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HARMONIC MAPS AND THE GAUSS-BONNET FORMULA

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ABSTRACT. Let $f:M\to N$ be a non-constant harmonic map between compact Riemann surfaces M,N, N orientable and equipped with a chosen real-analytic Riemann metric. We prove: $\{$total curvature of $f(M)\} \geqslant 2\pi\{$Euler characteristic of $f(M)\}$. We give this result for a slightly larger class of mappings, and we give some related results showing that if the codomain N has negative curvature, then such a mapping cannot exhibit certain types of "redundant" folding.

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1-INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let M,N be C^∞ connected Riemannian manifolds without boundary whose C^∞ metrics are denoted by g,h respectively. A C^∞ mapping $f:M\to N$ is said to be harmonic if its tension field $\tau(f)$ [1] satisfies:

(1) $\tau(f) = 0$

In local C^∞ coordinates $(x^1,\dots,x^m)$ for M, $(u^1,\dots,u^n)$ for N (1) reads:

(2) $\Delta u^\gamma + g^{ij}L_{\alpha\beta}^{\gamma}u_i^\alpha u_j^\beta = 0 \quad (\gamma=1,\dots,n)$

Here, $L_{\alpha\beta}^{\gamma}$ denotes the Christoffel symbols on N, Δ denotes the Laplacian on M and $u_i^\alpha = \partial u^\alpha/\partial x^i$.

If dim(M) = dim(N) and Mand N are orientable, then we may consider the larger class of C^∞ mappings $f:M\to N$ which satisfy:

(3) $\tau(f) = Dv(f)$.

Here, D is a prescribed C^∞ vector field on N and v(f) = volume magnification factor of f with + or - sign according to whether f is orientation preserving or reversing with respect to fixed chosen orientations on M and N. In local oriented coordinates $(x^1,\dots,x^m)$ on M, $(u^1,\dots,u^n)$ on N,

(4) $v(f) = \sqrt{\{\det(g^{ij})\,.\,\det(h_{\alpha\beta})\}}\det(u_i^\alpha)$

If dim(M) = dim(N) = 2, we may regard M and N as Riemann surfaces. For every choice of Riemann metrics g,h on M,N we may discuss harmonic maps $M\to N$ or maps satisfying (3). It is easily shown that the property of being harmonic or satisfying (3) is independent of the particular choice of Riemann metric g; indeed, in isothermal coordinates (x^1,x^2) on M, (3) becomes:

(5) $$\frac{\partial^2 u^\gamma}{\partial x^{1^2}} + \frac{\partial^2 u^\gamma}{\partial x^{2^2}} + L_{\alpha\beta}^{\gamma}\{u_1^\alpha u_1^\beta + u_2^\alpha u_2^\beta\} = D^\gamma\sqrt{\det(h_{\alpha\beta})}\,\det(u_i^\alpha)$$

If $f:M\to N$ is harmonic with respect to a real-analytic Riemann metric on N or satisfies (3) with D a prescribed real-analytic vector field on N, f is real-analytic.

Our main result is:

THEOREM 1. Let M,N be compact Riemann surfaces, N with chosen real-analytic Riemann metric. Let $f:M\to N$ be a non-constant mapping which is either harmonic or satisfies (3) for some real-analytic vector field D on N. Then the total curvature $K(f(M))$ of $f(M)$ is related to the Euler characteristic $\chi(f(M))$ of $f(M)$ by the inequality:

(6) $$K(f(M)) \geqslant 2\pi\chi(f(M))$$

COROLLARIES:

With hypotheses as in theorem 1:

COROLLARY 1. If $f(M)$ is contractible, then $K(f(M)) \geqslant 2\pi$

COROLLARY 2. If N = the Riemann sphere with its standard metric, then if $f(M)$ is contractible, $f(M)$ must cover at least half the surface area of N.

COROLLARY 3. If the metric on N has non-positive curvature, then $f(M)$ cannot be contractible.

COROLLARY 4. If the metric on N has strictly negative curvature on a dense subset of N, then if $f(M)$ lies in a tubular neighbourhood V of a closed geodesic γ of N with V diffeomorphic via the exponential map to $\gamma\times(-r,r)$ for some r, $0<r<\infty$, $f(M)$ must be contained in γ.

