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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
MIRAMARE - P.O.B. 586 - 34100 TRIESTE (ITALY) - TELEPHONES: 224281/2/3/4/5/6 - CABLE: CENTRATOM

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AUTUMN WEEK ON

GEOMETRY OF THE LAPLACE OPERATOR

27 September - 3 October 1976

GEOMETRY OF THE LAPLACIAN

An Introduction

J. Eells

University of Warwick

3. The Laplacian for maps.

(A) Let (M,g) and (M',g') be Riemannian manifolds, and $f: M \to M'$ a map. For any point $p \in M$ the differential of f is a linear map $df(p): M_p \to M'_{f(p)}$ between tangent spaces. Once again, we form the Dirichlet integral (or energy)

$$E(f) = \tfrac{1}{2} \int_M |df(p)|^2 dp.$$

The operator which plays the role of the Laplacian is now a non-linear elliptic operator called the tension field of f. To each point $p \in M$ it assigns a vector $\tau_f(p) \in M'_{f(p)}$. In terms of coordinates (x_i) and (y_α) on M and M' we can represent $g = (g_{ij})$ and $g' = (g'_{\alpha\beta})$. Then the energy and tension fields assume the forms

$$(3) \qquad E(f) = \tfrac{1}{2} \int_M g^{ij} \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j} g'_{\alpha\beta}\, dp,$$

$$(4) \qquad \tau_f^\gamma(p) = g^{ij} \left\{ \frac{\partial^2 f^\gamma}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial f^\gamma}{\partial x_k} + \Gamma'^\gamma_{\alpha\beta} \frac{\partial f^\alpha}{\partial x_i} \frac{\partial f^\beta}{\partial x_j} \right\}.$$

A map $f: M \to M'$ for which the tension field $\tau_f \equiv 0$ is called harmonic.

(B) Example 1. If (M, g) is the unit circle S^1, then (*) takes the form

$$\tau_f^\gamma = \frac{d^2 f^\gamma}{dt^2} + \Gamma'^{\gamma}_{\alpha\beta} \frac{df^\alpha}{dt} \frac{df^\beta}{dt} ;$$

otherwise said, the harmonic maps $f: S^1 \to (M', g')$ are just the closed geodesics of (M', g'), suitably parametrized.

Remark. The configuration space of a conservative dynamical system determines a manifold N; its potential energy a function $f: N \to \mathbb{R}$; and its kinetic energy a positive definite quadratic form $g(p)$ at each point $p \in M$. The classical paths are just the geodesics of the Riemannian manifold (N, g).

Example 2. If $\dim M = 2$, then the harmonic maps $f: (M, g) \to (M', g')$ correspond (roughly speaking) to soap bubbles of (M', g'); for τ_f is essentially the tension field for that case.

Geometry of the Laplacian: An Introduction.

J. Eells (ICTP. Sept. 1976)

1. The Laplacian in Euclidean spaces.

(A) Let V be a real finite dimensional Hilbert space with inner product denoted by $\langle\ ,\ \rangle$. If U is an open subset of V and $f : U \to \mathbb{R}$ is a function (we assume that all functions and maps have sufficient differentiability), then its <u>differential at a point</u> $p \in U$ is defined as that linear form $df(p) : V \to \mathbb{R}$ given by

$$df(p)(v) = \lim_{t \searrow 0} \frac{f(p+tv) - f(p)}{t}.$$

We can now form the Hilbert space (of infinite dimension) $\mathcal{L}_1^2(U, \mathbb{R}) = H^1(U)$ obtained by completion with respect to the inner product

$$\langle f, g \rangle_{\mathcal{L}_1^2(U,\mathbb{R})} = \int_U f(p) g(p)\, dp + \int_U \langle df(p), dg(p) \rangle\, dp$$

