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AUTUMN WEEK ON

GEOMETRY OF THE LAPLACE OPERATOR

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# BASIC FACTS ON HARMONIC MAPS

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<u>Introduction</u> We formulate harmonic maps in the real and complex cases and discuss some elementary properties and examples. No mention of existence problems is made. For full proofs see cited references

References are numbered according to "BIBLIOGRAPHY FOR HARMONIC MAPS —(1976)"

Preliminary lecture notes for Autumn Week on "Geometry of the Laplace Operator".

# 1 PRELIMINARIES ON RIEMANNIAN GEOMETRY

(1) (A) <u>Tensors and Forms</u> Let $M$ be a manifold of class $C^\infty$ and $TM$ its tangent bundle. Let $T_pM$ denote the fibre of $TM$ over $p \in M$, a member of $T_pM$ is called a <u>tangent vector at $p$</u>. A smooth section of $TM$ is called a <u>vector field</u>. Let $T^*M$ denote the dual bundle of $TM$, we can form the bundle $\underbrace{TM \otimes \ldots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{s \text{ times}}$; a smooth section of this is called a <u>tensor field</u> ($r$ times contravariant, $s$ times covariant). We shall denote the space of smooth sections of any bundle $E$ over $M$ by $C^\infty(E)$, $C^\infty(M)$ will denote the ring of smooth functions $M \to \mathbb{R}$. If $r = 0$, a tensor field $F$ gives an $s$-linear map on each tangent space $T_pM$, if this map is alternating we call $F$ an <u>$s$-form</u>.

More generally, let $E$ be any vector bundle over $M$. A smooth section of $\underbrace{TM \otimes \ldots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{s \text{ times}} \otimes E$ is called an <u>$E$-valued tensor</u> ($r$ times contravariant, $s$ times covariant). If $r=0$ we can interpret such a tensor as an $s$-linear bundle map from $TM$ to $E$. If this map is alternating on each tangent space we call the tensor an <u>$E$-valued $s$-form</u>

<u>Example</u> Let $f: M \to N$ be a smooth map between smooth manifolds. Let $E =$ the pull-back bundle $f^{-1}TN$. Then the derivative $df$ of $f$ can be considered to be a smooth section of $T^*M \otimes E$.

<u>Local coordinates</u> If $(x^1, \ldots, x^m)$ are local coordinates for $M$, the vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$ give a (local) basis for $TM$ and their duals $dx^1, \ldots, dx^m$ give a basis for $T^*M$. All possible tensor products give bases for $TM \otimes \ldots \otimes TM \otimes T^*M \otimes \ldots \otimes T^*M$. In the example, a basis for $E = f^{-1}TN$ is given by pulling back a basis for $TN$; if $(u^1, \ldots, u^n)$ are local coordinates for $N$, then $\{ dx^i \otimes \frac{\partial}{\partial u^\alpha} \}$ gives a

(2)

(B) Metrics A (Riemannian) metric on a bundle $E$ over $M$ is a smooth section $g$ of $E^* \otimes E^*$ (where $E^*$ denotes the dual bundle of $E$) such that for each $p \in M$ $g$ gives a pairing $\langle\ \rangle_p : E_p \times E_p \to \mathbb{R}$ (here $E_p$ denotes the fibre of $E$ over $p$) with the properties:

$$\begin{aligned}
&(1)\quad \langle X, Y\rangle_p = \langle Y, X\rangle_p \qquad \forall X, Y \in E_p \quad \text{(symmetry)}\\
&(2)\quad \langle X+Y, Z\rangle_p = \langle X, Z\rangle_p + \langle Y, Z\rangle_p \quad \forall X, Y, Z \in E_p \quad \text{(bilinearity)}\\
&(3)\quad \langle aX, Y\rangle_p = a\langle X, Y\rangle_p \qquad \forall X, Y \in E_p\ \forall a \in \mathbb{R} \quad (\ \text{"}\ )\\
&(4)\quad \langle X, X\rangle_p > 0 \qquad \forall X \in E_p,\ X \neq 0 \quad \text{(positive definiteness)}
\end{aligned}$$

A metric on $TM$ is called a <u>Riemannian metric on $M$</u>; $M$ together with a chosen Riemannian metric is called a <u>Riemannian manifold</u>.

If $M$ is a compact Riemannian manifold and $E$ a bundle over $M$ with a Riemannian metric, all the bundles $TM \otimes \ldots \otimes TM \otimes T^*M \otimes \ldots \otimes T^*M \otimes E$ "inherit" metrics in a canonical way from the metrics on $TM$ and $E$. Using this metric the "<u>global inner product</u>" of two $E$-valued tensors ~~is def~~ $F_1, F_2$ is defined as the integral

$$(\!(F_1, F_2)\!) = \int_M \langle F_1, F_2\rangle_p \, v_M \qquad \text{where}$$

$v_M$ is the volume form of $M$

<u>Local coordinates</u> If $(x^1, \ldots, x^m)$ are local coordinates for $M$ then $\{dx^i \otimes dx^j\}$ is a basis for $T^*M \otimes T^*M$. Thus a metric $g$ on $M$ can be written locally as

$$g = g_{ij}\, dx^i \otimes dx^j \qquad \text{(double summation convention)}$$

where $g_{ij} = g_{ji}$. The tensor product sign may be omitted.

<u>Example</u> Recall a basis for $T^*M \otimes E$ is given by $\{dx^i \otimes \frac{\partial}{\partial u^\alpha}\}$ in the case $E = f^{-1}TN$, and thus the derivative $df$ of $f$ can be written as $f_i^{\ \alpha}\, dx^i \otimes \frac{\partial}{\partial u^\alpha}$. Then

$$\langle df, df\rangle_p = g^{ij} h_{\alpha\beta} f_i^{\ \alpha} f_j^{\ \beta} \qquad \text{where}$$

$h_{\alpha\beta}\, du^\alpha \otimes du^\beta$ is the metric on $N$ and $g^{ij} = (g_{ij})^{-1}$ (inverse matrix).

(C) <u>Connections on Manifolds and Bundles</u> Let $M$ be a ~~Riemannian~~ smooth manifold. A <u>connection</u> or <u>covariant differentiation</u> $D$ on $M$ is a real-linear map

$$D : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(TM)$$
$$(Z, X) \longmapsto D_Z X$$

satisfying:

$$(1) \quad D_{fZ}(X) = f D_Z X \qquad \forall f \in C^\infty(M)$$
$$(2) \quad D_Z(gY) = g D_Z(Y) + (\nabla_Z g) Y \quad \forall g \in C^\infty(M)$$

Here $\nabla_Z g$ denotes the <u>directional derivative of $g$ in direction $Z$</u> which may be defined as $dg(Z)$ where $dg : TM \to \mathbb{R}$ is the derivative of $g$. ~~The connection~~ If $M$ is a Riemannian manifold, the connection $D$ is s.t.b. <u>Riemannian</u> if $\forall X, Y, Z \in C^\infty(TM)$

$$(3) \quad \nabla_Z \langle X, Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle$$
$$(4) \quad D_Z X - D_X Z = [X, Z] \qquad (\text{where } [\ ,\ ]$$

is the <u>bracket</u> of two vector fields ) i.e. $D$ is "<u>torsion free</u>". There exists a unique torsion free Riemannian connection on a Riemannian manifold called the <u>Levi-Civita connection</u>.