NOTES. (1) Some of these results have similarites with results proved by convex function methods [2][9].

(2) Theorem 1 does not hold in general if M and N are not compact for take M= N = a Riemann surface with Riemann metric which satisfies the Cohn-Vossen inequality: $K(N)<2\pi\chi(N)$, and take $f:M\to N$ to be any onto mapping (e.g. take M=N ,f=identity map), then inequality (6) does not hold as it is in contradiction to the Cohn-Vossen inequality.

The proof of theorem 1 consists of three stages: Firstly, in §2, we show that if $f:M\to N$ is a C^{ω} map between connected compact C^{ω} surfaces whose Jacobian is not identically zero, then $f(M)$ can be described as a subcomplex in some triangulation of N; then, in §3, we show how to apply the Gauss-Bonnet formula to $f(M)$ using this description; lastly, assuming now that f is harmonic or satisfies (3) we use the Maximum Principle [6] to convert the

Gauss-Bonnet formula into the inequality (6).

DESCRIPTION OF $f(M)$ FOR A C^{ω} MAP

Let M be a connected C^{ω} surface. A subset A of M is called semi-analytic [5] if for every point $p \in A$, there exists a neighbourhood U of p and real-analytic functions $r_1, \dots, r_s$ on U such that $A \cap U$ is a finite union of finite intersections of sets of the form:

$$\{q \in U:\ r_j(q) > 0\},\ \{q \in U:\ r_j(q) = 0\}$$

We say that a point p of a semi-analytic set $A \subset M$ is regular of dimension k [5] if for some neighbourhood U of p, $A \cap U$ is an analytic submanifold of M of dimension k. The dimension of a semi-analytic set is the maximum dimension of its regular points.

Now suppose that M is second countable. A locally finite (resp. finite) analytic triangulation of M [4] is a locally finite (resp. finite) simplicial complex K together with a homeomorphism $T: |K| \xrightarrow{\text{onto}} M$ such that:

(7) for any $\sigma \in K$, $T|_{\sigma}: \sigma \to T(\sigma)$ is an analytic isomorphism onto an analytic submanifold of M.

If $\nu = T(\sigma)$ where σ is a simplex of K of dimension r, we shall call ν a simplex of M of dimension r.

Now let $\{A_{\alpha}\}$ be a locally finite collection of semi-analytic subsets. Then an analytic triangulation $T: |K| \to M$ is said to be compatible with the $\{A_{\alpha}\}$ if

(8) for any $\sigma \in K$ and for any A_{α}, $T(\sigma) \subset A_{\alpha}$ or $T(\sigma) \subset M - A_{\alpha}$.

Lojasiewicz [4] proves that for any such collection $\{A_{\alpha}\}$ there exists a locally finite analytic triangulation of M compatible with the A_{α}. If M is compact, such a triangulation must be finite. Using this:

LEMMA 1. Let M,N be connected real-analytic surfaces, N second countable, and let $f: M \to N$ be a real-analytic map. Then there exists a locally finite analytic triangulation of M such that

(9) for each simplex ν of M, $f|_{\nu}$ has constant rank.

PROOF. Let $\Sigma_i = \{p \in M : \dim \ker df(p) = i\}$, $(i=0,1,2)$, let $\Sigma = \Sigma_1 \cup \Sigma_2$, note that Σ is the set of singular points of f; let $\Sigma_{11} = \{p \in \Sigma_1 : p$ is

regular of dimension 1 and $\nabla_w f(p) = 0$ for w tangent to Σ_1 at p .
Then it is easy to show [9] that $M-\Sigma$, $\Sigma_1-\Sigma_{11}$, Σ_{11}, Σ_2 are semi-analytic sets, further they are mutually disjoint and have union M. By [4] there exists a locally finite analytic triangulation of M compatible with this collection of subsets; by a case-by-case study it is easy to show that this triangulation has the required property.

