Radiation (-Damping) in a Universe with Topologically Closed Space Sections+

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Abstract:

A soluble model for an interacting oscillator-field system discussed previously is reconsidered with Minkowski space replaced by a Lorentzian static manifold with topologically closed space-sections. It is shown that the general solution to the dynamical equations is - in the technical sense - almost periodic in time. In the case of the Einstein universe a more detailed discussion is presented, containing especially a study of the "thermodynamic limit" in which the radius of the universe tends to infinity.

I. Introduction

This work concerns itself with the dynamics of a system in which a harmonic one-dimensional oscillator interacts linearly with a scalar massless field in 3+1 - dimensional spacetime. It grew out of a study of an analogous system in Minkowski space [(1)]. There it was shown that, provided the Cauchy data have finite energy, the system, though reversible, has a certain "dissipative" property. The energy of the oscillator vanishes in the limit as time $|t| \to \infty$. The finite-energy solutions may hence be described by saying that they correspond to an incident wave which excites an initially (with $t \to -\infty$) quiescent oscillator which then reradiates the energy thus acquired to infinity [(2)].

If in this set-up one replaces Minkowski space by a spacetime with topologically closed (in particular compact) space-slices, a new situation arises. Firstly, the total energy is now always finite. Secondly, related to firstly, energy cannot "escape to infinity". It is hence to be expected, that the dissipative behaviour in the sense mentioned above will strictly be absent here.

Preliminary attempts to integrate the equations in special cases by a rather "brute force" method, as outlined in the appendix, suggested (but failed to prove) that the system might behave quite analogously to a quantum system in a finite volume the wave function of which, due to the discreteness of the energy spectrum, is an almost periodic function of time. (This is, of course, the quantum analogue of Poincaré recurrence) (see e.g. [3]).

Since that is a typical Hilbert space result, it seemed natural to tackle the present problem in a Hilbert space formulation which is introduced in Sec.II. Sec.III contains the proof of almost periodicity. In Sec. IV the solution to the Cauchy problem is written down more explicitly. To obtain more concrete information, we specialise the underlying manifold to be the Einstein universe in Sec V. We show that, letting the radius $\mathcal{R}$ of the universe tend to infinity, one recovers the time evolution in Minkowski space (provided an obvious identification between the fields in the two models is made). Considering the asymptotic behaviour of the oscillator in the two cases, this especially implies non-interchangeability of the limits $t \to \infty$, $\mathcal{R} \to \infty$, a fact familiar from non-equilibrium statistical mechanics. From local causality arguments it is clear that the closed topology, though decisive for the asymptotic behaviour, cannot have any impact on the oscillator for times $t < \mathcal{R}\pi$. It is shown that its motion is (again with suitable modifications) in fact, equal to the one in flat space. This is not unexpected on grounds of the local conformal flatness of the Einstein universe, from which the absence of tails in the radiation follow. In an appendix a heuristic argument is given which should serve to show that the almost periodic motion of the oscillator is turned into damping if the universe contains matter which absorbs radiation. In view of the results in [4]

it seems likely that incorporation of the cosmic expansion in the model has a similar effect.

II. The Model

We consider as spacetime M a smooth manifold of the form $M = R^1 \times N$, where N is a closed, orientable 3-manifold. M is endowed with a line-element g of Lorentzian signature and a global timelike Killing vector field $\partial/\partial t$. In terms of the set of equations to be described shortly one can without loss of generality assume g to be conformally rescaled such that

$$(2.1) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - \gamma_{ij}(\vec{x}) dx^i dx^j$$
$$(\mu, \nu = 0, \ldots 3;\ i, j = 1, \ldots 3)$$

where $\gamma_{ij}(\vec{x})$ is a Riemannian metric on N. The model consists of a one-dimensional harmonic oscillator Q(t) coupled to a scalar massless field $\phi(\vec{x}, t)$. The equations are (c=m=1):

$$(2.2a) \quad \frac{d^2}{dt^2} Q(t) + \omega_0^2 Q(t) = \ddot{Q}(t) + \omega_0^2 Q(t) = \lambda \int dV \rho(\vec{x}) \phi(\vec{x}, t)$$

$$(2.2b) \quad \left(\Box + \frac{R}{6}\right)\phi(\vec{x}, t) = \left(\frac{\partial^2}{\partial t^2} - \Delta + \frac{{}^3R}{6}\right)\phi(\vec{x}, t) = \lambda \rho(\vec{x}) Q(t) \qquad \left(\int dV \rho = 1\right)$$

Here $dV = (\det \gamma_{ij})^{\frac{1}{2}} d^3x$ is the invariant volume element on N. $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$, $\Delta = \gamma^{ij} \nabla_i \nabla_j$, R is the scalar curvature on M, ω_0^2 the spring constant, λ a coupling constant, $\rho(\vec{x})$ a prescribed scalar $\in C^\infty(N)$. We further assume the 3-scalar curvature ${}^3R$ on N to be positive: $R = {}^3R > 0$ (5). The equations are invariant under conformal transformations which do not involve time.

