INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

SMR/23 - 10

AUTUMN COURSE ON APPLICATIONS OF ANALYSIS TO MECHANICS

22 September - 26 November 1976

Lecture III : Dynamical Invariants and Universal Equilibrium Theory (continued)

Lecture IV : Turbulent diffusion

Lecture V : Convection and diffusion of a magnetic field by turbulence

H.K. Moffatt
University of Cambridge
Cambridge, England

implies that

$$\frac{E(k)}{4\pi k^4}\left[k^2|\underset{\sim}{X}|^2 - |\underset{\sim}{k}\cdot\underset{\sim}{X}|^2\right] + i\,\underset{\sim}{k}\cdot\left(\underset{\sim}{X}\wedge\underset{\sim}{X}^*\right)\frac{F(k)}{8\pi k^4} \geq 0 \qquad (3.13)$$

for all complex $\underset{\sim}{X}$. Let $\underset{\sim}{X} = \underset{\sim}{p} + i\underset{\sim}{q}$ with $\underset{\sim}{p}\cdot\underset{\sim}{k} = \underset{\sim}{q}\cdot\underset{\sim}{k} = 0$, and $\underset{\sim}{p}$ and $\underset{\sim}{q}$ real, so that

$$i\,\underset{\sim}{X}\wedge\underset{\sim}{X}^* = 2\,\underset{\sim}{p}\wedge\underset{\sim}{q} \qquad (3.14)$$

Choosing $\underset{\sim}{p}, \underset{\sim}{q}$ so that $(\underset{\sim}{p}, \underset{\sim}{q}, \underset{\sim}{k})$ is an orthogonal triad with $|\underset{\sim}{p}| = |\underset{\sim}{q}|$, we then find

$$|F(k)| \leq 2k\,E(k) \qquad (3.15),$$

a condition which, like the condition $E(k) \geq 0$, may be described as a realisability condition. If $F(k) = \pm 2k\,E(k)$, then we may describe the turbulence as being in a state of maximal helicity. Physically the condition (3.15) means simply that we cannot inject helicity into a flow without injecting energy also. Now, suppose that $E(k)$ is given by (3.6) in the inertial range and that

$$F(k) = 2k\,E(k) = 2C\epsilon^{2/3}\,k^{-2/3} \qquad (k_0 \ll k \ll k_\nu) \qquad (3.16)$$

then

$$\int_0^\infty F(k)\,dk \sim \epsilon^{2/3}\,k_\nu^{\frac{1}{3}} \sim \epsilon^{\frac{3}{4}}\,\nu^{-\frac{1}{4}} \qquad (3.17)$$

and so

$$\mathcal{H} = \frac{\langle\underset{\sim}{u}\cdot\underset{\sim}{\omega}\rangle}{\langle\underset{\sim}{u}^2\rangle^{\frac{1}{2}}\langle\underset{\sim}{\omega}^2\rangle^{\frac{1}{2}}} \sim \frac{\epsilon^{\frac{3}{4}}\,\nu^{-\frac{1}{4}}}{u_0(\epsilon/\nu)^{\frac{1}{2}}} \sim \left(\frac{\nu}{u_0 l_0}\right)^{\frac{1}{4}} \sim R^{-\frac{1}{4}} \ll 1 \qquad (3.18)$$

The spread of energy over a wide range of length-scales therefore necessarily leads to a situation in which the relative helicity $\mathcal{H}$ is small.

Even the estimate (3.18) however is probably an overestimate of $\mathcal{H}$ in any situation in which statistical equilibrium has been established. To see this, consider a situation in which, at time $t = 0$, $E(k) = E_0\,\delta(k - k_0)$ and $F(k) = 2k\,E(k)$ so that $\mathcal{H} = 1$ and suppose that

Viscous effects will then be negligible during some time $t \lesssim T$ say during which $\langle \underset{\sim}{u}^2 \rangle$ and $\langle \underset{\sim}{u} \cdot \underset{\sim}{\omega} \rangle$ remain constant, but $\langle \underset{\sim}{\omega}^2 \rangle$ rises by a factor $O(R)$ (since $\langle \underset{\sim}{\omega}^2 \rangle = \epsilon/\nu = R(u_0/\ell_0)^2$ in the ultimate statistical equilibrium that is established). Hence $\mathcal{H}$ decreases by a factor $O(R^{1/2})$, i.e.

$$\mathcal{H} = O(R^{-1/2}) \qquad (3.20)$$

in the ultimate statistical equilibrium. This estimate is clearly an order of magnitude smaller than (3.18) and provides an indication that the condition (3.16) of maximal helicity cannot be preserved in time under evolution governed by the Navier-Stokes equations.