LEMMA 2. Let M,N be connected real-analytic surfaces, M second countable, and let $f:M\to N$ be a proper real-analytic map whose Jacobian is not identically zero on M. Then if M is given a locally finite analytic triangulation having property (9) above, then for each simplex ν of M, $f(\nu)$ is a semi-analytic subset of N.

PROOF. Case-by-case study (see [9]).

LEMMA 3. With hypotheses as in lemma 2, f(M) is a semi-analytic set of dimension 2 and its topological boundary, $\partial f(M)$ is empty or semi-analytic of dimension ≤ 1.

PROOF. The set f(M) = the union of the semi-analytic sets $f(\nu)$, ν varying over all simplexes of M. By properness of f, this union is locally finite and therefore ([4],[5]) f(M) is semi-analytic and clearly has dimension 2. By [5] it follows that $\partial f(M)$ is empty or semi-analytic of dimension ≤ 1.

PROPOSITION 1 (DESCRIPTION OF f(M)). Let M,N be connected compact C^ω surfaces and let $f:M\to N$ be a real-analytic mapping whose Jacobian is not identically zero on M. Then there exists a finite analytic triangulation of N compatible with $\{f(M), \partial f(M)\}$. In any such triangulation f(M) is a pure subcomplex of dimension 2, $\partial f(M)$ is empty or is a pure subcomplex of dimension 1. Further any 0-simplex of $\partial f(M)$ is the face of a non-zero even number of 1-simplexes of $\partial f(M)$. Any 1-simplex of $\partial f(M)$ is the common face of a 2-simplex of int(f(M)) and a 2-simplex of N-f(M). In particular, at any 0-simplex v of $\partial f(M)$, the 1-simplexes of $\partial f(M)$ with face v divide a neighbourhood of v like a pie into an even number of sectors alternately in int(f(M)) , N-f(M), (see FIG.1).

PROOF. The triangulation exists by Łojasiewicz, the rest of the proposition follows by simple topological considerations [9].

3-APPLYING THE GAUSS-BONNET FORMULA TO f(M)

Let M,N be compact real-analytic surfaces, N orientable, and let N be equipped with a real-analytic Riemann metric. Let $f:M\to N$ be a real-analytic map whose Jacobian is not identically zero on M. Triangulate N as indicated in proposition 1, then f(M) is a subcomplex of dimension 2 as described in that proposition. We wish to apply the Gauss-Bonnet formula to the region f(M), however f(M) is not necessarily bounded by a set of disjoint closed Jordan curves - for example $\partial f(M)$ might be in the form of a figure 8 - thus the Gauss-Bonnet formula cannot be applied immediately. However, we proceed as follows:

Firstly, orient N and orient each simplex of f(M) and $\partial f(M)$ accordingly. Next, partition the 1-simplexes of $\partial f(M)$ into subsets forming piecewise-analytic closed Jordan curves as follows: let α be a 1-simplex of $\partial f(M)$. The orientation of α determines one of its 0-faces as its initial point and the other as its final point and also determines a positive tangent direction at each point of α . Let v denote the final point of α. We define a 1-simplex $S\alpha$ of $\partial f(M)$ with initial point v , called the <u>successor</u> of α as follows; for each 1-simplex β of $\partial f(M)$ which has initial point v, let $\theta(\alpha,\beta)$ denote the signed angle through which we must turn the positive tangent to α at v to bring it into coincidence with the positive tangent of β, $-\pi\leq\theta(\alpha,\beta)\leq\pi$. The successor $S\alpha$ of α is defined to be the 1-simplex β for which this angle is algebraically least. Note that the sector 'between' α and $S\alpha$ lies in N-f(M) (see FIG.2). Now for any two 1-simplexes α,ω of $\partial f(M)$, write $\alpha\sim\omega$ if there is a sequence $\alpha,\beta,\dots,\psi,\omega$ with $\beta=S\alpha,\gamma=S\beta,\dots,\omega=S\psi$ or $\psi=S\omega,\dots,\beta=S\gamma,\alpha=S\beta$. It is easy to see that $\sim$ is an equivalence relation which thus partitions the 1-simplexes of $\partial f(M)$ into subsets which form piecewise-analytic closed Jordan curves. Now although these are not necessarily disjoint, we can apply the Gauss-Bonnet formula as follows:

