 
$$\lim_{s \to \mp \infty} f(s) = f(\mp \infty)$$

Then:

i) 
$$\lim_{t \to \infty} \Gamma_g(t) = \underline{b}^+ = \overline{b}^+ = b^+$$

ii) 
$$\lim_{t\to -\infty} r_g(t) = \underline{b} = \overline{b} = b$$
.

Now we state our main theorems:

Theorem 3. Assume H1,H2 and H3. Then for every  $g \in L^2$  there exist  $a_g \leq a_g$  with  $a_g \leq \min(b^+,b^-)$ ,  $a_g \geq \max(b^+,b^-)$  such that: if  $P_k g \in J_{\underline{a}g}$ ,  $a_g \in J_{\underline{a}g}$ ,  $a_g$ 

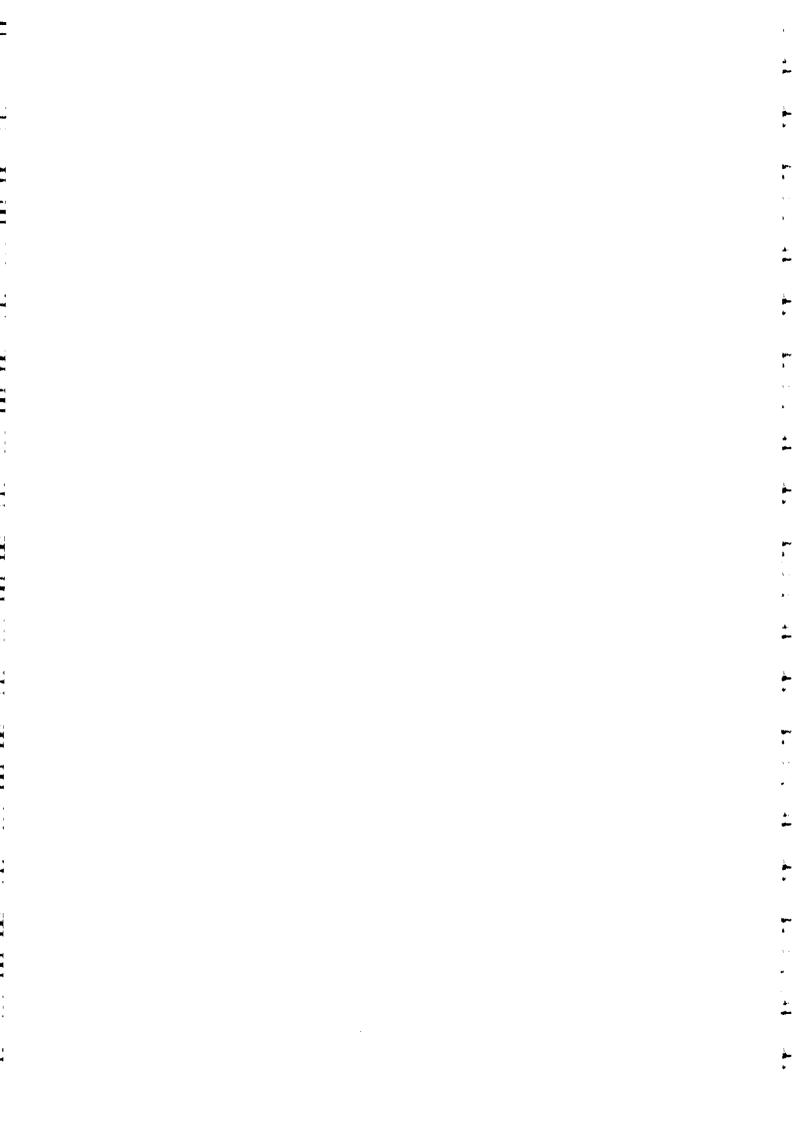
Theorem 4. Assume H1, H2 and H4. Then for every  $g \in L^2$  there exist  $\frac{a}{g} \leq \frac{a}{g}$  with  $\frac{a}{g} \leq \min(b^+, b^-)$ ,  $\frac{a}{g} \geq \max(b^+, b^-)$  such that the same results of Theorem 3 obtain.

The proofs are easy consequence of Lemmas 2 and 3 above, taking  $\underline{\underline{a}}_g$  as the minimum of  $\Gamma_g$  and  $\overline{\underline{a}}_g$  the maximum.