A direct demonstration of this fact has been provided by Kraichnan (J. Fluid Mech. 1973) who considered the evolution in time (under the N.S. equations) of a velocity field $\underset{\sim}{u}(\underset{\sim}{x}, t)$ which at time $t = 0$ is given by

$$\underset{\sim}{u}(\underset{\sim}{x}, 0) = u_0(\sin k_1 z, \cos k_1 z, 0) + u_0(0, \sin k_2 x, \cos k_2 x) \qquad (3.21)$$

i.e. a superposition of two pure helicity modes. Kraichnan showed that non-linear interactions of these modes generates new modes that do *not* satisfy the 'pure helicity' condition (3.12), the inference being that the condition $F(k) = 2k\,E(k)$, although possible as an *initial condition*, cannot in fact survive in time.

The fact that so $|\mathcal{H}| \ll R^{-1/4}$ implies that in statistical equilibrium $|F(k)| \ll 2k\,E(k)$ when $k \gg k_0$. Hence it may be anticipated that helicity has negligible effect on the energy cascade for $k \gg k_0$, and that it therefore behaves like a *passive scalar contaminant*; if this is so, then

$$F(k) = C'' \eta\, \epsilon^{-1/3} k^{-5/3} \qquad (k_0 \ll k \ll k_\nu) \qquad (3.22)$$

and C is a further universal constant.

injection at wave numbers of order k_0. [Clearly an inequality of the form $|\eta| \leq 2k_0 \epsilon$ must apply.] When $F(k)$ is given by (3.22), we have

$$\int_0^\infty F(k)\,dk \sim \eta\, \epsilon^{-1/3} k_0^{-2/3} \qquad (3.23)$$

and if the helicity injection rate is maximal ($\eta \sim 2k_0\epsilon$), then

$$\langle \underset{\sim}{u} \cdot \underset{\sim}{\omega} \rangle = \int_0^\infty F(k)\,dk \sim u_0^2/l_0 \qquad (3.24)$$

consistent with the description leading to (3.20).

Spectra of the form (3.6) and (3.22) have in fact recently been obtained in numerical work by André & Lesieur (1976 - private communication) on the basis of a particular "closure scheme" for the system of equations that can be constructed for nth order spectral tensors [$\Phi_{ij}(\underset{\sim}{k}, t)$ is a 2nd order spectral tensor.] It is a fairly straightforward matter to obtain from the N.S. equs. equations of the form

$$\frac{\partial E(k,t)}{\partial t} = T(k,t) - 2\nu k^2 E \qquad (3.25)$$

$$\frac{\partial F}{\partial t} = U(k,t) - 2\nu k^2 F \qquad (3.26)$$

where T and U represent transfer functions for energy and helicity respectively. Quantitative theories of turbulence seek to find ways of expressing T and U in terms of E and F, so that (3.25) and (3.26) can be integrated. All closure schemes so far studied are to some extent arbitrary. The scheme adopted by André & Lesieur is the so-called "Eddy-damped quasi-normal Markovianised" closure scheme of Orszag (J. Fluid Mech. 1970) which guarantees that the realizability conditions

$$E(k,t) \geq 0, \qquad F(k,t) \leq 2k\,E(k,t) \qquad (3.27)$$

forms of (3.25) and (3.26), shows the clear emergence of "$k^{-5/3}$-ranges" for both $E(k,t)$ and $F(k,t)$ as time proceeds. The result (3.22) was first proposed as one of two alternative possibilities by Brissaud et. al (Phys. Fluids, Letters 1971); the second alternative is in fact untenable, and it does seem that the above description of helicity effects in the context of the Universal Equilibrium Theory is qualitatively correct.

Although helicity has negligible effect for $k \gg k_0$, it does have a mild effect for $k \sim k_0$ in inhibiting the process of energy transfer. To see this, it is sufficient to write the N.S. equns. in the alternative form

$$\frac{\partial \underset{\sim}{u}}{\partial t} = -\nabla(p + \tfrac{1}{2}\underset{\sim}{u}^2) + \underset{\sim}{u} \wedge \underset{\sim}{\omega} + \nu \nabla^2 \underset{\sim}{u}. \qquad (3.28)$$

Clearly if $\underset{\sim}{u}$ is parallel to $\underset{\sim}{\omega}$, so that $\underset{\sim}{u} \wedge \underset{\sim}{\omega} = 0$, then there is no non-linear effect, and an energy cascade cannot occur. It is to be expected therefore that if $|\langle \underset{\sim}{u} \cdot \underset{\sim}{\omega} \rangle|$ is in some sense maximised so that ~~[illegible]~~ $\langle (\underset{\sim}{u} \wedge \underset{\sim}{\omega})^2 \rangle$ is minimised, then the energy cascade process will proceed more slowly than in the absence of helicity, and it will take longer to establish a $k^{-5/3}$ energy spectrum starting from an arbitrary initial form of $E(k,t)$. This effect of inhibiting energy transfer has also been identified in the numerical experiments of André & Lesieur.