Let $(X_1, \ldots, X_m)$ be a local basis of vector fields. A connection may be specified by giving a matrix of connection 1-forms $(\Phi_i{}^k)_{i,k=1,\ldots,m}$ defined by $D_Z X_i = \Phi_i{}^k(Z) X_k$.
The <u>Christoffel symbols</u> of the connection are the numbers $\Gamma_{ij}{}^k = \Phi_i{}^k(X_j)$

Now let $E$ be a vector bundle over $M$. A <u>connection</u> on $E$ is defined as a real-linear map

$$D' : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$$
$$(Z, \cancel{X}) \longmapsto D'_Z \cancel{X}$$

satisfying (1) and (2) above. If $E$ has a metric $\langle\ ,\ \rangle$, $D'$ is s.t.b. a <u>Riemannian connection</u> and $E$ is s.t.b. <u>Riemannian connected</u> if (3) and (4) above hold. Again, a connection on $E$ can be specified by a matrix of connection 1-forms w.r.t. a bases for ~~[illegible]~~ $E$. Note $D'$ can be considered as a real-linear map $D' : C^\infty(E) \to C^\infty(TM^* \otimes E)$

We shall now show how to define a sequence of covariant differentiations

$$C^\infty(E) \xrightarrow{\tilde{D} = D'} C^\infty(TM^* \otimes E) \xrightarrow{\tilde{D}} C^\infty(TM^* \otimes TM^* \otimes E) \xrightarrow{\tilde{D}} \cdots$$

(4) Firstly the Levi-Civita connection on $M$ can be used to define covariant differentiations $D_Z \,.\, C^\infty(TM \otimes \dots \otimes TM \otimes T^*M \otimes \dots \otimes T^*M) \circlearrowleft$ by insisting that the "product rule" holds viz:

$$D_Z(X \otimes Y) = D_Z X \otimes Y + X \otimes D_Z Y$$ for any tensor fields $X, Y$, for all $Z \in C^\infty(TM)$. In particular $D_Z : C^\infty(T^*M) \to C^\infty(T^*M)$ is found by solving

$$D_Z(w(Y)) = (D_Z w)(Y) + w(D_Z Y) \qquad \forall w \in C^\infty(T^*M)$$

$\forall Y, Z \in C^\infty(TM)$. It follows that if $D$ has connection 1-forms $\Phi_i^{\,k}$ w.r.t. a basis $\{X_1, \dots, X_m\}$ and if $\{X^1, \dots, X^m\}$ is the dual basis of 1-forms, $D_Z : C^\infty(T^*M) \to C^\infty(T^*M)$ is given by $D_Z X^k = -\Phi_i^{\,k}(Z)\, X^i$.

To define $\tilde{D} : C^\infty(T^*M \otimes E) \to C^\infty(T^*M \otimes T^*M \otimes E)$ we again require that the "product law" holds viz, if $w \in C^\infty(T^*M)$ $\quad e \in C^\infty(E)$:

$$\tilde{D}(w \otimes e) = \tilde{D}(w) \otimes e + w \otimes \tilde{D}(e)$$

or equivalently:

$$\tilde{D}_Z(w \otimes e) = \tilde{D}_Z(w) \otimes e + w \otimes \tilde{D}_Z(e) \qquad \forall Z \in C^\infty(TM)$$

Note that $\tilde{D}(w)$ ~~must~~ can be defined as $D(w)$ and $\tilde{D}(e)$ as $D'(e)$ so we have, as the definition of our $\tilde{D}$:

$$\tilde{D}_Z(w \otimes e) = D_Z(w) \otimes e + w \otimes D'_Z(e)$$

Another useful formula for $\tilde{D}_Z$ is, for $W \in C^\infty(T^*M \otimes E)$:

$$(\tilde{D}_{Z_1} W)(Z_2) = D'_{Z_1}\{W(Z_2)\} - W\{D_{Z_1} Z_2\}$$

$(Z_1, Z_2 \in C^\infty(TM)$ so that $W(Z_2) \in C^\infty(E))$

<u>Divergence</u> Define the <u>(generalised) divergence</u> $\tilde{\delta}$: $C^\infty(T^*M \otimes E) \to C^\infty(E)$ as the composition:

$$C^\infty(T^*M \otimes E) \xrightarrow{\tilde{D}} C^\infty(T^*M \otimes T^*M \otimes E) \xrightarrow{Tr} C^\infty(E)$$

(5) where if $\beta \in C^\infty(T^*M \otimes T^*M \otimes E)$, $\operatorname{Tr}\beta$ may be defined as $\operatorname{Tr}\beta = \sum_i \beta(X_i, X_i)$ where $X_i$ is an orthonormal basis.

<u>Proposition</u>: $-\tilde{\delta}$ is the <u>adjoint</u> of $\tilde{D} : C^\infty(E) \to C^\infty(T^*M \otimes E)$ i.e.

$$\int_M \langle \tilde{D}\alpha, \beta \rangle = -\int_M \langle \alpha, \tilde{\delta}\beta \rangle$$

$\forall \alpha \in C^\infty(E)$, $\beta \in C^\infty(T^*M \otimes E)$ where $\langle \ \rangle$ denotes the metric on $T^*M \otimes E$ (l.h.side), $E$ (r.h.side).

<u>Proof</u> See [16], [34].

<u>Case when $E = f^{-1}TN$</u>

Let $f : M \to N$ be a smooth map between Riemannian manifolds. Let $E = f^{-1}TN$. Then $E$ inherits a metric from that of $TN$ and can also be given a connection by pulling back the Levi-Civita connection on $TN$. Specifically, if $e_1, \dots, e_n$ is a local basis for $TN$ which is pulled back to a local basis (which we still denote by $e_1, \dots, e_n$) for $E$ <u>define</u> the connection 1-form of $E$ w.r.t. this basis by:

$$\theta_\alpha{}^\gamma(Z) = (\nabla_Z f)\left(\Phi_\alpha{}^\gamma(df(Z))\right) \qquad Z \in C^\infty(TM)$$

where $\Phi_\alpha{}^\gamma$ are the connection 1-forms for $N$

$$= (\nabla_Z f^\beta)\, L_{\alpha\beta}{}^\gamma$$

where $L_{\alpha\beta}{}^\gamma$ are the Christoffel symbols for $N$.

Note that since $TN$ is Riemannian connected it easily follows that $E$ is.