PROPOSITION 2. (GAUSS-BONNET FORMULA FOR f(M)) Let M,N be compact connected real-analytic surfaces, N oriented, and let N be given a real-analytic Riemannian metric. Let $f:M \rightarrow N$ be a real-analytic map whose Jacobian is not identically zero on M. Then:

$$K(f(M)) = 2\pi\chi(f(M)) - \sum_{\alpha}\int_{\alpha} k\,ds - \sum_{\alpha}\theta(\alpha, S\alpha) \tag{10}$$

where the summations extend over all 1-simplexes α of $\partial f(M)$.

Here, k is the signed geodesic curvature of α at a point of α, and $\theta(\alpha, S\alpha)$ is the signed "corner" angle as defined earlier.

PROOF. Partition the 1-simplexes of $\partial f(M)$ into subsets forming piecewise analytic closed Jordan curves $c_1, \ldots, c_r$ as described above. We now apply a construction of Kreysig [3] to deform these into C^1-smooth closed Jordan curves $c_1^{\varepsilon}, \ldots, c_r^{\varepsilon}$ - specifically, at each 0-simplex v of $\partial f(M)$, for each pair of 1-simplexes $(\alpha, S\alpha)$ of $\partial f(M)$, replace the corner at v by a geodesic arc G_{ε}=AB of small radius whose tangents at the endpoints A and B coincide with those of α and $S\alpha$ respectively (see FIG. 3). The curves $c_1^{\varepsilon}, \ldots, c_r^{\varepsilon}$ so defined are disjoint and bound a region $f(M)^{\varepsilon}$ slightly larger than f(M). For this region:

$$K(f(M)^{\varepsilon}) = 2\pi\chi(f(M)^{\varepsilon}) - \sum_{i}\int_{c_i^{\varepsilon}} k\,ds$$

Now f(M) is a strong deformation retract of $f(M)^{\varepsilon}$ and thus has equal Euler characteristic. Further it is easy to see that :

$$\lim_{\varepsilon \to 0}\int_{c_i^{\varepsilon}} k\,ds = \int_{c_i} k\,ds + \sum_{\alpha}(\alpha, S\alpha)$$ where we sum over all 1-simplexes α

of c_i. The desired result follows by letting $\varepsilon \rightarrow 0$.

4-OBTAINING THE INEQUALITY FROM THE GAUSS-BONNET FORMULA BY USE OF THE MAXIMUM PRINCIPLE

THEOREM 2 (MAXIMUM PRINCIPLE) (Sampson). Let M,N be connected Riemann surfaces, N equipped with a smooth Riemann metric. Let $S \subset N$ be a submanifold of dimension 1 with non-zero geodesic curvature at b=f(p). where $p \in M$. If $f: M \to N$ is a C^{∞} map which is harmonic or satisfies (3), then no neighbourhood of p is mapped entirely to the concave side of S.

PROOF. SEE [6] and [8],[9].

We now apply the maximum principle to give the

PROOF OF THEOREM 1. Firstly we dispose of the case that the Jacobian of f is identically zero on M. In this case, [6], [9] f(M) is a closed geodesic of N. Thus $K(f(M)) = 0$, $\chi(f(M)) \leq 0$ and thus (6) holds.

Thus we may assume that the Jacobian of f is not identically zero on M and we may apply the Gauss-Bonnet formula as in Proposition 2 equation (10). We now show that the last two terms of (10) are non-positive.