4 Turbulent diffusion

4.1 The two-scale approach

We return now to the problem of convection and diffusion of a scalar field $\theta(\underset{\sim}{x},t)$ by turbulence, and we suppose that the ensemble average of θ is non-zero: i.e. suppose that

$$\langle\theta(\underset{\sim}{x},t)\rangle = \Theta(\underset{\sim}{x},t), \qquad \theta(\underset{\sim}{x},t) = \Theta(\underset{\sim}{x},t) + \theta'(\underset{\sim}{x},t) \qquad (4.1)$$

Evidently the θ-field is now inhomogeneous in that its statistical properties now vary with $\underset{\sim}{x}$. We shall however suppose that this variation is slow compared with the spatial variation of $\underset{\sim}{u}(\underset{\sim}{x},t)$; i.e. let L be a scale characteristic of $\Theta(\underset{\sim}{x},0)$ (and of any other statistical properties of $\theta(\underset{\sim}{x},0)$) and let l_0 (as usual) be the length-scale characterising the most energetic ingredients of the $\underset{\sim}{u}$-field; then we suppose that

$$L \gg l_0, \qquad (4.2)$$

and in this circumstance an ensemble average can in fact be identified with a spatial average over any intermediate scale $\underset{\sim}{a}$ satisfying

$$L \gg a \gg l_0. \qquad (4.3)$$

Equ. (3.10) for θ may be written in the form

$$\frac{\partial\theta}{\partial t} = -\nabla\cdot(\underset{\sim}{u}\,\theta) + \kappa\nabla^2\theta \qquad (4.4)$$

and the ensemble average of this equation is

$$\frac{\partial\Theta}{\partial t} = -\nabla\cdot\underset{\sim}{F} + \kappa\nabla^2\Theta \qquad (4.5)$$

where
$$\underset{\sim}{F} = \langle\underset{\sim}{u}\,\theta\rangle_{a} = \langle\underset{\sim}{u}\,\theta'\rangle. \qquad (4.6)$$

F evidently represents the 'heat' flux due to the turbulence and evidently we must aim to determine $\underset{\sim}{F}$ as a function or functional of $\Theta(\underset{\sim}{x}, t)$, so that (4.5) may be integrated.

Subtracting (4.5) from (4.4), we have

$$\frac{\partial \theta'}{\partial t} = -\underset{\sim}{u} \cdot \underset{\sim}{G} - \nabla \cdot (\underset{\sim}{u}\theta' - \langle \underset{\sim}{u}\theta' \rangle) + \kappa \nabla^2 \theta' \qquad (4.7)$$

where $\underset{\sim}{G} = \nabla \Theta$. This equation evidently establishes a <u>linear relation</u> between θ' and $\underset{\sim}{G}$ and so between $\underset{\sim}{F} = \langle \underset{\sim}{u}\theta' \rangle$ and $\underset{\sim}{G}$; the most general form that this linear relationship can take is

$$F_i = -D_{ij} G_j - E_{ijk} \frac{\partial G_j}{\partial x_k} - F_{ijkl} \frac{\partial^2 G_j}{\partial x_k \partial x_l} + \cdots \qquad (4.8)$$

where the coefficient tensors $\underset{\approx}{D}$, $\underset{\approx}{E}$, ... are determined (in principle) by the statistical properties of the turbulence and by the diffusion parameter κ (which enters via the solution of eqn. (4.7)). [The minus signs in (4.8) are introduced simply for convenience; note that we could in principle include also terms involving $\partial\Theta/\partial t$ in (4.8) but these can always be replaced by spatial derivatives of Θ using iteration based on (4.5)]. Substitution of (4.8) in (4.5) gives

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x_i}\left(D_{ij} \frac{\partial \Theta}{\partial x_j}\right) + \kappa \nabla^2 \Theta + \ldots \qquad (4.9)$$

where the terms represented by --- involve derivatives of Θ of 3rd and higher order. Evidently D_{ij} is an <u>eddy diffusion tensor</u> associated with the action of the turbulence on the field Θ. D_{ij} is uniform and constant only insofar as the turbulence is homogeneous (in $\underset{\sim}{x}$) and stationary (in t).

If the turbulence is isotropic then D_{ij} (being itself a statistical

(4.9) becomes

$$\frac{\partial \Theta}{\partial t} = (\kappa + D) \nabla^2 \Theta + \cdots \qquad (4.10)$$

When the scale L of Θ is sufficiently large, the terms ... involving higher derivatives have negligible influence on the evolution of Θ.

