INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

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SMR/26 - 6

AUTUMN WEEK ON

GEOMETRY OF THE LAPLACE OPERATOR

27 September - 3 October 1976

SOME RECENT RESULTS IN SPECTRAL GEOMETRY

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Lecture notes for Lecture One

"Some recent results in spectral geometry"

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Section One

Let M be a compact Riemannian manifold of dimension m without boundary and let $x = (x_1, \ldots, x_m)$ be a system of local coordinates on M. The metric tensor is given by:

$$ds^2 = g_{ij}\, dx_i dx_j \quad \text{(we sum over repeated indices)}$$

where the matrix g_{ij} is positive definite and symmetric. Let

$$g = (\det(g_{ij}))^{1/2}.$$

The Riemannian element of volume is given by

$$|dvol| = g\, dx_1 \cdots dx_m.$$

We use the notation $|dvol|$ to emphasize that it is a measure and not an m-form; $dx_1 \cdots dx_m$ is the ordinary Lebesgue measure on $\mathbb{R}^m$. $|dvol|$ is intrinsic; it only depends on the metric and is independent of the coordinate system chosen.

The Laplacian, Δ, is defined by:

$\Delta(f) = -g^{-1}$ [redacted] $\partial/\partial x_i \{ g^{ij} g\, \partial/\partial x_j (f)\}$

$$= -g^{ij}\{\partial/\partial x_i \, \partial/\partial x_j (f) - \Gamma_{ij}^{k}\, \partial f/\partial x_k\}$$

The Laplacian is intrinsic; it is a positive self-adjoint elliptic operator. Note that we have taken the opposite sign from the Laplacian used in physics. Since Δ is self adjoint, we can take a spectral resolution of Δ into a complete orthonormal set of eigenfunctions ϕ_i and corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \ldots$. If $\phi(x)$ is any function on M, we can expand

$$\phi = \sum a_i \phi_i.$$

The ϕ_i are smooth functions of x; ϕ is a C^∞ function if the coefficients a_i decay rapidly enough.

We illustrate this phenonum with the following example: Let $M = S^1$ be the unit circle in the plane and let θ be arc length on S^1. We can define local coordinates on M by restricting the domain of θ to any interval of length less than 2π. The Laplacian is

given by: $\Delta(f) = -\partial^2 f/\partial\theta^2$. The eigenfunctions of Δ are the trignometric polynomials $\{e^{in\theta}\}$ with eigenvalue n^2 as $n \in \mathbb{Z}$. The expansion described above is just the familiar expansion of a function in terms of Fourier series and the a_i are just the Fourier coefficients.
If $M = S^1 \times S^1$ is the two dimensional torus, let θ_1 & θ_2 be arc length on each factor. Then $\Delta(f) = -\partial^2 f/\partial\theta_1^2 - \partial^2 f/\partial\theta_2^2$ and the eigen functions of Δ are the functions $\{e^{in_1\theta_1} e^{in_2\theta_2}\}$ with corresponding eigenvalues $n_1^2 + n_2^2$. If spec (M) denotes the set of eigenvalues, counted with multiplicity, then

$$\mathrm{spec}\,(S^1) = \{n^2\}_{n\in\mathbb{Z}} = \{0,1,1,4,4,9,9,16,16,\ldots\}$$

$$\mathrm{spec}\,(S^1 \times S^1) = \{n_1^2+n_2^2\} = \{0,1,1,1,1,2,2,2,2,4,4,4,4,5,5,5,5,\ldots\}$$

The eigenfunctions and corresponding eigenvalues of the two dimensional sphere, S^2, can be determined using spherical harmonics.

The question that we shall be discussing in the next 3 lectures is the following: to what extent is the geometry of M reflected by the spectrum of M, or putting the question slightly differently, can we obtain information about the spectrum of M from its geometry. In general, the spectrum of a manifold is quite dificult to compute and has been worked out for relatively few examples. One might conjecture that the spectrum completely determines the geometry of the manifold, but alas this is false. There is an example, due to Milnor, of two manifolds of dimension 17 with the same spectrum which are <u>not</u> isometric. To a certain extent, however, this example is misleading since many geometrical properties are reflected by the spectrum.