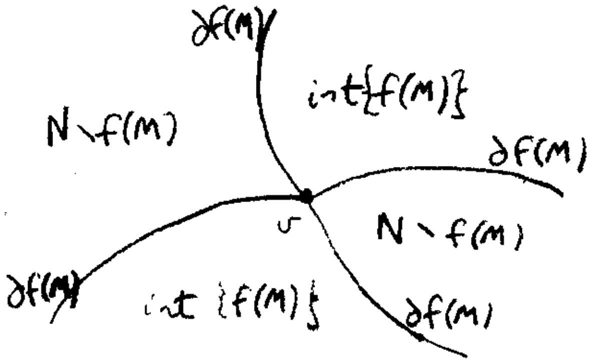
(1) Let b be a point on a 1-simplex α of $\partial f(M)$. By the Maximum Principle f(M) must lie to the convex side of α. Thus

(11) $k \leq 0$

(2) Let b=f(p) be a 0-simplex of $\partial f(M)$, and let α be a 1-simplex with final point b. As remarked in §3, the sector between α and its successor $S\alpha$ lies in N-f(M). Now if $\theta(\alpha, S\alpha) > 0$, then we could construct a 1-submanifold S such that f(M) lies to the concave side of S (see FIG.4). Then some neighbourhood of p would map to the concave side of S contradicting the maximum principle. Thus

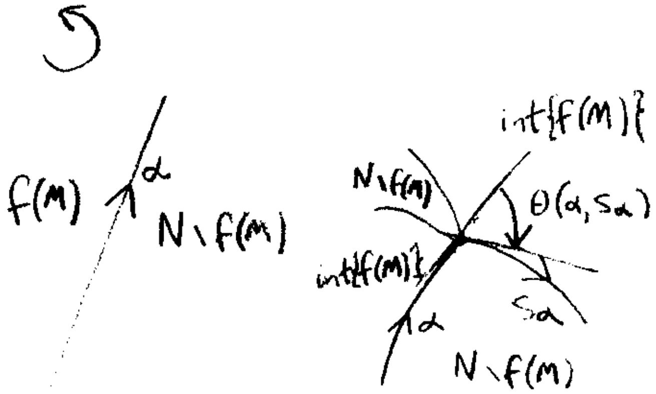
(12) $\theta(\alpha, S\alpha) \leq 0$

Equations (11) and (12) show that the last two terms of (10) are non-positive and theorem 1 follows immediately.



f(M) and its boundary

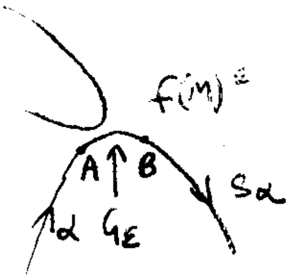
FIGURE 1.



Orientation Conventions

Corner Angle θ(α, Sα) (Here negative)

FIGURE 2.

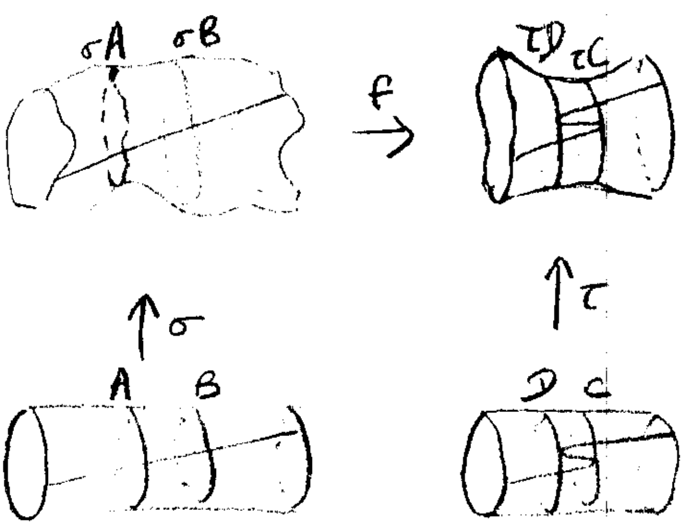


Construction of Kreyszig (c.f. FIG. 2)

FIGURE 3.

