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AUTUMN COURSE ON APPLICATIONS OF ANALYSIS
TO MECHANICS

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MATHEMATICAL ASPECTS OF FLUID FLOW

Part.2

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These are preliminary lecture notes intended for participants only.
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room 112.

Part 2. Simple properties of the Navier-Stokes system

Here and in Part 3, we shall use the notation

$$N(\underset{\sim}{v}, \underset{\sim}{w}) \quad = \quad (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{w} \quad = \quad v_j \frac{\partial w_i}{\partial x_j} \qquad \text{(summation over } j = 1,2,3\text{)}$$

Hence the fluid domain $D \subset \mathbb{R}^3$ is taken to be <u>fixed</u> (and always D is a bounded open set with boundary ∂D).

Applying to a Newtonian fluid, the Navier-Stokes system is

$$\left.\begin{aligned} \frac{\partial \underset{\sim}{u}}{\partial t} + R\, N(\underset{\sim}{u}, \underset{\sim}{u}) &= -\nabla p + \Delta \underset{\sim}{u}, \\ \nabla \cdot \underset{\sim}{u} = 0, \qquad \underset{\sim}{u}\big|_{\partial D} &= \underset{\sim}{q}. \end{aligned}\right\} \qquad (1)$$

Here $\underset{\sim}{q}$ is a prescribed vector field on the boundary ∂D, and for consistency with $\nabla \cdot \underset{\sim}{u} = 0$ it must satisfy

$$\int_{\partial D} \underset{\sim}{q} \cdot \underset{\sim}{n} \; dx \quad = \quad 0.$$

(<u>Exercise</u>. Prove why this condition is necessary and explain what it means in physical terms.)

For a <u>steady flow</u> corresponding to a time-independent $\underset{\sim}{q}$, say $\underset{\sim}{u} = \underset{\sim}{U}(\underset{\sim}{x})$ with $p = P(\underset{\sim}{x})$, the equations are

$$\left.\begin{aligned} R\, N(\underset{\sim}{U}, \underset{\sim}{U}) &= -\nabla P + \Delta \underset{\sim}{U}, \\ \nabla \cdot \underset{\sim}{U} = 0, \qquad \underset{\sim}{U}\big|_{\partial D} &= \underset{\sim}{q}. \end{aligned}\right\} \qquad (2)$$

Part 3 will outline the famous existence theory due to Leray, Hopf and Ladyzhenskaya, which shows that this nonlinear boundary-value problem has at least one solution for all (finite) values of the Reynolds number R. It can also be shown that, provided the boundary is reasonably smooth (all practical cases being covered), this solution has Hölder-continuous second derivatives, being thus a classical solution of the problem. (By this we mean that the p.d.e.'s in (2) are satisfied pointwise everywhere in $\bar{D}$). As will be discussed later, however, the system may have many different solutions when R is not small. Some of the solutions always represent <u>unstable</u> steady flows, but others may be stable. Furthermore, at values of R that are sufficiently large, the system has (turbulent) time-dependent solutions, even though the boundary values q_i are independent of time.

A simpler but very important proposition will now be proved: If $R > 0$ is sufficiently small (but <u>not</u> infinitesimal), the solution $\underset{\sim}{U}(\underset{\sim}{x}; R)$ of (2) is <u>unique</u> and is <u>globally stable</u> in the sense that solutions of (1) will converge to it as $t \to \infty$, whatever the initial velocity field $\underset{\sim}{u}(\underset{\sim}{x}, 0)$ on D.

Putting $\underset{\sim}{u} = \underset{\sim}{U}(\underset{\sim}{x}; R) + \underset{\sim}{v}(\underset{\sim}{x}, t)$, $\quad p = P + p'$ in (1), we obtain

$$\left.\begin{aligned} &\frac{\partial \underset{\sim}{v}}{\partial t} + R\{L\underset{\sim}{v} + N(\underset{\sim}{v}, \underset{\sim}{v})\} = -\nabla p' + \Delta \underset{\sim}{v}, \\ &\nabla \cdot \underset{\sim}{v} = 0, \qquad\qquad \underset{\sim}{v}\big|_{\partial D} = 0, \end{aligned}\right\} \qquad (3)$$

where $L\underset{\sim}{v} = N(\underset{\sim}{U}, \underset{\sim}{v}) + N(\underset{\sim}{v}, \underset{\sim}{U})$. For the time being, assume that $\underset{\sim}{v} = C^\infty(\bar{D} \times \mathbb{R} \to \mathbb{R}^3)$. Hence it is seen that

$$\int_D \underset{\sim}{v} \cdot N(\underset{\sim}{v}, \underset{\sim}{v})\, dx = \int_D v_i v_j \frac{\partial v_i}{\partial x_j}\, dx$$

$$= \int_D v_j \frac{\partial(\frac{1}{2} v_i^2)}{\partial x_j}\, dx = 0,$$

since $v_i|_{\partial D} = 0$ and $\partial v_j / \partial x_j = 0$. Also,

$$\left| \int_D \underset{\sim}{v} \cdot L \underset{\sim}{v}\, dx \right| = \left| \int_D v_i \left\{ \underbrace{U_j \frac{\partial v_i}{\partial x_j}}_{= 0} + v_j \frac{\partial U_i}{\partial x_j} \right\} dx \right|$$

$$= \left| \int_D v_i v_j \frac{\partial U_i}{\partial x_j}\, dx \right| \leq \max_{\substack{\underset{\sim}{x} \in \bar{D} \\ i, j \in \{1,2,3\}}} |\partial U_i / \partial x_j| \cdot 3 \left| \int_D v_i^2\, dx \right|.$$

(<u>EXERCISE</u>. Confirm the last inequality.)