By (Cauchy, initial) data to the system (2.2 a,b) we

mean a quadruple $|f\rangle = |Q, \dot{Q} = P, \phi(\vec{x}), \dot{\phi}(\vec{x}) = \pi(\vec{x})\rangle$, where $\phi, \pi \in C^\infty(N)$ and which, for later convenience, is allowed to attain complex values.

Use will be made of the following (Sobolev-) spaces: $H^0(N) = L^2(N)$, consisting of complex-valued square integrable functions $u(\vec{x})$ on N: $(u|u) = \int dV |u(\vec{x})|^2 < \infty$, $H^i(N)$ (i = 1, 2) of functions whose weak derivatives up i-th order are in $L^2(N)$. The following known properties of the positive operator $L = -\Delta + \frac{R}{6}$ will be used in a crucial way (see e.g. (5)):

A) L with the domain $D(L) = H^2(N)$ is self-adjoint in $H^0(N)$

B) L^{-1} is a compact operator which implies that the spectrum of L consists of isolated eigenvalues ν_i with $0 < \nu_1 < \nu_2 < \dots$ of finite multiplicity.

To introduce a norm into the space H of Cauchy data $|f\rangle = |Q, P, \phi, \pi\rangle$, the total conserved energy seems to be a good candidate:

$$(2.3)\quad E(f) = \frac{1}{2}\left[\omega_0^2 |Q|^2 + |P|^2\right] + \frac{1}{2}\left[(\phi|L\phi) + (\pi|\pi)\right] - \frac{\lambda}{2}\left[Q(\rho|\phi) + \bar{Q}(\phi|\rho)\right]$$

Let $\tilde{\phi} = \phi - \lambda Q L^{-1}\rho$. Assume that $\omega_0^2, \lambda, \rho(\vec{x})$ are arranged in such a way that

$$(2.4)\quad \bar{\omega}^2 = \omega_0^2 - \lambda^2 (\rho | L^{-1}\rho) > 0$$

(For given $\lambda, \rho(\vec{x})$) this can always be achieved by making the "bare" spring constant ω_0^2 sufficiently large). E can now be written

$$(2.5)\quad E(f) = \frac{1}{2}\left[\bar{\omega}^2 |Q|^2 + |P|^2\right] + \frac{1}{2}\left[(\tilde{\phi}|L\tilde{\phi}) + (\pi|\pi)\right]$$

and is hence $\geqslant 0$ and = if and only if $|f\rangle = 0$. Henceforth we write $E^{\frac{1}{2}}(f) = \| f \|$. $\| \;\|$ constitutes a norm in H which is obviously induced by

(2.6) $$\langle f_1 | f_2 \rangle = \frac{1}{2}[\omega_0^2 \bar{Q}_1 Q_2 + \bar{P}_1 P_2] + \frac{1}{2}[(\phi_1 | L \phi_2) + (\pi_1 | \pi_2)] - -\frac{\lambda}{2}[\bar{Q}_1 (\rho | \phi_2) + Q_2 (\phi_1 | \rho)]$$

The completion of H in the norm $\| \ \|$ is given by $R^2 \oplus H^1(N) \oplus H^0(N)$ which will be also denoted by H.

We now write (2.2 a,b) in "first order form"

(2.7) $$\frac{\partial f}{\partial t} = iAf$$

where (in obvious notation)

(2.8) $$A = \frac{1}{i}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -\omega_0^2 & 0 & \lambda(\rho| & 0 \\ \hline 0 & 0 & 0 & 1 \\ \lambda|\rho) & 0 & -L & 0 \end{array}\right)$$

Energy conservation implies that A is symmetric with respect to $\langle \ \rangle$ in $D(A) = R^2 \oplus H^2(N) \oplus H^1(N)$. (Contrary to quantum mechanics, which (2.7) reminds of, A has nothing to do with the energy of the system. It would rather be appropriate to call it the "frequency operator").

III. Dynamics

Theorem 1: A is self-adjoint.