(B) The <u>Dirichlet integral</u> of $f \in \mathcal{L}_1^2(U, \mathbb{R})$ is

$$E(f) = \tfrac{1}{2} \int |df(p)|^2 dp .$$

Thus $E : \mathcal{L}_1^2(U, \mathbb{R}) \to \mathbb{R}$ is a quadratic function whose differential we can calculate at $f \in \mathcal{L}_1^2(U, \mathbb{R})$ in the direction $v \in \mathcal{L}_1^2(U, \mathbb{R})$:

$$dE(f)(v) = \int_U \langle df(p), dv(p) \rangle dp.$$

If we restrict attention to $v \in \mathcal{L}_1^2(U, \mathbb{R})$ which vanish off a compact subset of U, we can write

$$dE(f)(v) = \int_U \langle \Delta f(p), v(p) \rangle dp,$$

for a unique $\Delta f \in \mathcal{L}^2(U, \mathbb{R})$ called the <u>Laplacian</u> of f.

(C) <u>Example</u>. Take $V = \mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ with inner product $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$. Then the differential of $f : U \to \mathbb{R}$ can be written

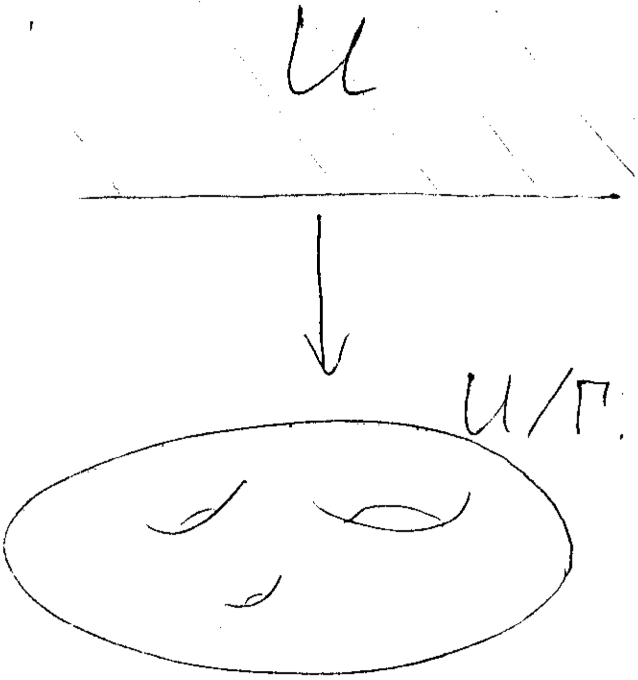
$$df(p) = \sum_{i=1}^n \frac{\partial f(p)}{\partial x_i} dx_i,$$

where $dx_i : \mathbb{R}^n \to \mathbb{R}$ assigns x_i to $(x_1, \ldots, x_i)$. Then $\Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

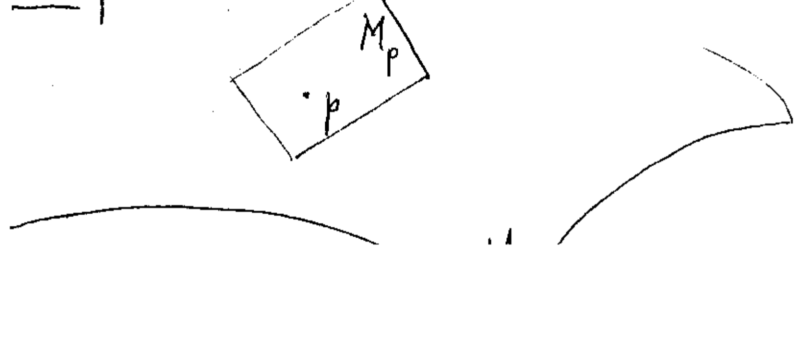
(D) It is convenient and easy to generalize these notions as follows: Start with an open U of an

<u>Example 3</u>. The n-<u>sphere</u> $S^n=\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1} : \sum x_i^2=1\}$. The product $S^{n_1}\times\dots\times S^{n_k}$ of such spheres.

<u>Example 4</u>. If $\mathcal{U}=\{(x_1,x_2)\in\mathbb{R}^2 : x_2>0\}=\{z\in\mathbb{C} : \operatorname{Im} z>0\}$ and Γ is a discrete subgroup of $\left\{\frac{\alpha z+\beta}{\gamma z+\delta} : \alpha,\beta,\gamma,\delta\in\mathbb{R}\right.$ and $\left.\det\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=1\right\}$, then Γ acts on $\mathcal{U}$, and the space of orbits $\mathcal{U}/\Gamma$ is a <u>Riemann surface</u> (a manifold of dimension 2).