4.2 The strong diffusion limit

There is no known technique for evaluating D_{ij} for arbitrary values of κ; we can however find the asymptotic forms when $Pe = u_0 l_0/\kappa \to 0$ (the strong diffusion limit) or $Pe \to \infty$ (the weak diffusion limit). (Pe is the Péclet number.) Suppose first that $Pe \ll 1$. Then, as in §3.3, Equ. (4.7) may be approximated by [Corrigendum: in (3.15) replace $-\underset{\sim}{u}\cdot\underset{\sim}{G}$ by $+\underset{\sim}{u}\cdot\underset{\sim}{G}$]

$$\kappa \nabla^2 \theta' \simeq \underset{\sim}{u} \cdot \underset{\sim}{G} \qquad (4.11)$$

with Fourier transform

$$-\kappa k^2 \tilde{\theta}' = \tilde{\underset{\sim}{u}} \cdot \underset{\sim}{G} \qquad (4.12)$$

Here we regard $\underset{\sim}{G}$ as constant on all relevant scales characteristic of the turbulence (although of course it varies on a much larger scale L). Previously we deduced the spectrum of θ'; now however we need to calculate

$$\underset{\sim}{F} = \langle \underline{u}\,\theta' \rangle = \iint \langle \tilde{\underset{\sim}{u}}(\underset{\sim}{k},t)\,\tilde{\theta}'(\underline{k}',t) \rangle e^{-i(\underline{k}+\underline{k}')\cdot\underline{x}}\, d\underline{k}\, d\underline{k}' \qquad (4.13)$$

or

$$F_i = -\iint \langle \tilde{u}_i(\underset{\sim}{k},t)\,\tilde{u}_j(\underline{k}',t) \rangle e^{-i(\underline{k}+\underline{k}')\cdot\underline{x}} (\kappa k^2)^{-1} G_j\, d\underline{k}\, d\underline{k}' \qquad (4.14)$$

which is evidently of the required form $F_i = -D_{ij} G_j$. In fact, using equs. (2.4), (2.9) and integrating w.r.t. $\underline{k}'$, we have

$$D_{ij} = \kappa^{-1} \int k^{-2}\, \Phi_{ij}(\underline{k})\, d\underline{k} \;,$$

In the isotropic case, $D_{ij} = D\,\delta_{ij}$ where

$$D = \tfrac{1}{3} D_{ii} = \frac{1}{3\kappa}\int k^{-2}\,\Phi_{ii}(\underline{k})\,d\underline{k} = \frac{1}{3\kappa}\int_0^\infty k^{-2}E(k)\,dk \qquad (4.16)$$

(using the form (2.23), (2.24) for Φ_{ij}).

Although this derivation is interesting from a methodological point of view, the result is not in fact of any physical importance; the reason is that, for any 'normal' energy spectrum $E(k)$, (4.16) gives

$$D \sim \frac{1}{3\kappa}\, l_0^2 u_0^2 \sim (Pe)^2\,\kappa \qquad (4.17)$$

and so $D \ll \kappa$ when $Pe \ll 1$ (as assumed). Hence the turbulent diffusion effect is weak in this limit compared with the molecular diffusion. It is nevertheless of some theoretical interest that D, as given by (4.16) is certainly positive. We shall find that this type of result does not continue to hold for the more complex problem of diffusion of a passive solenoidal vector field (see chapter 5).

4-3 Weak diffusion limit

Suppose now that $Pe \gg 1$; in this limit we may expect the turbulent diffusivity to dominate over molecular diffusivity. In fact let us be quite brutal, and put $\kappa = 0$. The solution of the equation

$$\frac{D\theta}{Dt} \equiv \frac{\partial\theta}{\partial t} + \underline{u}\cdot\nabla\theta = 0 \qquad (4.18)$$

is then evidently (in Lagrangian notation)

$$\theta(\underline{x}, t) = \theta(\underline{a}, 0) \qquad (4.19)$$

where $\underline{a}$ is the initial position of the fluid particle that finds itself