Proof: Define for $\Im m\, z \neq 0$ the following quantities

(3.1) $$|\mu(z)) = (z^2 - L)^{-1} |\rho)$$

(3.2) $$D(z) = z^2 - \omega_0^2 - \lambda^2 (\rho | \mu(z)) = z^2 - \bar{\omega}^2 - \lambda^2 z^2 (\mu(0) | \mu(z))$$

(3.3) $$B(z) = D(z)(z^2 - L)^{-1} + \lambda^2 |\mu(z))(\overline{\mu(\bar{z})}|$$

A straightforward calculation gives for the resolvent $R(z) = (A - z)^{-1}$ of A:

$$(3.4)\qquad \frac{1}{i}R(z)=(D(z))^{-1}\left(\begin{array}{cc|cc} iz & 1 & -\lambda iz(\overline{\mu(z)}| & -\lambda(\overline{\mu(z)}| \\ -z^2+D(z) & iz & \lambda z^2(\overline{\mu(z)}| & -\lambda iz(\overline{\mu(z)}| \\ \hline -\lambda iz|\mu(z)) & \lambda|\mu(z)) & izB(z) & B(z) \\ \lambda z^2|\mu(z)) & -\lambda iz|\mu(z)) & -z^2B(z)+D(z) & izB(z) \end{array}\right).$$

Using self-adjointness of L with $D(L) = H^2(N)$ in $H^0(N)$, inspection of the structure of R(z) shows that it maps H into $D(A) = R^2 \oplus H^2(N) \oplus H^1(N)$ for $\mathrm{Im}\, z \neq 0$ which implies[7] that A is self-adjoint.

q.e.d.

Self-adjointness implies that A generates a strongly continuous one-parameter group of unitary transformations $U(t) = e^{iAt}$ [8]. The unique solution (in Hilbert space) to the Cauchy problem is therefore given by

$$(3.5)\qquad |f(t)\rangle = e^{iAt}|f(0)\rangle$$

This can also be written as an inverse Laplace transform [8]

$$(3.6)\qquad |f(t)\rangle = \lim_{c\to\infty} \frac{1}{2\pi}\int_{\epsilon-ic}^{\epsilon+ic} ds\, e^{st} R(-is)|f(0)\rangle \qquad (t>0,\ \epsilon>0)$$

which will be useful later on.

It is still to be seen that the time evolution maps C^∞-data into themselves. However, a result of that sort is known for the inhomogeneous wave equation with more general source $\rho(\vec{x},t)$ [9]. In our case $\rho(\vec{x},t)\in C^\infty(N)$ for all t. Also, as a consequence of the strong continuity of the evolution, $\rho(\vec{x},t)=\rho(\vec{x})Q(t)$ is itself continuous in t. Under these conditions the assumptions of [9] are valid.

<u>Theorem 2</u>: The spectrum of A consists of isolated eigenvalues of finite multiplicity and is symmetric with respect to the origin $\ldots \omega_{-2} < \omega_{-1} < 0 < \omega_1 < \omega_2 < \ldots$ with $\omega_{-i} = -\omega_i$.

Proof: Consider $R(0) = A^{-1}$. For what follows it is convenient to set $\lambda = 0$ in the inner product (2.6) and observe that this yields an equivalent norm on H (though, of course, A is now no longer symmetric). We now write A^{-1} in the form

$$(3.7)\quad A^{-1} = \left(\begin{array}{c|c} \times & \times \\ \hline \times & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \times \end{array}\right)$$

It is easily seen that both terms in (3.7) are bounded operators in the "$\lambda = 0$ - norm" and hence in the energy norm. The first term in (3.5) is of finite rank and therefore compact. The second term reads

$$(3.8)\quad i(\omega_0^2)^{-1}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 0 & -L^{-1} \\ 1 & 0 \end{matrix} \end{array}\right) \quad + \quad \text{finite-rank}$$

One has to worry only about the first term in (3.8). It is bounded since

$$(3.9)\quad \begin{aligned} (L^{-1}\pi | L L^{-1}\pi) + (\phi|\phi) &\le \|L^{-1}\|_{L^2}(\pi|\pi) + C(\phi|L\phi) \le \\ &\le D[(\phi|L\phi) + (\pi|\pi)] \end{aligned}$$

It is also self-adjoint. Furthermore its square is

$$(\omega_0^2)^{-2}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} L^{-1} & 0 \\ 0 & L^{-1} \end{matrix} \end{array}\right)$$

which is compact by statement B) in Sec. II. By a well-known theorem this implies compactness of this operator itself. Hence A^{-1}, as a sum of compact operators, is compact, and the spectrum of A is purely discrete and of finite multiplicity. The symmetry of the spectrum with respect to the origin is a reflection

of the equations (2.2 a,b) being real and follows from inspection of the resolvent(3.4) or else from the eigenvalue equations to be investigated later.

q.e.d.