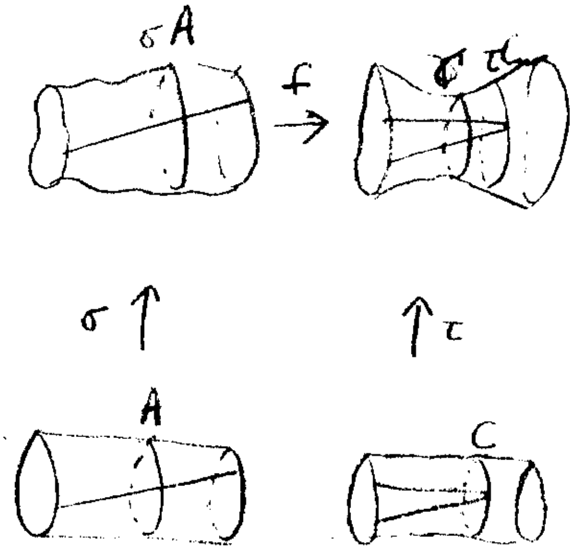
Maximum Principle violated when θ(α, Sα) is positive.

FIGURE 4.



Redundant pair of folds

FIGURE 5.



Folding past a closed geodesic

FIGURE 6.

5- REDUNDANT FOLDS

We now present another application of the Gauss-Bonnet formula and the Maximum Principle to harmonic maps between surfaces. Our first result shows that for a harmonic map mapping into a codomain with negative curvature, a pair of "opposite" folds which could be "pulled out" cannot occur.

Let M be a C^∞ surface.

DEFINITION. A <u>handle</u> of M is a C^∞ submanifold-with-boundary, H, of M which is C^1-diffeomorphic to $S^1\times I$, i.e. there exists a C^1-diffeomorphism, C^1 up to the boundary: $(S^1\times I, S^1\times\partial I)\rightarrow(H,\partial H)$. (Here S^1 = unit circle, I = closed unit interval.)

DEFINITION. Let $f:M\rightarrow N$ be a C^∞ map between C^∞ surfaces. The <u>singular set of f</u> $=\{p\in M : J_f(p)=0\}$ where J_f denotes the Jacobian determinant of f. A <u>fold line</u> of f may be defined as a C^∞ 1-submanifold F of M such that at each point $p\in F$, there exist C^∞-smooth coordinates (x,y) centred on p, (u,v) centred on f(p) such that, in a neighbourhood of p, f has the form: $u=x^2, v=y$.

Fold lines can and do occur for harmonic maps [8] [9].

THEOREM 3. Let $f:M\rightarrow N$ be a C^∞ map which restricts to a map $f|:(H,\partial H)\rightarrow(K,\partial K)$ between handles of M and N. Suppose that this map is such that there exist C^1-diffeomorphisms, C^1 up to the boundary:

$$\sigma:(S^1\times I, S^1\times\partial I)\rightarrow(H,\partial H)\ ,\ \tau:(S^1\times I, S^1\times\partial I)\rightarrow(K,\partial K)$$

such that the singular set of $f|_H$ consists of two closed fold lines $\sigma A, \sigma B$ where $A=S^1\times\{a\}$, $B=S^1\times\{b\}$, $0<a<b<1$, whose images are two closed C^1-smooth curves $\tau C, \tau D$ respectively, where $C=S^1\times\{c\}$, $D=S^1\times\{d\}$, $0\leq d<c\leq 1$.

Then, if N has a Riemann metric of strictly negative curvature on a dense subset of N, f cannot be harmonic or satisfy equation (3). (See FIG.5)

PROOF. We apply the Gauss-Bonnet formula to the region $T=\tau(S^1\times[d,c])$:

$$\int_T K*1 = -\int_{\tau C} k\,ds - \int_{\tau D} k\,ds$$

Suppose f is harmonic. Then by the Maximum Principle, $k\leq 0$ on τC and τD. But by hypothesis, $\int K*1 < 0$, thus we have a contradiction.

NOTE. the result is not true if we remove the curvature restriction on N. For example, R.T. Smith [7] constructs maps from the torus to the sphere exhibiting any number of pairs of "opposite" folds.

Our second result concerns a harmonic map $f:M\to N$ which restricts to a map $f|:(H,\partial H)\to(K,\partial K)$ of handles with a single fold. We show that if N has negative curvature, folding cannot occur "past" a closed geodesic "around" K.