The following formulae follow from the product rule:

(a) If $e \in C^\infty(E)$, $Z \in C^\infty(TM)$; if $e$ has components $(e^1, \dots, e^n)$ w.r.t. some local basis for $E$:

$$\{\tilde{D}_Z e\}^\gamma = \nabla_Z(e^\gamma) + \theta_\alpha{}^\gamma(Z)\, e^\alpha$$

(6) (b) If $W \in C^\infty(T^*M \otimes E)$, $Z \in C^\infty(TM)$ and $W$ has components $W_j^{\alpha}$ w.r.t. basis $\{dx^j \otimes \frac{\partial}{\partial u^\alpha}\}$ for $T^*M \otimes E$ which is obtained from local coordinates $(x^1, \ldots, x^m)$ on $M$, $(u^1, \ldots, u^n)$ on $N$, then

$$\{\tilde{D}_{Z_i}(W)\}_j^{\alpha} = \nabla_{Z_i}(W_j^{\alpha}) - \Gamma_{ji}^{k} W_k^{\alpha} + (\nabla_{Z_i} f^\beta) L_{\alpha\beta}^{\gamma} W_j^{\alpha}$$

where $\Gamma_{ij}^{k}$ $L_{\alpha\beta}^{\gamma}$ are the Christoffel symbols for $M$, $N$ respectively and $f^\beta$ denotes the $\beta^{th}$ component of $f$ wrt the coordinates on $N$.

<u>Notes</u> This brief description may be supplemented by

[Hi] [KN1] basic differential geometry

[11] [16] pull-back bundles etc.

## 2 THE CONCEPT OF A HARMONIC MAP

(A) Energy  Let $M, N$ be connected $C^\infty$ Riemannian manifolds of dimensions $m, n$ respectively and let $f: M \to N$ be a $C^\infty$ map. Form the pull-back bundle $E = f^{-1}TN$ over $M$. This bundle together with the bundles $T^*M \otimes E$, $T^*M \otimes T^*M \otimes E$, etc. inherit metrics from those of $M$ & $N$ as described earlier. In particular the for derivative $df \in C^\infty(T^*M \otimes E)$, we have the inner product at each point $p \in M$

$$\langle df, df \rangle_p = \sum_i \langle df(X_i), df(X_i) \rangle_{E_p} \quad \text{where}$$

$\{X_i\}$ is an orthonormal basis for $T_pM$; the number $e(f)(p) = \frac{1}{2}\langle df, df\rangle_p$ is called the energy density of $f$ at $p$, ~~the number~~ if $M$ is compact, the number

$$E(f) = \int_M e(f)\, v_M \qquad \text{is called the } \underline{\text{energy of } f}$$

Intuitively, the energy density measures how much the map "stretches" at a point $p$, and the energy is a measure of the total amount of "stretching" of the map.

Local coordinates  if $(x^1, \ldots, x^m)$, $(u^1, \ldots, u^n)$ are local coordinates for $M, N$ respectively, then $df = f_i^{\alpha}\, dx^i \otimes \dfrac{\partial}{\partial u^\alpha}$

where $f_i^\alpha =$ the partial derivative $\dfrac{\partial u^\alpha}{\partial x^i}$. Write

$g_{ij}\, dx^i dx^j$, $h_{\alpha\beta}\, du^\alpha du^\beta$ for the metrics on $M, N$ respectively then

$$e(f)(p) = \frac{1}{2} g^{ij} f_i^{\alpha} f_j^{\beta} h_{\alpha\beta} \qquad g^{ij} = (g_{ij})^{-1} \text{ (inverse matrix)}$$

Definition If $M$ is compact $f: M \to N$ is harmonic if $f$ is a stationary point for the energy functional $E$. w.r.t. all smooth variations of $f$.

Note If $M$ is not compact, we can define the energy $E(f, D) = \int_D e(f)\, v_M$ of $f$ over any compact region $D$ we say $f$ is harmonic if it renders all energy functionals $E(\cdot\ ,D)$ stationary wrt all smooth variations of $f$ that are fixed on $\partial D$.

(B) <u>Tension</u> Given $M, N, f$ as above we define the <u>tension field</u> of $f$ as follows. Recall that using the ~~connection on~~ Levi-Civita connections on $M,N$ we can define covariant differentiations:

$TM^* = T^*M$ !

$$\tilde{D}: C^\infty(E) \to C^\infty(TM^* \otimes E)$$
$$\tilde{D}: C^\infty(TM^* \otimes E) \to C^\infty(TM^* \otimes TM^* \otimes E)$$

In particular $\tilde{D}\,df \in C^\infty(T^*M \otimes T^*M \otimes E)$ is called the <u>second fundamental form</u> of $f$ and $\mathrm{Tr}(\tilde{D}\,df) \in C^\infty(E)$ is called the <u>tension field</u> of $f$ denoted $\tau(f)$. Note the following formulae: which follows from §1.

$$\tau(f) = \mathrm{Tr}\,\tilde{D}\,df = \sum_i \tilde{D}\,df(X_i, X_i) \quad \text{where } \{X_i\} \text{ is an orthonormal basis for } TM$$

$$\tilde{D}\,df\,(X_i, X_j) = D'_{X_j}\left(df(X_i)\right) - df\{D_{X_j}X_i\}$$

where $D'$ is the connection on $E$, & $D$ is the connection on $TM$.

$$\text{hence } \tau(f) = \sum_i D'_{X_i}\left(df(X_i)\right) - \sum_i df\{D_{X_i}X_i\}$$

where $\{X_i\}$ is an orthonormal basis for $TM$

In local coordinates (same notation as ~~last~~ section A) $\tilde{D}\,df$ has components

$$f_{;ij}{}^{\gamma} = f_{ij}{}^{\gamma} - \Gamma_{ij}{}^{k} f_k{}^{\gamma} + L_{\alpha\beta}{}^{\gamma} f_j{}^{\alpha} f_i{}^{\beta}$$

where

$$f_{ij}{}^{\gamma} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} \quad \& \ \Gamma_{ij}{}^{k},\ L_{\alpha\beta}{}^{\gamma} \text{ are Christoffel symbols for } M, N$$

$\tau(f)$ has components

$$\tau(f)^\gamma = g^{ij} f_{;ij}{}^{\gamma}$$

$$= g^{ij} f_{ij}{}^{\gamma} - g^{ij}\Gamma_{ij}{}^{k} f_k{}^{\gamma} + g^{ij} L_{\alpha\beta}{}^{\gamma} f_i{}^{\alpha} f_j{}^{\beta}$$

(9) (c) Tie-up – First Variation of the Energy. Let $f_t : M \times I \to N$ be a $C^\infty$ with $f_0 = f$. We calculate

$$\left.\frac{dE(f_t)}{dt}\right|_{t=0} = \frac{1}{2}\int_M \sum_i \langle df(X_i), df(X_i)\rangle_E \qquad \{X_i\} \text{ orthonormal basis for } TM$$

$$= \int_M \sum_i \langle \frac{D'}{\partial t} df(X_i), df(X_i)\rangle_E$$

since $E$ is Riemannian connected and $\langle\ ,\ \rangle$ is symmetric – ~~~~

$$= \int_M \langle D'_{X_i} \frac{\partial f}{\partial t}, df(X_i)\rangle_E \qquad \text{since } \frac{\partial}{\partial t} \text{ and } e_i \text{ clearly have zero bracket on } M \times (-\varepsilon, \varepsilon)$$

$$= \int_M \langle \tilde{D} w, df\rangle_{TM^* \otimes E}$$

where $w = \left.\frac{\partial f}{\partial t}\right|_{t=0}$

$$= -\int_M \langle w, \tau(f)\rangle \qquad \text{since adjoint of } \tilde{D} \text{ is } -\mathrm{Tr}\,\tilde{D}$$

Thus $E$ is stationary for all smooth variations $\iff \tau(f) \equiv 0$. We thus now adopt the definition

<u>Definition</u> $f : M \to N$ is <u>harmonic</u> if $\tau(f) \equiv 0$.