Hence

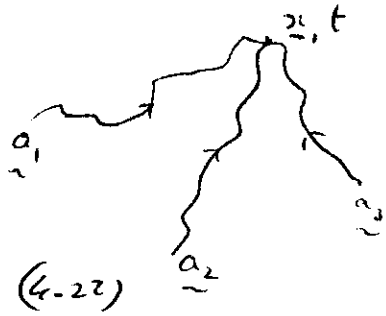
$$\underset{\sim}{F} = \langle \underset{\sim}{u}\,\theta' \rangle = \langle \underset{\sim}{u}(\underset{\sim}{x},t)\,\theta(\underset{\sim}{x},t) \rangle$$

$$= \langle \underset{\sim}{v}(\underset{\sim}{a},t)\,\theta(\underset{\sim}{a},0) \rangle \qquad (4.20)$$

where $\underset{\sim}{v}(\underset{\sim}{a},t) = \underset{\sim}{u}(\underset{\sim}{x}(\underset{\sim}{a},t),t)$. Now, let us suppose that $\theta'(\underset{\sim}{x},0) \equiv 0$, i.e. that $\theta(\underset{\sim}{x},0) = \Theta(\underset{\sim}{x},0)$, the same function in each realisation of the ensemble. Then

$$\underset{\sim}{F}(\underset{\sim}{x},t) = \langle \underset{\sim}{v}(\underset{\sim}{a},t)\,\Theta(\underset{\sim}{a},0) \rangle \qquad (4.21)$$

and the quantity that should be regarded as random here is $\underset{\sim}{a}$ which (for given $\underset{\sim}{x},t$) varies from one realisation to another.



Now to calculate D_{ij} in (4.8), we may suppose that $\nabla\Theta$ is uniform so that

$$\Theta(\underset{\sim}{a},0) = \Theta(\underset{\sim}{x},0) - (\underset{\sim}{x}-\underset{\sim}{a})\cdot\nabla\Theta \qquad (4.22)$$

substitution in (4.21) gives then

$$F_i = -\langle v_i(\underset{\sim}{a},t)\,X_j(\underset{\sim}{a},t) \rangle\, G_j \qquad (4.23)$$

where

$$\underset{\sim}{X}(\underset{\sim}{a},t) = \underset{\sim}{x} - \underset{\sim}{a} = \int_0^t \underset{\sim}{v}(\underset{\sim}{a},\tau)\,d\tau \qquad (4.24)$$

Hence finally $F_i = -D_{ij}G_j$ where

$$D_{ij}(t) = \int_0^t \langle v_i(\underset{\sim}{a},t)\,v_j(\underset{\sim}{a},\tau) \rangle\, d\tau \qquad (4.25)$$

If the turbulence is homogeneous and stationary (statistically) in time, then

$$\langle v_i(\underset{\sim}{a},t)\,v_j(\underset{\sim}{a},\tau) \rangle = R_{ij}(t-\tau), \qquad (4.26)$$

the Lagrangian velocity correlation, depends only on $t-\tau$. Putting $t_1 = t-\tau$ and letting $t \to \infty$ in (4.25) gives

$$D_{ij} \sim \int_0^\infty R_{ij}(t_1)\,dt_1 \qquad (4.27)$$

V Convection and diffusion of a magnetic field by turbulence

The magnetic field $\underset{\sim}{B}(\underset{\sim}{x},t)$ in an electrically conducting fluid (e.g. the liquid core of the earth, or the ionised gas of the sun, or the very diffuse ionised gas of the galactic disc) satisfies the equations

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \wedge (\underset{\sim}{u} \wedge \underset{\sim}{B}) + \lambda \nabla^2 \underset{\sim}{B}, \qquad \nabla \cdot \underset{\sim}{B} = 0 \tag{5.1}$$

where λ is the magnetic diffusivity of the field. If $\underset{\sim}{u}(\underset{\sim}{x},t)$ is again turbulent, but with known statistical properties, and if it is supposed that $\underset{\sim}{B}$ is dynamically passive, then we can apply the techniques of Chapter IV equally to this situation.

Thus, suppose that at time $t = 0$, $\underset{\sim}{B} = \underset{\sim}{B}_0(\underset{\sim}{x}, 0)$ varying on a large scale L, and for $t > 0$, let

$$\langle \underset{\sim}{B} \rangle = \underset{\sim}{B}_0(\underset{\sim}{x},t), \qquad \underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{b}(\underset{\sim}{x},t) \tag{5.2}$$

Then the ensemble average of (5.1) gives

$$\frac{\partial \underset{\sim}{B}_0}{\partial t} = \frac{\partial \langle \underset{\sim}{B} \rangle}{\partial t} = \nabla \wedge \underset{\sim}{\mathcal{E}} + \lambda \nabla^2 \underset{\sim}{B}_0 \tag{5.3}$$

where

$$\underset{\sim}{\mathcal{E}} = \langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle \tag{5.4}$$

(assuming as always that $\langle \underset{\sim}{u} \rangle = 0$).