The major tool in studying the spectrum of a Riemannian manifold is the heat equation. Let $\phi(x)$ be a distribution of heat on the manifold M. The equation determining the evolution of the system is given by:

$$\frac{\partial u}{\partial t}(x,t) + \Delta u(x,t) = 0 \quad \text{with initial condition } u(x,0)=\phi(x).$$

(Note that with our sign convention for the Laplacian that the equation is $\partial/\partial t + \Delta$ and not $\partial/\partial t - \Delta$). We can solve this equation as follows: express

$$\phi = \Sigma\, \alpha_i \phi_i \quad \text{where} \quad \alpha_i = \int_M \phi \cdot \bar{\phi}_i\, |dvol|$$

are the Fourier coefficients. Then

$$u(x,t) = \Sigma\, e^{-t\lambda_i} \phi_i(x) \cdot \alpha_i$$

is the solution describing the evolution of this dynamical system. Let

$$k(t,x,y) = \Sigma\, e^{-t\lambda_i} \phi_i(x)\, \bar{\phi}_i(y).$$

Then:

$$\begin{aligned} u(x,t) &= \Sigma\, e^{-t\lambda_i} \phi_i(x) \cdot \alpha_i \\ &= \Sigma\, e^{-t\lambda_i} \phi_i(x) \int \phi(y)\, \bar{\phi}_i(y)\, |dvol|(y) \\ &= \int k(t,x,y)\, \phi(y)\, |dvol|(y). \end{aligned}$$

$k(t,x,y)$ is the fundamental solution of the heat equation. It is a smooth function of (t,x,y) for $t>0$. As $t\to\infty$, it tends to the constant 1. As $t \to 0$, it tends to the delta function. For $d(x,y) \geq c$, it is smooth and exponentially small as $t\to 0$.

In our example, if $M=S^1$, then

$$k(t,x,y) = \sum_n e^{-tn^2}\, e^{in(x-y)}$$

We can recover information about the spectrum of M from the kernel of the heat equation as follows: let

$$f(t) = \Sigma\, e^{-t\lambda_i};$$

if the spectrum is known, $f(t)$ is determined. Conversly if $f(t)$ is known, we can recover the spectrum. Since $\int \phi_i \bar{\phi}_i = 1$,

$$\int k(t,x,x)\, |dvol| = \Sigma\, e^{-t\lambda_i} \int \phi_i(x)\, \bar{\phi}_i(x) = f(t).$$

A great deal is known about the asymptotic behavior of the kernel function as $t \to 0$. One of the earliest results is the following:

$$k(t,x,x) = (4\pi t)^{-m/2} + O(t^{-(m-2)/2}) \quad \text{as } t \to 0^+$$

this implies that

$$f(t) = (4\pi t)^{-m/2}\,\text{volume}(M) + O(t^{-m/2+1}) \text{ as } t \to 0^+.$$

we can translate this estimate to obtain information about the growth of the eigenvalues: let spec(M) = $\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots\}$. Then $\lambda_n \sim n^{2/m} \cdot C_m^{\mathrm{vol}(M)}$ as $n \to \infty$. C_m is a universal constant which only depends on the dimension of the manifold. Thus for example, if $m=1$, $\lambda_n \sim n^2$; if $m=2$, $\lambda_n \sim n$ and so on.

It is natural to ask if there are further terms in the asymptotic expansion of $k(t,x,x)$. The answer is yes

<u>THEOREM</u>:

$$k(t,x,x) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} a_i(x)\, t^i$$

i.e. for any N, $(4\pi t)^{m/2}\{k(t,x,x)\} - \sum_{n=0}^{N} a_i(x)\, t^i = O(t^{N+1})$ as $t \to 0$.

The first coefficient in this asymptotic expansion is the constant 1; $a_0(x) = 1$. The other terms are <u>local</u> invariants of the geometry of M and can be computed in terms of curvature and its covariant derivatives.

If $a_i = \int a_i(x)\,|d\mathrm{vol}|$, then $f(t) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} a_n t^n$. Consequently, the <u>numbers</u> a_i are invariants of the spectrum of M. It should be noted that we <u>cannot</u> recover $f(t)$ and thereby spec(M) from the a_i's.