<u>Notes</u> For a better explanation of above calculation see [173], [15] [16] [17], for the second variation of the energy see [34] [49].

(10) (D) Interpretation of the Condition $\tau(f) = 0$ [187]

Definitions: By a (parametrised) curve on $N$ through $q \in N$ we mean a smooth map $c: (-\varepsilon, \varepsilon) \to N$, $t \mapsto c(t)$ with $c(0) = q$. The acceleration vector of $c$ at $q$ is the vector $\left.\frac{D'}{\partial t}\frac{\partial c}{\partial t}\right|_{t=0}$. This is a tangent vector at $q$. $c$ is a geodesic if its acceleration vector at each point is zero, more generally if the parameter $t$ measures arc length (i.e. $c$ is an isometry) the acceleration vector of $c$ at $q$ is merely the geodesic curvature of $c$ at $q$.

Now let $f: M \to N$ be a smooth map, let $p \in M$ and let $\{X_1, \ldots, X_m\}$ be an orthonormal basis of tangent vectors at $p$. Let $\gamma_i$ be the unique geodesic ~~of~~ $M$ through $p$ tangent to $X_i$

Claim:

$\tau(f)(p) =$ sum of acceleration vectors at $q = f(p)$ of the parametrised curves $f \circ \gamma_i$

NB Here, and elsewhere, rather than considering $\tau(f)$ as a section of the bundle $f^{-1}TN$, it is convenient to consider $\tau(f)$ as a smooth map $\tau(f): M \to TN$ with $\tau(f)(p) \in T_{f(p)}N$. Such a map is called a vector field along $f$.

Proof of Claim From the formulae in §B

$$\tau(f) = \sum_i D'_{X_i}(df(X_i)) - \sum_i df\{D_{X_i}X_i\}$$

~~The second term is zero at p.~~ Choose normal coordinates $(x_1, \ldots, x_m)$ for $M$ centred on $p$ and choose the basis $X_1, \ldots, X_m$ by $X_i = \frac{\partial}{\partial x_i}$. Then since the "axes" through $p$ are geodesics, $D_{X_i}X_i = 0$ and the second term vanishes. The first term can be written $\sum_i \frac{D'}{\partial x_i}\frac{\partial f}{\partial x_i}$ and the claim follows. Thus:

Proposition A smooth map $f: M \to N$ is harmonic $\iff$ at all points $p \in M$ and for all sets of geodesics $\{\gamma_1, \ldots, \gamma_m\}$ of $M$ through $p$ whose tangents at $p$ form an orthonormal set, the sum of the acceleration vectors of the curves $f \circ \gamma_i$ is zero.

Thus a harmonic map is one that maps geodesics into "a set of curves

(11) 3 SPECIAL TYPES OF HARMONIC MAPS

(A) Totally geodesic maps [11] Defn $f: M \to N$ is s.t.b. totally geodesic if its second fundamental form $\tilde{D}df$ is identically zero. Lemma: $f$ is totally geodesic $\iff$ it maps geodesics to geodesics linearly.

Proof In normal coordinates ~~centred~~ $(x^1, \ldots, x^m)$ centred on a point $p \in M$ and normal coordinates $(u^1, \ldots, u^n)$ centred on $f(p) \in N$ ~~[illegible]~~ we have the simple expression for the components of $\tilde{D}df$ ~~[illegible]~~

$$f_{;ij}{}^{\gamma} = \frac{\partial^2 f^{\gamma}}{\partial x^i \partial x^j}$$

The lemma follows.

Clearly, a totally geodesic map is harmonic.

(B) Geodesics [11] If $M$ is 1-dimensional, to be specific, suppose $M =$ open subset of $\mathbb{R}$ we have, (as before):

$\tau(f) =$ acceleration vector of $f: M \to N$. Hence

$f$ is harmonic $\iff$ $f$ ~~defines~~ is a geodesic.

This is a special case of (A).

(C) Harmonic Functions [11] If $N$ is 1-dimensional, to be specific, suppose $N = \mathbb{R}$ then: ~~found~~

$$\tau(f) = \Delta f = \text{the Laplacian of the function } f: M \to \mathbb{R}$$

This can be seen from the expression in §2B for $\tau(f)$ in local coordinates: which reduces to

$$\tau(f) = g^{ij} f_{ij} - g^{ij} \Gamma_{ij}^{k} f_k$$

which is a well-known expression for the Laplacian of $f$.

Note The Laplacian of $f$ may be defined as:

$$\Delta f = \delta df \quad \text{where} \quad \delta = \mathrm{Tr} \circ D$$

Hence $f: M \to \mathbb{R}$ is a harmonic map $\iff$ $f$ is a harmonic function

Thus the concept of harmonic map is a generalisation of the concept of harmonic function.

(D) Minimal Immersions [11] $f: M \to N$ is called a Riemannian immersion if $df$ is an isometry on each tangent space. The mean normal curvature field $H$ of such a map may be defined as follows: ~~choose~~ given $p \in M$, choose geodesics $\gamma_i$ in $M$ through $p$ such that the tangent vectors of the $\gamma_i$ form an orthonormal basis for the tangent space at $p$, then $H(p) = \sum_i$ acceleration vectors of $f \circ \gamma_i$ at $f(p)$. Then clearly we have: (by §2D):

Lemma For a Riemannian immersion: $\tau(f) = H$

Note also:

Lemma For a Riemannian immersion $\tau(f)(p)$ is perpendicular to $T_pM$

This can be proved in local coordinates [11] but is also obvious from the interpretation of $\tau(f)(p) = H(p)$ as the sum of acceleration vectors.

We call a Riemannian immersion minimal if its mean normal curvature field $= 0$.

Proposition [11] A Riemannian immersion is harmonic $\Leftrightarrow$ it is minimal.

(E) ~~Harmonic Gauss Maps~~ The Gauss Map of a Minimal Immersion [44]

Given a Riemannian immersion $i: M \to \mathbb{R}^{m+r}$ ($M$ $m$-dimensional) we define its Gauss map ~~by~~ $g: M \to G(m,r)$ = Grassmann manifold of $m$ planes in $\mathbb{R}^{m+r}$ by $g(p)$ = tangent plane $i_* T_pM$. Now we may form the mean normal curvature vector field $H$ of $i$, which by the second Lemma of §D is a cross-section of $N(M)$, the normal bundle of $M$. Using the covariant derivative on $M$ we may form the covariant derivative $\nabla H$ of $H$ in $N(M)$, $\nabla H \in C^\infty(T^*M \otimes NM)$

Ruh & Vilms [44] show that the bundle $T^*M \otimes NM$ may be identified with the bundle $g^{-1}TG(m,r)$ and

Lemma Under this identification $\nabla H = \tau(g)$. Hence:

Theorem [44] If $i: M \to \mathbb{R}^{m+r}$ is a Riemannian immersion with ~~parall~~ mean normal curvature vector parallel in the normal bundle, its Gauss map is harmonic

Corollary The Gauss-map of a minimal submanifold of Euclidean space is harmonic

Corollary The Gauss-map of a surface $M$ in $\mathbb{R}^3$ with constant scalar mean normal curvature is harmonic. Note since $N(M)$ is 1-dimensional the mean normal curvature field $H$ has only one component.