Corollary If we assume also (5) then the necessary and sufficient condition of Theorem 2 obtain.

Remark. If it results  $\underline{a} < \min(b^{\dagger}, b^{-})$ , namely if (4) has solution for same g, with  $P_k g < \min(b^{\dagger}, b^{-})$ , then  $\Lambda(4)$  possesses a more solution. We shall see these kinds of multeplicity results in examples below.

The method above applies to a large variety of situations, and in



\*\*many cases, under additional hypotheses on f, the theorems 3 and 4 can be considerably streightened. Among others, we expose here tree examples:

#### Example 1. The problem

$$\Delta u + \lambda_1 u + \frac{u}{\sqrt{1+u^2}} = g \text{ on } \Omega$$
,  $u = 0 \text{ on } \partial \Omega$ 

has exactly one solution if and only if  $|\int gv_1| < \int v_1$ .

## Example 2. Consider the problem

(8) 
$$\Delta u + \lambda_1 u + \frac{u}{1+u^2} = g \quad \text{on } \Omega$$
,  $u = 0 \quad \text{on } \partial \Omega$ 

Then for every  $g \in L^2$  there exist  $\frac{a}{g} < 0 < \frac{a}{g}$  such that:

- i) (8) has solutions if and only if  $\int gv_1 \in [\underline{a}_g, \overline{a}_g]$ ;
- ii) if  $\int gv_1 = \frac{1}{g}$ , O[U]0,  $a_g$  then (8) has at least two distinct solutions.

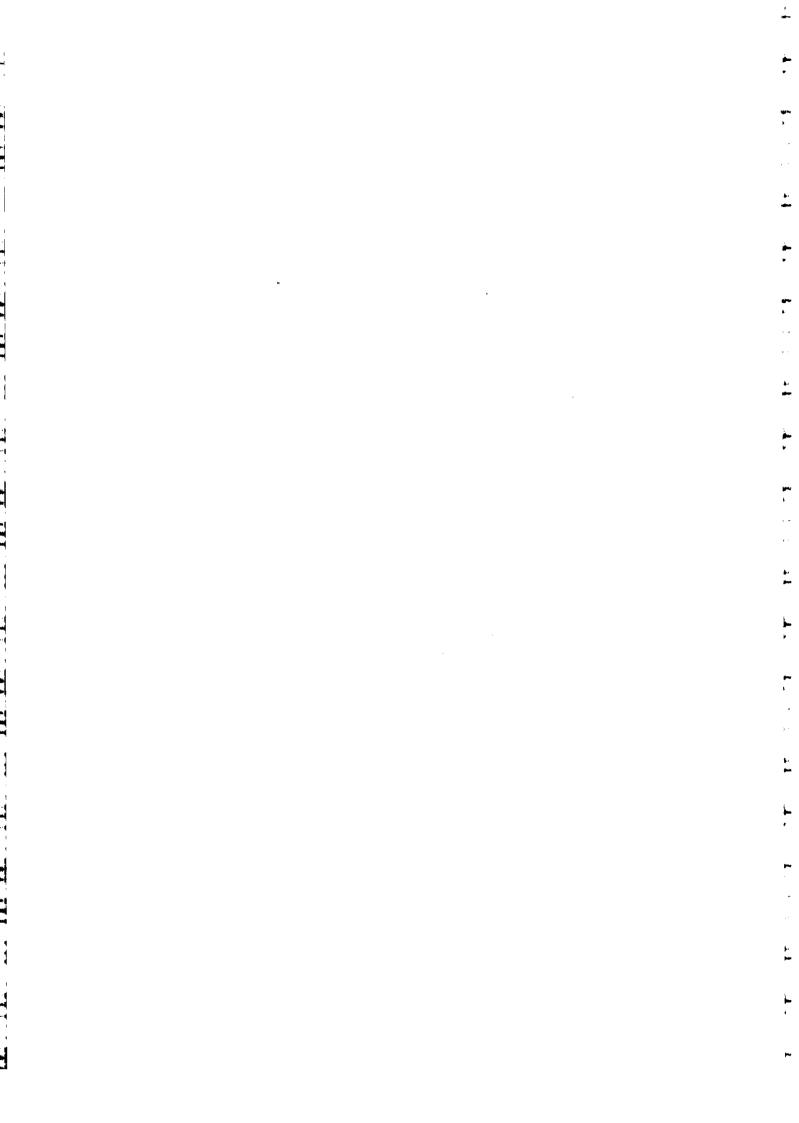
## Example 3. Consider the problem

(9) 
$$\Delta u + \lambda_1 u + \frac{1}{1+u^2} = g \text{ on } \Omega$$
,  $u = 0 \text{ on } \partial \Omega$ 