Subtraction of (5.3) from (5.1) then gives

$$\frac{\partial \underset{\sim}{b}}{\partial t} = \nabla \wedge (\underset{\sim}{u} \wedge \underset{\sim}{B}_0) + \nabla \wedge (\underset{\sim}{u} \wedge \underset{\sim}{b} - \langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle) + \lambda \nabla^2 \underset{\sim}{b}, \tag{5.5}$$

and equation which, together with the initial condition $\underset{\sim}{b}(\underset{\sim}{x},0) = 0$, establishes a linear relation between $\underset{\sim}{b}$ and $\underset{\sim}{B}_0$ and so between $\underset{\sim}{\mathcal{E}} = \langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle$ and $\underset{\sim}{B}_0$. Since the scale of $\underset{\sim}{B}_0$ is large, this linear relation may in principle be developed as a (rapidly converging) series of the form (cf. 4.8)

$$\mathcal{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \gamma_{ijkl} \frac{\partial^2 B_{0j}}{\partial x_k \partial x_l} + \cdots \tag{5.6}$$

where however it is now important to note that α_{ij}, β_{ijk}, --- are <u>pseudo-tensors</u> ($\underset{\sim}{\mathcal{E}}$ being a polar vector and $\underset{\sim}{B_0}$ an axial vector), which again must be determined exclusively by the statistical properties of the $\underset{\sim}{u}$-field and the parameter λ.

In particular, if the $\underset{\sim}{u}$-field is isotropic, then (cf. $D_{ij} = D\delta_{ij}$)

$$\alpha_{ij} = \alpha\,\delta_{ij}\ , \quad \beta_{ijk} = +\beta\,\epsilon_{ijk}\ , \quad --- \tag{5.7}$$

where α is a pseudo-scalar (cf. $\langle \underset{\sim}{u}.\underset{\sim}{\omega} \rangle$), β is a pure scalar, and so on. Substitution of (5.7) in (5.6) gives

$$\underset{\sim}{\mathcal{E}} = \alpha\,\underset{\sim}{B_0} - \beta\left(\nabla\wedge\underset{\sim}{B_0}\right) + --- \tag{5.8}$$

and so eqn (5.3) becomes (regarding α & β as constants)

$$\frac{\partial \underset{\sim}{B_0}}{\partial t} = \alpha\,\nabla\wedge\underset{\sim}{B_0} + (\lambda+\beta)\,\nabla^2\underset{\sim}{B_0} + --- . \tag{5.9}$$

Here, β clearly plays the role of an eddy diffusivity for a magnetic field, an effect that might have been anticipated. The term $\alpha\,\nabla\wedge\underset{\sim}{B_0}$ however represents a dramatically new effect first identified by Steenbeck, Krause & Rädler 1966, and described by them as the 'α-effect'). It is <u>dramatic</u> in that, provided $\alpha \neq 0$, the term $\alpha\,\nabla\wedge\underset{\sim}{B_0}$ is of dominant importance for the evolution of large-scale field structures.

In fact (5.9) admits solutions that grow exponentially in time: suppose that $\underset{\sim}{B_0}(\underset{\sim}{x},0)$ has a spatial structure satisfying

$$\nabla\wedge\underset{\sim}{B_0} = K\,\underset{\sim}{B_0} \tag{5.10}$$

for example the circularly polarised field

$$\underset{\sim}{B_0} = B_{00}\,(\sin Kz,\ \cos Kz,\ 0) \tag{5.11}$$

satisfies this condition. This structure is evidently preserved by eqn. (5.9), and we have in fact

$$\underset{\sim}{B_0}(\underset{\sim}{x},t) = \underset{\sim}{B_0}(\underset{\sim}{x},0)\,e^{\omega t} \tag{5.12}$$

where
$$\omega = \alpha K - (\lambda+\beta)K^2 \tag{5.13}$$

and we have exponential growth of $\underset{\sim}{B_0}$ provided

$$\alpha K \cancel{} > (\lambda+\beta)K^2$$

i.e. provided α and K have the same sign, and $|K|$ is sufficiently small. This type of magnetic field excitation underlies all modern dynamo theory (i.e. the theory of generation of magnetic fields in cosmic bodies), although of course in particular contexts, boundary effects must also be taken into consideration.

It remains of course to determine α and β in terms of the statistical properties of the turbulence and the parameter λ. In the following sections we obtain expressions for α in the strong diffusion and weak diffusion limits, and simply quote the corresponding results for β.