(F) Harmonic Fibre Maps [11] [173] For any smooth map $f: M \to N$ we may write $TM = V \oplus H$ where $V_p = \{ X \in T_pM : df(X) = 0 \}$ and $H_p$ is its orthogonal complement. Vectors in $V$ are called vertical, vectors in $H$ horizontal. $f$ is called a Riemannian fibration if $df|_{H_p}$ is an isometry $\forall p \in M$. Such a map is a locally trivial fibre map [108]

Lemma $\tau(f)(p) = -df \circ \tau(i)(p)$ where $i$ is the inclusion of the fibre containing $p$ into $M$.

Proof Choose coordinates $(x^1, \dots, x^m)$ for $M$ centred on $p$ such that: (1) the vectors $X_i = \frac{\partial}{\partial x_i}$ form an orthonormal base for $T_pM$ with $X_{n+1}, \dots, X_m$ vertical and $X_1, \dots, X_n$ horizontal, (2) the axes $x^1, \dots, x^n$ are the horizontal lifts of geodesics in $N$ through $f(p)$ — by [108] they must also be geodesics. Then by the composition law (see §3)

$$\tau(f \circ i) = df \circ \tau(i) + \sum_{i=n+1}^{m} \tilde{D}\,df\,(di(X_i), di(X_i))$$

But $f \circ i$ is constant so $\tau(f \circ i) = 0$. Also the last term

$$= \sum_{i=n+1}^{m} \tilde{D}\,df(X_i, X_i) = \sum_{i=1}^{m} \tilde{D}\,df(X_i, X_i) = \tau(f)$$ since

$$\tilde{D}\,df(X_i, X_i) = 0 \quad i = 1, \dots, n$$ by choice of coordinates.

Proposition A Riemannian fibration is harmonic $\Leftrightarrow$ all the fibres are minimal submanifolds

Proof By Lemma, recalling $\tau(i) \in H$ (see §D) and $\tau(i) =$ mean normal curvature of fibre in $M$, and remembering $df$ is an isometry on $H$.

(G) Holomorphic Maps [11] Holomorphic and antiholomorphic maps between Kähler manifolds are harmonic - see §4.

## (H) Mappings of Spheres [173] [47] [48] [188]

(1) $S^2 \to S^2$ Treating $S^2$ as the Riemann sphere, the maps $z \mapsto z^k$ $k = 0, \pm 1, \pm 2, \ldots$ provide harmonic maps of each (topological) degree $k$.

(2) Spherical Harmonics [173] A spherical harmonic $f$ of degree $k$ on $\mathbb{R}^m$ is a homogeneous harmonic polynomial of algebraic degree $k$ on $\mathbb{R}^m$. We examine $f|_{S^{m-1}}$. By an easy computation

$$\Delta_{S^{m-1}} f|_{S^{m-1}} = \Delta_{\mathbb{R}^m} f\big| - \frac{\partial^2 f}{\partial r^2} - (n-1)\frac{\partial f}{\partial r}$$

$$= -k(k+n-2) f|_{S^{m-1}}$$

(Thus $f|_{S^{m-1}}$ is an eigenfunction of $\Delta_{S^{m-1}}$. All eigenfunctions of $\Delta_{S^{m-1}}$ arise in this way [BGM] )

Note The definition of $\Delta_{S^{m-1}}$ $\Delta_{\mathbb{R}^m}$ may differ in sign from the one we have adopted. This will not trouble us.

Lemma Suppose $f : S^{m-1} \to S^{n-1}$ is the restriction of a homogeneous harmonic polynomial map $\mathbb{R}^m \to \mathbb{R}^n$, each polynomial $\mathbb{R}^m \to \mathbb{R}$ being of degree $k$. Then $f$ is a harmonic map.

Proof By §J (last prop^n): $\tau(f) =$ proj^n of $\{\Delta_{S^{m-1}} f$ where $f$ is considered as map $S^{m-1} \to \mathbb{R}^n\}$. But by computation above $\Delta_{S^{m-1}} f$ is normal to $S^{n-1}$.

Harmonic maps of this type may be constructed from

(3) Orthogonal Multiplications An orthogonal multiplication $F : \mathbb{R}^m \times \mathbb{R}^{m'} \to \mathbb{R}^n$ is a bilinear map s.t. $|F(x,y)| = |x|\,|y|$ $\forall x \in \mathbb{R}^m, y \in \mathbb{R}^{m'}$. Given an orthogonal multiplication it is easily seen that the following maps are harmonic:

(1) the restriction $F : S^{m-1} \times S^{m'-1} \to S^{n-1}$

(2) if $m = m'$ the map $H : S^{2m-1} \to S^{n}$ given by

$H(x,y) = \left(|x|^2 - |y|^2,\ 2F(x,y)\right)$ where $(x,y)$ are Euclidean coords for $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ ("Hopf construction")

(3) given two second degree harmonic polynomial maps $f : S^{m-1} \to S^{q-1}$, $g : S^{m'-1} \to S^{q'-1}$ the map $h : S^{m+m'-1} \to S^{q+q'+n-1}$

$h(x,y) = \left(f(x),\ g(y),\ \sqrt{2}\,F(x,y)\right)$

where $(x,y)$ are Euclidean coordinates for $\mathbb{R}^{m+m'} = \mathbb{R}^m \times \mathbb{R}^{m'}$.

(1!)

Examples (1) $F: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ $m = 2, 4, 8$, given by multiplication of complex numbers, quaternions, Cayley numbers give the Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ when the "Hopf construction" above is applied. These maps are all harmonic.

(2) Real tensor product is an orthogonal multiplication $F: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m^2}$ yielding harmonic maps $S^{2m-1} \to S^{m^2-1}$.

Many other examples are given in [173]

(4) Eikonals [188] Let $P: \mathbb{R}^m \to \mathbb{R}$ be a spherical harmonic of degree $k+1$. If

$$\left(\frac{\partial P}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial P}{\partial x_m}\right)^2 = (x_1^2 + \dots + x_m^2)^k \qquad (*)$$

then the gradient of $P$. $\left(\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_m}\right): \mathbb{R}^m \to \mathbb{R}^m$

restricts to $S^{m-1} \to S^{m-1}$ and is of course harmonic.

$(*)$ is called the Eikonal equation of geometric optics and was studied by Cartan. The most general solution for $k=1$ is $P = \frac{1}{2}(x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_m^2)$ which is harmonic $\iff r = s$. The resulting harmonic maps $\left(\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_m}\right): S^{m-1} \to S^{m-1}$ are isometries.

Much more interesting are the solutions for larger $k$. R. Wood shows that we get: $(k=2)$ harmonic maps $S^4 \to S^4$ $S^8 \to S^8$ $S^{13} \to S^{13}$ $S^{25} \to S^{25}$ of degree 2, $(k=3)$ harmonic maps $S^5 \to S^5$ degree 1 $S^9 \to S^9$ degree ?.