Each of the spaces M in these examples can be given as a subset of some Euclidean space $\mathbb{R}^k$, and with each point $p\in M$ we have an n-dimensional linear subspace M_p of $\mathbb{R}^k$ which is <u>tangent to</u> M <u>at</u> p.



Furthermore, each tangent space M_p has an associated positive definite quadratic form g_p, inherited from $\mathbb{R}^k$. Will call these (M, g) Riemannian manifolds. The energy (= dirichlet integral) and Laplacian of $f : M \to \mathbb{R}$ are defined by (1) and (2).

(C) There is a substantial difference in the qualitative aspects of those Riemannian manifolds which are compact and those which are not. For instance, from spectral theory of elliptic operators we obtain the

Theorem. Let (M, g) be a compact Riemannian manifold. Then there is a canonical decomposition of the Hilbert space $\mathcal{L}^2(M, \mathbb{R})$ into finite dimensional mutually orthogonal Hilbert subspaces:

$$\mathcal{L}^2(M, \mathbb{R}) = \sum_{k \geq 0} \oplus\, L_k .$$

With each k there is a number λ_k:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots \to \infty,$$

and $L_k = \{ f \in \mathcal{L}^2(M, \mathbb{R}) : \Delta f = \lambda_k f \}$.

The set (λ_k) with their multiplicities constitutes the spectrum of Δ (or, equivalently, of (M, g)).

Example. If $S^n \subset \mathbb{R}^{n+1}$ then the functions in L_k are the spherical harmonics.

Note also for any compact M $\langle \Delta f, f \rangle = \|df\|^2 \geq 0$

n-dimensional vector space V, but now assign to each point $p \in U$ a positive definite quadratic form $g(p)$. Then we can form the integral

$$(1) \qquad E(f) = \tfrac{1}{2} \int_U g(p)\,(df(p), df(p))\, dp,$$

and its associated Laplacian Δf.

Again, if $V = \mathbb{R}^n$ and we write $g(p) = (g_{ij}(p))$, with inverse matrix $(g^{ij}(p))$, then

$$(1) \qquad E(f) = \tfrac{1}{2} \int_U g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j}\, dp, \qquad \text{(summation convention)}$$

where $dp = \sqrt{\det g(p)}\, dx_1 \ldots dx_n$;
and Δf takes the form

$$(2) \qquad \Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k} \right),$$

for suitable smooth functions Γ_{ij}^k on U. (In fact (U, g) is a Riemannian manifold, and (Γ_{ij}^k) define the linear connection associated with g.)

Thus Δ is a <u>self-adjoint linear elliptic differential operator</u> with smooth coefficients. We continue to refer to it as the <u>Laplacian on the manifold</u> (U, g).

A function $f : U \to \mathbb{R}$ for which $\Delta f = 0$ is

2. The Laplacian on manifolds

(A) If the boundary ∂U of U is compact and sufficiently regular, then much of the geometry of U and ∂U can be determined from the operator Δ and its spectrum. See for instance:

(1) R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol I and II. Interscience.

(2) M. Kac, Can one hear the shape of a drum? Amer. Math. Monthly 73 (1966), 1–23.

(3) H. Weyl, Ramifications, old and new, of the eigenvalue problem. Bull. Amer. Math. Soc. 56 (1950), 115–139.

(B) For us to have a good perspective we should broaden our considerations to include general Riemannian manifolds. A good reference for our purposes is S. Robertson, Riemannian geometry, Global Analysis and its Applications Vol I, I.C.T.P. (1974). However, for this lecture (and for much of our Autumn Week) it is sufficient and representative to consider the following examples:

Example 1. The open set (U, g) of the preceding section.

Example 2. A circle $S^1 = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \}$

The product $S^1 \times \cdots \times S^1$ of n circles; this is called an n-dimensional torus. The cylinders