<u>NOTATION</u>. 1. $(\underset{\sim}{v}, \underset{\sim}{w})_0 = \int_D \underset{\sim}{v} \cdot \underset{\sim}{w}\, dx = \int_D v_i w_i\, dx.$

This is the inner product for $L_2(D \to \mathbb{R}^3)$.

$$\|\underset{\sim}{v}\|_0 = (\underset{\sim}{v}, \underset{\sim}{v})^{\frac{1}{2}} > 0 \quad \text{if} \quad \underset{\sim}{v} \neq 0 \text{ a.e. on } D.$$

2. $(\underset{\sim}{v}, \underset{\sim}{w})_1 = \int_D \nabla \underset{\sim}{v} : \nabla \underset{\sim}{w}\, dx = \int_D \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial w_i}{\partial x_j}\, dx$ (SUMMATION CONVENTION TWICE).

$$\|\underset{\sim}{v}\|_1 = (\underset{\sim}{v}, \underset{\sim}{v})^{\frac{1}{2}}.$$

Note that this is <u>not</u> the norm for $H_1(D) = W_{1,2}(D)$.

However, if $\underset{\sim}{v} \in C^\infty(D \to \mathbb{R}^3)$ and $\underset{\sim}{v}|_{\partial D} = 0$ (or, more generally if $\underset{\sim}{v} \in \overset{\circ}{W}_{1,2}(D)$ or if $\underset{\sim}{v} \in \mathcal{H} \subset \overset{\circ}{W}_{1,2}(D)$, to be described later), then

$$\|\underset{\sim}{v}\|_0 \leq c\,\|\underset{\sim}{v}\|_1 \qquad \text{(Poincaré's inequality).}$$

Here c is a constant independent of $\underset{\sim}{v}$, depending <u>only</u> on the shape and size of the domain D. (<u>Exercise</u>. Show that for a rectangular domain with sides of length ℓ_1, ℓ_2, ℓ_3, the best estimate of c is

$$c = \pi\left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} + \frac{1}{\ell_3^2}\right)^{\frac{1}{2}}.$$)

Note also that
$$\int_D \underset{\sim}{v}\cdot\Delta\underset{\sim}{v}\,dx = -\|\underset{\sim}{v}\|_1^2$$

and
$$\int_D \underset{\sim}{v}\cdot\nabla\phi\,dx = 0 \qquad \forall\quad \phi \in C^1(\bar{D}\to\mathbb{R}).$$

(<u>Exercise</u>. Verify these equalities for $\underset{\sim}{v} \in C^\infty(\bar{D}\to\mathbb{R}^3)$, $\underset{\sim}{v}|_{\partial D} = 0$.)

Now, take the scalar product of $\underset{\sim}{v}$ and each term term of the p.d.e. satisfied by $\underset{\sim}{v}$. Using results just noted, we obtain

$$(\underset{\sim}{v}, \partial\underset{\sim}{v}/\partial t)_0 + R(\underset{\sim}{v}, L\underset{\sim}{v})_0 + \|\underset{\sim}{v}\|_1^2 = 0;$$

hence

$$0 = \frac{d}{dt}\left(\tfrac{1}{2}\|\underset{\sim}{v}\|^2\right) + R(\underset{\sim}{v}, L\underset{\sim}{v})_0 + \|\underset{\sim}{v}\|_1^2 \geq$$
$$\geq \qquad '' \qquad + (c^2 - Rb)\,\|\underset{\sim}{v}\|_0^2,$$

where $b = 3\max|\partial U_i/\partial x_j|$. This differential inequality, i.e.

$$\frac{1}{2}\,\frac{d(\|\underset{\sim}{v}\|_c^2)}{dt} + \gamma\,\|\underset{\sim}{v}\|^2 \leq 0 \tag{4}$$

with $\gamma = c^2 - Rb$, has the solution

$$\|\underset{\sim}{v}\|(t) \leq \|\underset{\sim}{v}\|(0)\,e^{-\gamma t}.$$

But $\gamma > 0$ if $R < c^2/b$. In this case it follows that (i) the solution $\underset{\sim}{U}$ is <u>unique</u> (because the existence of a second solution implies a $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x}) \neq 0$ and then, with $\gamma > 0$, the inequality (4) is contradictory), and (ii) the solution $\underset{\sim}{U}$ is <u>unconditionally</u> <u>stable</u> with respect to the L_2 norm.

'By continuity', these conclusions extend to <u>any</u> $\underset{\sim}{v} \in L_2(D \to \mathbb{R})$, not necessarily a continuous function. That is, we can take infinite sequences of C^∞ vector fields which are solenoidal ($\nabla \cdot \underset{\sim}{v} = 0$) and vanish on ∂D, and the above inequalities remain valid in the limit, which may not be a C^∞ function. Note that the sufficient condition $R < c^2/b$ for stability has been established rigorously, without approximation of any kind and without limitation to particular forms of the boundary ∂D.