(5) Joining Polynomial Maps of Spheres R.T. Smith [47] shows that two homogeneous harmonic polynomial maps $f: S^{m-1} \to S^{n-1}$ $g: S^{m'-1} \to S^{n'-1}$ of degree $\ell$ and $k$ may be "joined" to give a harmonic map $F: S^{m+m'-1} \to S^{n+n'-1}$ of the form $F(u,v) = \left(\frac{g(u)}{|u|^k} \cos\alpha(t), \frac{f(v)}{|v|^\ell} \sin\alpha(t)\right)$ where $\alpha: \mathbb{R} \to [0, \frac{\pi}{2}]$ is a smooth monotone function with $\alpha(t) \to 0$ as $t \to -\infty$ $\alpha(t) \to \frac{\pi}{2}$ as $t \to \infty$ and $t = \ln(|u|/|v|)$, provided

Examples (1) Suspending a map is achieved by joining it with the ~~constant~~ identity map $S^0 \to S^0$. Suspending $S^2 \to S^2$ $z \mapsto z^k$ gives harmonic maps of degree $k$ $S^n \to S^n$ for $n = 2, \ldots, 7$ (for larger $n$ conditions for joining are no longer satisfied)

(2) Suspending the Hopf maps $S^3 \to S^2$ $S^7 \to S^4$ yields essential harmonic ~~representatives~~ maps $S^{n+1} \to S^n$ $n = 3, \ldots, 8$, $S^{n+3} \to S^n$ $n = 5, \ldots, 10$.

(I) Maps Into $S^1$ and Tori [11] [32] [187] Let $M$ be a connected compact oriented Riemannian manifold. Given any harmonic 1-form $\gamma$ on $M$ with integral periods, i.e. such that $\int_c \gamma$ is integral for all smooth closed curves $c$, choose a base point $p_0$ and set

$$f(p) = \int_{p_0}^{p} \gamma$$

This defines a harmonic map into $\mathbb{R}/\mathbb{Z} = S^1$

(Note A harmonic 1-form may be defined as a 1-form which is locally of the form $dg$ for some harmonic function $g$ defined on part of $M$)

Harmonic maps in every homotopy class $M \to S^1$ may be constructed in this way, for the set of homotopy classes $[M, S^1]$ is canonically isomorphic to $H^1(M, \mathbb{Z})$, and any cohomology class in $H^1(M, \mathbb{Z})$ can be canonically represented by a harmonic 1-form on $M$ with integral periods.

For a flat torus $T = \mathbb{R}^n/\Gamma$ where $\Gamma$ is a lattice generated by $\underline{a}_1, \ldots, \underline{a}_n$, say, for any $n$ harmonic 1-forms $\gamma_1, \ldots, \gamma_n$:

$$f : M \to \mathbb{R}^n/\Gamma = T \quad \text{defined by}$$

$$p \longmapsto \underline{a}_1 \int_{p_0}^{p} \gamma_1 + \ldots + \underline{a}_n \int_{p_0}^{p} \gamma_n$$

provides a harmonic map in any desired homotopy class

Maps from a surface into a torus are studied in [187].

(5) Compositions

Let $f: M \to N$ and $k: N \to P$ be smooth maps between connected Riemannian manifolds. If $f$ and $k$ are harmonic, it does not necessarily follow that $k \circ f$ is harmonic. In fact note the following formulae for the second fundamental form and tension field of $k \circ f$

(5 1) $$\tilde{D} d(k \circ f)(X,Y) = dk\{\tilde{D}df(X,Y)\} + \tilde{D}dk(df(X), df(Y))$$

where $X, Y \in TM$, or in local coordinates:

(5 2) $$(k \circ f)_{;ij}{}^{a} = f_{;ij}{}^{\gamma} k_{\gamma}{}^{a} + k_{;\alpha\beta}{}^{a} f_i{}^{\alpha} f_j{}^{\beta}$$

(5 3) $$\tau(k \circ f) = dk \circ \tau(f) + \sum_i \tilde{D}dk(df(X_i), df(X_i))$$

where $\{X_i\}$ is orthonormal basis in $TM$

or in local coordinates:

(5 4) $$\tau(k \circ f)^{a} = \tau(f)^{\gamma} k_{\gamma}{}^{a} + k_{;\alpha\beta}{}^{a} g^{ij} f_i{}^{\alpha} f_j{}^{\beta}$$

where $g_{ij}$ is metric on $M$.

From these formulae we deduce:

(a) Proposition [11] If $f$ is harmonic and $k$ is totally geodesic, then $k \circ f$ is harmonic.

(b) Proposition [187] If $f$ is a weakly conformal map between surfaces and $k$ is harmonic then $k \circ f$ is harmonic

(c) Proposition [173] If $f$ is a Riemannian fibration with fibres minimal submanifolds and $k$ is harmonic then $k \circ f$ is harmonic.

Proofs (a) $\tilde{D}dk \equiv 0$ ... (b) A weakly conformal map $f$ is one such that $df$ is conformal or trivial on each tangent space. Such a map must be holomorphic or antiholomorphic and $\therefore$ harmonic by §4C. Further, ~~$\tilde{\nabla}dk(df(X_i), df(X_i)) = \lambda \tilde{\nabla}dk$~~ if we write $df(X_i) = \lambda Y_i$ where $Y_i$ are unit vectors in $TN$ then ~~$\tilde{\nabla}dk(df(X_i)$~~ the $Y_i$ must be an orthonormal basis whence $\sum_i \tilde{\nabla}dk(df(X_i), df(X_i)) = \lambda^2 \sum_i \tilde{\nabla}dk(Y_i, Y_i)$

$= \lambda^2 \tau(k) = 0$ ... (c) Similar proof. (see [173]).

(18)

Note also the useful

<u>Proposition [11]</u> If $k$ is a Riemannian immersion, $\tau(f) =$ projⁿ of $\tau(k\circ f)$ onto $TN$.

<u>Proof</u> The formula (J3) gives a decomposition of $\tau(k\circ f)$ into components in tangent bundle of $N$ and normal bundle of $N$

~~and~~

(k) <u>Compositions with Convex Functions [21] [187] ~~cf.~~ [53]</u>

A function $k: U \to \mathbb{R}$ defined on an open subset of $N$ is s.t.b. convex (resp. strictly convex) if its second fundamental form $\tilde{\nabla} dk$ is positive ~~semidefinite~~ (resp. positive definite).

<u>Propn</u> If $f: M\to N$ is harmonic and $k: U \to \mathbb{R}$ is convex where $U \supseteq f(M)$, then $k\circ f$ is subharmonic i.e. $\Delta(k\circ f) \geq 0$.

<u>Proof</u> By (J3) $\tau(k\circ f) = \Delta(k\circ f) = \sum_i \tilde{\nabla} dk(df(X_i), df(X_i))$

($X_i$ orthonormal basis for $TM$) ...

<u>Corollary</u> (1) If $M$ is compact $k\circ f$ must be constant

(2) If further, $k$ is strictly convex then $f$ must be constant

<u>Proof</u> (1) By maximum principle for subharmonic functions. (2) $\Delta(k\circ f) = \sum_i \tilde{\nabla} dk(df(X_i), df(X_i)) = 0 \Rightarrow df(X_i) = 0 \ \forall X_i \Rightarrow df \equiv 0$.