Since $a_0(x) = 1$, $a_0(M) = \mathrm{vol}(M)$. This implies that both the dimension of M and its volume are spectral invariants. In other words, if (M,G) + (M,G') are two compact Riemannian manifolds, then $\mathrm{spec}(M,G) = \mathrm{spec}(M,G')$ implies $\mathrm{Dim}\, M = \mathrm{Dim}\, M'$ and $\mathrm{vol}(M) = \mathrm{vol}(M')$. We will be interested in what other geometrical properties of M are reflected by the spectrum. |

The Gaussian scalar curvature is the simplest local geometric invariant. We shall restrict to the case $m=2$ for notational + conceptual simplicity. Let x be a local system of coordinates. Let $g_{ij/k} = \partial/\partial x_k (g_{ij})$, $g_{ij/kl} = \partial/\partial x_k \, \partial/\partial x_l (g_{ij})$ ect. We define

$$K = (2g)^{-1} \left\{ \left(\frac{g_{12}\, g_{11/2} - g_{11}\, g_{22/1}}{g\, g_{11}} \right)_{/1} + \left(\frac{2 g_{11}\, g_{12/1} - g_{11} g_{11/2} - g_{12} g_{11/1}}{g \cdot g_{11}} \right)_{/2} \right\}$$

This can be shown to be intrinsic and to be independent of the coordinate system chosen. For example, if $M = S^2$ with the standard metric, $K=1$. If $M = S^1 \times S^1$ with the "flat" product metric, $K=0$. If we embed the torus in R^3, the curvature is positive on the outer half and negative on the inner half.

Algebraically, when we expand K, it will contain ~~linear~~ quadratic terms in the first derivatives of the metric and linear terms in the second derivatives of the metric. Such an invariant will be said to be of second order in the derivatives of the metric. In a simialiar way, we define homogenity of arbitrary order. For example, on a 2-dimensional manifold, any invariant of fourth order can be written as a linear combination of the invariants K^2 and ΔK.

One can prove that $a_n(X)$ is homogeneous of order $2n$ in the derivatives of the metric. By using computer time generously supplied by IBT, we have computed:

Let $\mathrm{Dim}(M)=2$ then:

<u>Theorem</u>

$$a_0 = 1$$
$$a_1 = \frac{1}{2\cdot 3}\{2K\}$$
$$a_2 = \frac{1}{4\cdot 3\cdot 5}\{4\Delta K + 4K^2\}$$
$$a_3 = \frac{1}{8\cdot 3\cdot 5\cdot 7}\{6\Delta^2 K + 8\Delta(K^2) + 12K\cdot\Delta(K) + {}^{32}/_3\, K^3\}$$
$$a_4 = \frac{1}{16\cdot 3\cdot 5\cdot 7\cdot 9}\{8\Delta^3 K + 12\Delta^2(K^2) + 24\Delta(K\cdot\Delta K) + 8(\Delta K)\cdot(\Delta K) + 16K\cdot\Delta^2(K) + 40\,\Delta(K^3) + 72\,K^2\Delta(K) + 48K^4\}$$
$$a_n = \frac{1}{2^n\cdot 1\cdot 3\cdots(\frac{4n}{2}+1)}\{2n\,\Delta^{n-1}K + \text{lower order terms}\}$$

We can draw the following consequences from these calculations:

$$\mathrm{vol}(M),\ \int_M K\,|d\mathrm{vol}|,\ \int_M K^2\,|d\mathrm{vol}|$$

are all spectral invariants. We say that M has constant scalar curvature if there exists a constant c such that $K \equiv c$. This is equivalent to the condition:

$$0 = \int_M (K-c)^2\,|d\mathrm{vol}| = \int_M K^2\,|d\mathrm{vol}| - 2c\int_M K\,|d\mathrm{vol}| + c^2\,\mathrm{vol}(M).$$

Consequently, if $\mathrm{spec}(M,g) = \mathrm{spec}(M',g')$ and M has constant scalar curvature, so does M'. The only 2-dimensional manifold with scalar curvature 1 and volume 4π is the 2-sphere S^2 with the standard metric. This yields the isospectral result:

<u>THEOREM</u> Let $M = S^2$ with the standard metric. If $\mathrm{spec}(M,g) = \mathrm{spec}(M',g')$ then (M',g') is isometric to the standard sphere with the standard metric.

There are many other similiar results concerning geometric properties reflected by the spectrum in higher dimensions. Unfortunately time does not permit a detailed discussion so we refer the interested reader to the references in the bibliography. In the next section we shall discuss the Gauss-Bonnet-Von Dyck theorem for the Euler characteristic using heat equation methods.