5.2 Strong diffusion limit

Since α_{ij} in eqn. (5.6) is independent of $\underset{\sim}{B_0}$, it may be calculated on the simplifying assumption that $\underset{\sim}{B_0}$ is uniform. When λ is very large, the diffusion term $\lambda\nabla^2\underset{\sim}{b}$ dominates over all other terms linear in $\underset{\sim}{b}$, and the leading approximation to (5.5) is (cf. 4.11)

$$\lambda\nabla^2\underset{\sim}{b} = -\nabla\wedge(\underset{\sim}{u}\wedge\underset{\sim}{B_0}) = (\underset{\sim}{B_0}\cdot\nabla)\underset{\sim}{u}. \tag{5.14}$$

The Fourier transform of this is

$$-\lambda k^2\,\underset{\sim}{\tilde{b}} = i(\underset{\sim}{B_0}\cdot\underset{\sim}{k})\,\underset{\sim}{\tilde{u}} \tag{5.15}$$

and we may immediately construct

$$\langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle = \iint \langle \tilde{\underset{\sim}{u}}(\underset{\sim}{k},t) \wedge \tilde{\underset{\sim}{b}}(\underset{\sim}{k}',t) \rangle e^{i(\underset{\sim}{k}+\underset{\sim}{k}')\cdot \underset{\sim}{x}} d\underset{\sim}{k}\, d\underset{\sim}{k}'$$

or
$$\mathcal{E}_i = \epsilon_{ijk} \int i k_\ell\, \Phi_{jk}(\underset{\sim}{k}) (\lambda k^2)^{-1} B_{0\ell}\, d\underset{\sim}{k} \qquad (5.16)$$

This has the expected form $\mathcal{E}_i = \alpha_{i\ell} B_{0\ell}$ where

$$\alpha_{i\ell} = \lambda^{-1} i \epsilon_{ijk} \int k^{-2} k_\ell\, \Phi_{jk}(\underset{\sim}{k})\, d\underset{\sim}{k} \qquad (5.17)$$

In particular the quantity $\alpha = \frac{1}{3}\alpha_{ii}$ is given, ~~in the isotropic case,~~ via the definition (2.21), by

$$\alpha = -\frac{1}{3\lambda} \int_0^\infty k^{-2} F(k)\, dk. \qquad (5.18)$$

This result brings out the intimate relation between the helicity spectrum and the α-effect; evidently α is non-zero only if the turbulence lacks reflexion symmetry.

The result for β corresponding to (5.18) is

$$\beta = \frac{1}{3\lambda} \int_0^\infty k^{-2} E(k)\, dk. \qquad (5.19)$$

Hence, if $\lambda = \kappa$, $\beta = D$ (cf. equ. (4.16)) in the strong diffusion limit, a result that certainly does not have general validity.

5.3 The weak diffusion limit

In this more subtle limit we follow the steps of §(4.3), again supposing that $\underset{\sim}{B}_0$ is uniform in order to facilitate the calculation of α_{ij}. The Lagrangian solution of (5.1) when $\lambda = 0$ is (cf. Cauchy's "solution" of the vorticity equation when $\nu = 0$)

$$B_i(\underset{\sim}{x},t) = B_j(\underset{\sim}{a},0)\, \partial x_i / \partial a_j \qquad (5.20)$$

(with notation as in §4-3). Hence

$$\mathcal{E}_i(\underset{\sim}{x},t) = \langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle_i = \langle \underset{\sim}{u} \wedge \underset{\sim}{B} \rangle_i$$

$$= \epsilon_{ijk} \langle v_j(\underset{\sim}{a},t)\, B_l(\underset{\sim}{a},0)\, \partial x_k/\partial a_l \rangle \qquad (5.21)$$

And since $B_l(\underset{\sim}{a},0) = B_{0l}$, this has the expected form $\mathcal{E}_i = \alpha_{il} B_{0l}$ where

$$\alpha_{il}(t) = \epsilon_{ijk} \langle v_j(\underset{\sim}{a},t)\, \partial x_k(\underset{\sim}{a},t)/\partial a_l \rangle$$

$$= \epsilon_{ijk} \int_0^t \langle v_j(\underset{\sim}{a},t)\, \partial v_k(\underset{\sim}{a},\tau)/\partial a_l \rangle \, d\tau \qquad (5.22)$$

which may be compared with (4.25). In the isotropic case this becomes $\alpha_{il} = \alpha\,\delta_{il}$ where

$$\alpha = \tfrac{1}{3}\alpha_{ii} = -\int_0^t \langle \underset{\sim}{v}(\underset{\sim}{a},t) \cdot \nabla_{\underset{\sim}{a}} \wedge \underset{\sim}{v}(\underset{\sim}{a},\tau) \rangle \, d\tau \qquad (5.23)$$

Again, helicity makes its appearance, but this time via a rather curious "Lagrangian helicity correlation". There are actually some doubts about the convergence of integrals such as (5.23) as $t \to \infty$ (and it is the asymptotic value of α for large t that is relevant for the long-term evolution of the mean field). Numerical experiments (involving an average over 20000 distinct realisations!) have however recently been carried out by Kraichnan (1976 - private communication), and he has ~~and of maximal helicity~~ found that when the statistics of the $\underset{\sim}{u}$-field are Gaussian and of maximal helicity (which is not a bad first approximation) the integral does in general converge to a value of order $\mp 0.6\, u_0$ according as the mean helicity is positive or negative.