These corollaries tell us much about the image of a harmonic map [21] [187] [53] for example: (1) every point of $N$ has a neighbourhood on which there may be defined a strictly convex function. Then no harmonic map can have its image in this neighbourhood unless it is constant [21].

(2) <u>Sampson's Maximum Principle</u> [45] <u>Let $S$ denote a submanifold of $N$ of codimension 1 passing through a point $Q = f(P)$. Suppose $S$ has definite fundamental form at $Q$. Then if $f: M\to N$ is harmonic, no neighbourhood of $P$ is mapped entirely ~~to~~ the concave side of $S$</u>.

This is proved by constructing [187] a strictly convex function $k$ which changes sign when crossing $S$ and then using the maximum principle for the subharmonic function $\Delta(k\circ f)$. It can be used to show [53] that <u>for a non-constant harmonic map $f: M\to N$ $M, N$ compact connected $C^\omega$ Riemannian mfds $N$ oriented:</u>

(19) 4 FORMULATION OF HARMONICITY FOR MAPS BETWEEN COMPLEX MANIFOLDS

(A) Preliminaries on Complex Manifolds (1) Let $M$ be a differentiable manifold of even dimension $2m$. An almost complex structure on $M$ is a tensor field $J \in C^\infty(T^*M \otimes TM)$ such that for each $p \in M$ $J: T_pM \to T_pM$ is an endomorphism satisfying $J^2 = -I$. A manifold with a given almost complex structure is called an almost complex manifold. For example $\mathbb{R}^{2m}$ has standard almost complex structure $J$ given on its trivial tangent bundle by $J(x_1, y_1, \dots, x_m, y_m) = (-y_1, x_1, \dots, -y_m, x_m)$.
A mapping $f: M \to N$ between two almost complex manifolds is called holomorphic if $df \circ J = J' \circ df$ and antiholomorphic if
$df \circ J = -J' \circ df$ where $J'$ denotes the almost complex structure on $N$. If there exists an atlas of charts $c_\alpha: U_\alpha \to \mathbb{R}^{2m}$ which are biholomorphic (i.e. $c_\alpha$ and $c_\alpha^{-1}$ are holomorphic) we say the complex structure is integrable and the manifold is called a complex manifold. Such charts define ~~a~~ local complex coordinates. Conversely, given any complex manifold $M$ we may define an almost complex structure $J: T_pM \to T_pM$ by choosing a chart $c: U \to \mathbb{R}^{2m}$ and defining $J = \{dc(p)\}^{-1} \circ J_{\mathbb{R}^{2m}} \circ dc(p)$ where $J_{\mathbb{R}^{2m}}$ is the standard complex structure on $\mathbb{R}^{2m}$.

(2) A Hermitian metric $g$ on an almost complex manifold $M$ is a Riemannian metric which is invariant by the ~~complex~~ almost complex structure $J$ i.e. $g(JX, JY) = g(X, Y)$ $\forall X, Y \in T_pM$ $\forall p \in M$.
An (almost) complex manifold equipped with a Hermitian metric is called an (almost) Hermitian manifold.

Let $M = (M, g, J)$ be an almost Hermitian manifold and let $D$ be the Levi-Civita connection on $M$. We say that $g$ is a Kähler metric and $M$ is a Kähler manifold if
$$D_X(JY) = J D_X Y \qquad \forall X, Y \in C^\infty(TM)$$

It can be shown (Newlander & Nirenberg) - that a Kähler manifold is always integrable and that a ~~[illegible]~~ 1-complex dimensional almost Hermitian ~~complex~~ manifold is always Kähler.

On a Kähler manifold curvature tensors and Ricci tensors ~~have~~ commute with $J$ see [KN2]

(3) Now let $M$ be an almost complex manifold. It is useful to complexify the tangent bundle to give a bundle $T^cM = TM \otimes_{\mathbb{R}} \mathbb{C}$.
We then identify $v \in TM$ with $v \otimes 1$ and write $v \otimes i$ as $vi$ or $iv$. We extend $J$ to $T^cM$ by requiring it to be linear over $\mathbb{C}$. Now the complexified tangent bundle $T^cM$ can be written as a direct sum of complex bundles
$T^cM = T^{1,0}M \oplus T^{0,1}M$ where

$$T^{1,0}M = \{Z \in T^cM : JZ = iZ\} = \{X - iJX : X \in TM\}$$
$$T^{0,1}M = \{Z \in T^cM : JZ = -iZ\} = \{X + iJX : X \in TM\}$$

Conjugation $\bar{\ }: T^cM \to T^cM$ defined by $X + iY \mapsto X - iY$, $X, Y \in TM$ gives a real-linear isomorphism between $T^{1,0}M$ and $T^{0,1}M$.

We define similarly $T^*_cM = T^*M \otimes_{\mathbb{R}} \mathbb{C}$, this is the dual of $T^cM$, we then define $T^*_{1,0}M$ $T^*_{0,1}M$ similarly to the above — note that

$$T^*_{1,0}M = \{w \in T^*_cM : w(X) = 0 \quad \forall X \in T^{0,1}M\}$$
$$T^*_{0,1}M = \{w \in T^*_cM : w(X) = 0 \quad \forall X \in T^{1,0}M\}$$

The tensor product $T^cM \otimes \ldots \otimes T^cM \otimes T^*_cM \otimes \ldots \otimes T^*_cM$ can be decomposed similarly e.g.

$$T^cM \otimes T^cM = T^{2,0}M \oplus T^{1,1}M \oplus T^{0,2}M$$

where $T^{2,0}M = T^{1,0}M \otimes T^{1,0}M$, $T^{0,2}M = T^{0,1}M \otimes T^{0,1}M$

$$T^{1,1}M = T^{1,0}M \otimes T^{0,1}M \oplus T^{0,1}M \otimes T^{1,0}M.$$

(4) Now let $M$ be an almost Hermitian manifold with metric denoted by $g$. We extend $g$ to $T^cM$ by insisting it be bilinear over $\mathbb{C}$.
Then $g$ satisfies:

$$g(Z, iW) = g(iZ, W) = i\,g(Z, W)$$
$$g(Z, JW) = -g(JZ, W)$$
$$g(\bar{Z}, \bar{W}) = \overline{g(Z, W)}$$

Writing $Z = X + iY$, $X, Y \in TM$ $\quad g(Z, \bar{Z}) = g(X, X) + g(Y, Y) \geq 0$

BASES (5) Let $Z_1, \ldots, Z_n$ be a (local) basis for $T^{1,0}M$, then $\bar{Z}_1, \ldots, \bar{Z}_n$ is a basis for $T^{0,1}M$, and together they form a basis for $T^cM$

It is convenient to denote $\overline{Z_k}$ by $Z_{\bar{k}}$ and adopt the convention that small letters such as $k, \ell$ may range over the indices $1, 2, \ldots$ whereas large letters may range over $1, 2, \ldots, \bar{1}, \bar{2}, \ldots$
It is easily seen that

$$g_{j\bar{k}} = g(Z_j, \overline{Z_k}) = \overline{g_{k\bar{j}}}$$
$$g_{jk} = g(Z_j, Z_k) = 0$$