THEOREM 4. Let $f:M\to N$ be a C^∞ map which restricts to a map $f|:(H,\partial H)\to(K,\partial K)$ where H and K are handles. Suppose that N has a Riemann metric of strictly negative curvature on a dense subset of N and suppose that f is such that there exist C^1-diffeomorphisms, C^1 up to the boundary:

$$\sigma:(S^1\times I,S^1\times\partial I)\to(H,\partial H)\ ,\ \tau:(S^1\times I,S^1\times\partial I)\to(K,\partial K)$$

such that: (1) $f(\partial H)\subset\tau(S^1\times\{0\})$ i.e. both ends of H map to the same end of K; (2) the singular set of $f|_H$ is a closed fold line σA where $A = S^1\times\{a\}$, $0<a<1$, and $f(\sigma A)=\tau C$ where $C = S^1\times\{c\}$, $0<c<1$.

Then if f is harmonic or satisfies (3), int(f(H)) cannot contain a simple closed geodesic homotopic to τC, (see FIG.6).

PROOF. Suppose that int(f(H)) does contain a simple closed geodesic γ homotopic to τ C. Then let T be the region of K bounded by γ and τC; applying the Gauss-Bonnet to T we get a contradiction in a similar way to theorem 3.

ILLUSTRATIVE EXAMPLE OF THEOREMS 3 AND 4

Let M = infinite circular cylinder, C, $= \{(x,y,z)\in \mathbb{R}^3 : y^2+z^2=1\}$, let N = hyperboloid of revolution, B , $= \{(u,v,w)\in\mathbb{R}^3 : v^2+w^2=1+u^2\}$; give C and B the Riemann structures induced from $\mathbb{R}^3$, note that N then has a metric of strictly negative curvature except on the central geodesic u=0. Parametrise the cylinder by cylindrical coordinates (x,ϕ) where $\tan\phi = z/y$ and parametrise the hyperboloid by (u,ψ) where $\tan\psi = w/v$.

Let $f:C\to B$ be a C^∞ "axially symmetric" map i.e. a map of the form: $u=U(x)$, $\psi=\phi$, where $U:\mathbb{R}\to\mathbb{R}$ is C^∞. Then:

(1) if $U(x)\to\infty$ as $x\to\infty$ and $U(x)\to-\infty$ as $x\to-\infty$ then U must be monotonic increasing with $U'(x)>0\ \forall x\in(-\infty,\infty)$;

(2) if $U(x) \to -\infty$ as $x \to \pm\infty$ then there exists $\xi \in (-\infty, \infty)$ such that $U'(x) >, =, < 0$ for $x <, =, > \xi$ respectively, further $U(x) \leqslant 0 \ \forall x \in (-\infty, \infty)$.

This is easily seen from theorems 3 and 4 - note the result generalises to any surface of revolution N whose metric has strictly negative curvature on a dense subset.

In conclusion, I should like to thank J.Eells for help and encouragement, K.D.Elworthy for constructive criticism of my thesis and J.H.Sampson for his valuable results contained in [6].

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REFERENCES

[1] Eells,J. & Sampson,J.H., Harmonic Mappings of Riemannian Manifolds, Amer.J.Math.86(1964)109-160.

[2] Gordon,W.B., Convex Functions and Harmonic Maps, Proc.Amer.Math.Soc. 33(1972)433-437.

[3] Kreysig,E., Introduction to Differential Geometry and Riemannian Geometry, University of Toronto Press, 1968.

[4] Łojasiewicz,S., Triangulation of Semi-analytic Sets, Annali Scuola Normale Superiore, Pisa (3) 18(1964)449-474.

[5] Łojasiewicz,S., Sur les Ensembles Semi-analytiques, Actes Congr. Internat.Math. 1970,vol.2, 237-241.

[6] Sampson,J., Some Properties and Applications of Harmonic Maps, (to be published).

[7] Smith,R.T., Harmonic Mappings of Spheres, PhD. thesis, University of Warwick, 1971.

[8] Wood,J.C., Singularities of Harmonic Maps and other Solutions of Systems of Quasi-linear Elliptic Partial Differential Equations, (to be published).

[9] Wood,J.C., Harmonic Mappings between Surfaces, PhD. thesis, University of Warwick, 1974.

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