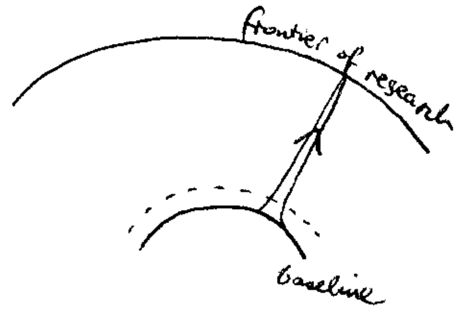
The formula for β corresponding to (5.23) is quite complicated:

$$\beta = \tfrac{1}{3}\int_0^t \langle \underset{\sim}{v}(t) \cdot \underset{\sim}{v}(\tau) \rangle \, d\tau + \tfrac{1}{6}\int_0^t\!\!\int_0^t \langle \underset{\sim}{v}(t) \cdot \underset{\sim}{v}(\tau_2)\, \nabla_{\underset{\sim}{a}} \cdot \underset{\sim}{v}(\tau_1) - \underset{\sim}{v}(t) \cdot \nabla_{\underset{\sim}{a}}\, \underset{\sim}{v}(\tau_1) \cdot \underset{\sim}{v}(\tau_2) \rangle$$

The numerical experiments of Kraichnan again indicate that generally in turbulence with Gaussian statistics the expression $\beta(t)$ converges as $t \to \infty$ (to a value of order $u_0 l_0$) whether the turbulence is reflexionally symmetric or not. This is surprising in the latter case, since the third integral in (5.24) certainly diverges as $t \to \infty$ if $\alpha(t) \to$ cst.; the indications are that this divergence is compensated by an equal and opposite divergence in the second term, a happy coincidence that may however be associated with the particular Gaussian statistics assumed in Kraichnan's treatment.

Limits of time do not permit me to treat various other extremely novel aspects of the magnetic diffusion problem — for example the fact that in reflexionally symmetric turbulence in which there are nevertheless large-scale fluctuations in the helicity density $\underset{\sim}{u} \cdot \underset{\sim}{\omega}$, the eddy diffusivity β may become negative! (Kraichnan 1976).
(J. Fluid Mech. - recent issue)

In a brief course of lectures on turbulence, one has the choice of advancing the baseline (a little) uniformly or making a concentrated dash for the frontiers of research in one specialised area. I have chosen to do the latter, because I think this more interesting for you (and for me!), and have tried to underline the importance of the concept of "lack of reflexional symmetry", as exemplified particularly by a non-zero helicity spectrum. As I have mentioned previously this situation arises naturally in a rotating system when turbulence is generated by some mechanism that distinguishes between the directions $\pm \underset{\sim}{\Omega}$ (where $\underset{\sim}{\Omega}$ is the rotation vector). The effects are of particular importance in geophysical and astrophysical contexts, but such results as conservation of mean



REFERENCES

Some books on turbulence:

Batchelor, G.K. (1953) Homogeneous Turbulence Camb. Univ. Press.

Monin A.S. & Yaglom A.M. (1971) Statistical fluid mechanics, Vols I & II MIT Press: Cambridge. Mass.

Townsend A.A. (1976) The Structure of Turbulent Shear Flow (2nd edn.) Camb. Univ. Press.

Articles referred to in text

Parker E.N. 1955 Hydromagnetic dynamo models, Astrophys. J. 121, 491-50

Steenbeck, M., Krause, F. & Rädler, K.H. (1966) A calculation of the mean electromotive force in an electrically conducting fluid in turbulent motion under the influence of Coriolis forces. (in German) Z. Naturforsch. A21, 369-376

[A translation of this and other papers by the same group is available as Tech. Note NCAR/TN/1A-60 by P.H. Roberts and M. Stix (1971).]

Braginskii S. (1964) Self excitation of a magnetic field during the motion of a highly conducting fluid Sov. Phys. JETP 20, 726-735.

Batchelor G.K. Howells, I.D. & Townsend A.A. (1959) Small scale variation of convected quantities like temperature in turbulent fluid, Part 2 - the case of large conductivity, J. Fluid Mech. 5, 134-139.

Kraichnan R.H. (1973) Helical turbulence and absolute equilibrium J. Fluid Mech. 59, 745-752.

Taylor G.I. (1921) Diffusion by continuous movements - See collected papers Vol II (1960) pp 172-184 Camb. Univ. Press.

Finally, for a review of the problem of generation of magnetic fields by fluid motion see

Moffatt H.K. (1976) Adv. Appl. Mech. Vol. 16 pp. 119-181 (to appear very soon!)

HK Moffatt