(6) <u>COORDINATES</u> Suppose now $M$ is a complex manifold. Let $(z^1, \ldots, z^m)$ be local complex coordinates for $M$, where $z^i = x^i + iy^i$. Setting

$$Z_i = \frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial x^i}\right) \qquad Z_{\bar{i}} = \frac{\partial}{\partial z^{\bar{i}}} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right)$$

gives a basis for $T^{1,0}M \oplus T^{0,1}M = T^cM$. The dual basis for $T^*_cM$ is easily seen to be

$$dz^i = dx^i + i\,dy^i \qquad d\bar{z}^i = dx^i - i\,dy^i$$

Now suppose $M = (M, g, J)$ is Kähler. Then w.r.t. the above basis the <u>Christoffel symbols</u> $\Gamma_{IJ}{}^{K}$ of $M$ have only <u>pure</u> components $\Gamma_{ij}{}^{k}$ $\Gamma_{\bar{i}\bar{j}}{}^{\bar{k}}$, other components being zero. (<u>Proof</u>: use cond$^n$ $D_X JY = JD_XY$)

<u>Note</u> It should first be mentioned that, as always, we extend the connection $D_X : C^\infty(T^cM) \to C^\infty(TM)$ to the complexified tangent bundle by requiring $D_XY$ to be <u>linear</u> in $X$ and $Y$ <u>over $\mathbb{C}$</u> and then $\Gamma_{IJ}{}^{K}$ = $K$'th component of $D_{Z_J}(Z_I)$.

(22)

(B) Harmonic Maps From a Complex Manifold. Let $M, N$ be Riemannian manifolds. Recall that, for a smooth map, $f: M \to N$, $\tau(f)$ is defined as $Tr\, \tilde{D}\, df$.

$$C^\infty(M,N) \xrightarrow{d} C^\infty(T^*M \otimes E) \xrightarrow{\tilde{D}} C^\infty(T^*M \otimes T^*M \otimes E) \xrightarrow{Tr} C^\infty(E)$$

where $E = f^{-1}TN$. We can evidently replace each bundle by its complexification and extend the definitions of $df$, $\tilde{D}$, $Tr$ by requiring them to be linear over $\mathbb{C}$:

$$\begin{tikzcd} C^\infty(M,N) \arrow[r, "d^c"] & C^\infty(T_c^*M \otimes E^c) \arrow[r, "\tilde{D}^c"] & C^\infty(T_c^*M \otimes T_c^*M \otimes E^c) \arrow[d, "Tr"] \\ & & C^\infty(E^c) \end{tikzcd}$$

Now suppose $M$ has an almost complex structure. Then we can decompose $T_c^*M = T_{1,0}^*M \oplus T_{0,1}^*M$, [redacted] and the maps decompose:

$$\begin{tikzcd} & & C^\infty(T_{2,0}^*M \otimes E^c) \arrow[dr, "Tr"] & \\ & C^\infty(T_{1,0}^*M \otimes E^c) \arrow[ur, "\tilde{\partial}"] \arrow[dr, "\tilde{\tilde{\bar\partial}}"] & & \\ C^\infty(M,N) \ni f \arrow[ur, "\partial"] \arrow[dr, "\bar\partial"'] & & C^\infty(T_{1,1}^*M \otimes E^c) \arrow[r, "Tr"] & C^\infty(E^c) \\ & C^\infty(T_{0,1}^*M \otimes E^c) \arrow[ur, "\tilde{\partial}"] \arrow[dr, "\tilde{\tilde{\bar\partial}}"] & & \\ & & C^\infty(T_{0,2}^*M \otimes E^c) \arrow[uur, "Tr"'] & \end{tikzcd}$$

and $\tau(f)$ is obtained by summing over the four routes.

But now suppose that $M$ is almost hermitian. Then $g(Z, W) = 0$ $\forall\, Z, W \in T^{1,0}M$, it follows that the top and bottom Trace maps are the zero maps. Also $\tilde{\tilde{\bar\partial}} \circ \partial = \tilde{\partial} \circ \bar\partial$ (basically since $\bar\partial f = \overline{\partial f}$ etc.) Thus

$$\underline{\tau(f) = 2Tr\, \tilde{\tilde{\bar\partial}} \circ \partial f = 2Tr\, \tilde{\partial} \circ \bar\partial f}$$

In local coordinates, with complex coordinates on $M$

$$\tau(f)^\gamma = 2 g^{i\bar{j}} \left( f_{i\bar{j}}{}^\gamma - \Gamma_{\bar{j}i}{}^k f_k{}^\gamma + L_{\alpha\beta}{}^\gamma f_{\bar{j}}{}^\alpha f_i{}^\beta \right)$$

Further, if $M$ is Kähler $\Gamma_{\bar{j}i}{}^k = 0$ and we thus see that harmonicity $M \to N$ depends only on the conformal equivalence class of the metric $g$.

(C) <u>Harmonic Maps Between Complex Manifolds</u> Let $M, N$ be Kähler manifolds. We can now split $E^c = E^{1,0} \oplus E^{0,1}$ and write $\partial = \partial^{1,0} + \partial^{0,1}$. We have a diagram, for [redacted] the $(1,0)$ case:

$$\begin{tikzcd}
 & C^\infty(T^*_{1,0}M \otimes E^{1,0}) \arrow[dr, "\tilde{\bar\partial}^{1,0}"] & & \\
C^\infty(M,N) \ni f \arrow[ur, "\partial^{1,0}"] \arrow[dr, "\bar\partial^{1,0}"'] & & C^\infty(T^*_{1,1}M \otimes E^{1,0}) \arrow[r, "Tr"] & C^\infty(E^{1,0}) \ni \tau^{1,0}(f) \\
 & C^\infty(T^*_{0,1}M \otimes E^{1,0}) \arrow[ur, "\tilde\partial^{1,0}"'] & &
\end{tikzcd}$$

We have
$$\tau^{1,0}(f) = 2\,\mathrm{Tr}\,\tilde{\bar\partial}^{1,0} \circ \partial^{1,0} f$$
$$= 2\,\mathrm{Tr}\,\tilde\partial^{1,0} \circ \bar\partial^{1,0} f$$
and $\tau^{0,1}(f) = \overline{\tau^{1,0}(f)}$

In local complex coordinates on $M, N$ $\tau^{1,0} f$ is given by
$$\tau(f)^\gamma = 2 g^{i\bar j}\left(f_{i\bar j}{}^\gamma + L_{\alpha\beta}{}^\gamma f_{\bar j}{}^\alpha f_i{}^\beta\right)$$
remembering only the pure Christoffel symbols of $M$ & $N$ are non-zero.

Note that $f$ is holomorphic $\Leftrightarrow$ $\bar\partial^{1,0} f = 0$
$f$ is antiholomorphic $\Leftrightarrow$ $\partial^{1,0} f = 0$
hence from expression for $\tau^{1,0}(f)$ we very easily see that a <u>holomorphic or antiholomorphic map between Kähler manifolds is harmonic</u>

<u>Note</u> ~~A More general types of~~ See [33] for improvement of this result.