Then for every  $g \in L^2$  there exist  $a_g > 0$  such that:

- i) (9) has solutions if and only if  $\int gv_1 \in \left]0, a_g\right]$ ;
- ii) if  $\int gv_1 \in \left]0, \overline{a}_g\right[$  then (9) has at least two distinct solutions.

Remarks. Example 1 is considered in [13]. Problem (8) is studied in a recent paper of KASDAN & WARNER [14]. Our results improves their statement. In general many of the results of [14] can be precised using the method above, which seems to be simpler too.



# 3. THE SUBLINEAR CASE: AN APPLICATION TO NONLINEAR ELASTICITY

The global Lyapunov-Schmidt method can be applied also when either  $f(s) \longrightarrow \infty$  as  $s \longrightarrow -\infty$  or  $f(s) \longrightarrow -\infty$  as  $s \longrightarrow \infty$  or both. In this case theorems as Theorem 3 or 4 can be proved but either  $a_g$  or  $a_g$  or both allow to be infinite. It can be useful to recall that if  $s^{-1}f(s) \longrightarrow -\infty$  for  $|s| \longrightarrow \infty$  then it is possible to use also the variational approach. Besides weakening the assumption at infinity, here we can study the existence of multiple solutions without any parity condition. To illustrate the sort of results which can be proved, we shall consider an application to nonlinear elasticity: the Van Karman equations for a clamped plate (we deal for semplicity with a plate, but the same arguments apply in the case of a shell sufficiently shallow)

We consider a thin plate whose shape is  $\Omega \subset \mathbb{R}^2$ . The equilibrium states of the plate subjected to forces along  $\partial \Omega$  and to external force q are the solutions of the following system

(10) 
$$\Delta^{2}\psi = -\frac{1}{2}[u,u]^{(')}$$
on  $\Omega$ 

$$\Delta^{2}u = [\psi,u] + q$$

$$u = u_{x} = u_{y} = 0, \quad \psi = \lambda\psi_{1}. \quad \frac{\partial\psi}{\partial n} = \lambda\psi_{2} \quad \text{on } \partial\Omega$$

In the equations above u is the deflection of the plate from the initial state,  $\psi$  is the Airy stress function,  $\psi_1$  and  $\psi_2$  are the edge stresses and  $\lambda$  is their magnitude. Finally the boundary conditions on u signify the plate is clamped and n is the outer normal at  $\partial \Omega$  which is supposed smooth. Let  $\psi_0$  be the solution of the Dirichlet problem  $\Delta^2 \psi_0 = 0$  on  $\Omega$  ,  $\psi_0 = \psi_1$  ,  $\frac{\partial \psi_0}{\partial n} = \psi_2$  on  $\partial \Omega$  (we assume  $\psi_1$   $\psi_2$  smooth). We pose  $\psi = \lambda \psi_0 + F$  and substitute in (10). It results

$$\Delta^{2}F = -\frac{1}{2}[u,u]$$
on  $\Omega$ 

$$\Delta^{2}u = \lambda[\Psi_{0},u] + [F,u] + q$$

$$u = \frac{\partial u}{\partial n} = 0, \quad F = \frac{\partial F}{\partial n} = 0 \quad \text{on } \partial\Omega$$

We look for solutions of (11) for  $\mu_1 < \lambda < \mu_2$ , where  $\mu_i$  are the eigenvalues of  $\Delta \omega$  with Dirichlet boundary conditions.

Using arguments similar to those of section 2 in connection with a global inversion theorem in presence of singularities (see [12]) it is possible to prove the following result:

Theorem 5. Assume  $\mu_1 < \lambda < \mu_2$ . Then (11) has at least one solution (u,F). Moreover there exists a set  $A \subset L^2$  such that if  $q \in A$  then (11) has at least tree distinct solutions.

For other results on the Van Karman equation we refer, among others, to BERGER & FIFE [15], BERGER [16], BERGER [17], RABINOWITZ [18], [19] NAUMANN [20], and [21].

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