From this theorem it follows that the general solution to (2.3 a,b) is of the form

$$(3.10) \qquad |f(t)\rangle = \sum_{i=-\infty}^{\infty} e^{i\omega_i t} \, a_i \, |f_i\rangle \, , \qquad a_i \in C$$

$|f_i\rangle$ being the complete set of eigenfunctions. The sum (3.10) converges uniformly in t in the norm $\| \; \|$. This implies uniform convergence of the corresponding sum in the expression for Q(t), $\dot{Q}$ (t) contained in (3.10) and, in the C^∞-case, for the ones for $\Phi(\vec{x},t)$, $\dot{\Phi}(\vec{x},t)$ almost everywhere in N. Thus, by a general theorem [10], Q(t), $\dot{Q}$(t) and, almost everywhere in N, $\Phi(\vec{x},t), \dot{\Phi}(\vec{x},t)$ are almost periodic functions of time.

This result exhibits the fact that the system - strictly speaking - does not have any tendency to reach a "state of equilibrium" for arbitrary large times where all the energy has been dissipated into the field degrees of freedom. We say: "strictly speaking", because it is physically obvious and will be shown in Sec. V in the case of the Einstein universe that there are means of escaping the implications of that result. Either one performs a limit, where the "radius" $\mathcal{R}$ of the universe (defined suitably) tends to infinity, <u>before</u> the limit $t \rightarrow \infty$. Or, more generally, one proves that, confining the support of Φ, $\dot{\Phi}$ at t = 0 to a region of length $\ell \ll \mathcal{R}$ which

contains the [illegible]

the system does come to equilibrium in the above sense if only observed on a time scale $\bar{t} < \mathcal{R}\pi$.

As a preparation we need more information on the spectral properties of A.

IV Eigenvalues and vectors of A

Writing down the eigenvalue equation for A implies two equations which can be immediately arrived at by formally inserting an exponential ansatz into (2.2 a,b): $Q(t)=e^{i\omega t}Q_\omega$, $\phi(\vec{x},t)=e^{i\omega t}\phi_\omega(\vec{x})$:

$$(4.1) \quad (-\omega^2+\omega_0^2)Q_\omega = \lambda(\rho/\phi_\omega)$$

$$(4.2) \quad (-\omega^2+L)/\phi_\omega) = \lambda Q_\omega /\rho)$$

and noting that

$$(4.3) \quad |f_\omega\rangle = |Q_\omega, i\omega Q_\omega, \phi_\omega(\vec{x}), i\omega\phi_\omega(\vec{x})\rangle$$

Due to compactness of L^{-1} one can apply Fredholm theory to (4.2). Let Λ_i be the subspace of $H^0(N)$ spanned by the eigenvectors $L/r_{ij}) = r_i/r_{ij})$. Then the following cases (listed in increasing generality) have to be considered:

A) $/\rho)$ is orthogonal to Λ_i for some i:

This implies that (4.2) may be solved for $\omega^2 = r_i$ to give

$$(4.4) \quad /\phi_{\pm\sqrt{r_i}}) = \lambda Q_{\pm\sqrt{r_i}} \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{(r_j/\rho)/r_j)}{r_j - r_i} + \mathrm{const}\,/r_i)$$

(We assume $\{/r_i)\}$ to be an orthonormal system in $H^0(N)$).

Eq. (4.1) now gives

$$(4.5) \quad \Big[-r_i+\bar{\omega}^2-\lambda^2 r_i \sum_{\substack{j=1\\ j\neq i}}^{\infty} \frac{|(r_j/\rho)|^2}{r_j(r_j-r_i)}\Big] Q_{\pm\sqrt{r_i}} = 0$$

(4.5) is clearly solved by $Q_{\pm\sqrt{r_i}} = 0$. If, by accident, the term in brackets in (4.5) vanishes for the considered i, then $Q_{\sqrt{r_i}}$ may be $\neq$ 0, and hence the eigenspaces $H_{\pm i} \subset H$ for eigenvalues $\omega = \pm\sqrt{r_i}$ have a degeneracy in addition to the one present in Λ_i .

B) $|\rho)$ is orthogonal to a proper subset of Λ_i .

Now $\pm\sqrt{r_i}$ appears as an eigenvalue of A as well, but necessarily $Q_{\pm\sqrt{r_i}} = P_{\pm\sqrt{r_i}} = 0$. This can be described by saying that the modes of the free field do not couple to the oscillator. They will, however, not occur at all in the coupled system in the general case where

C) $|\rho)$ is not orthogonal to any $|r_{i_j})$

Now we have to take $\omega^2 \neq r_i$. In this case (4.2) has the unique solution

(4.6) $$|\phi_\omega) = \lambda Q_\omega (L-\omega^2)^{-1} |\rho)$$

which, inserted into (4.1), gives

(4.7) $$\left[-\omega^2+\bar{\omega}^2-\lambda^2\omega^2(\rho|(L-\omega^2)^{-1}\rho)\right]Q_\omega = -D(\omega)Q_\omega = 0$$

Hence the eigenvalues ω with $\omega^2 \neq r_i$ are given by the solutions to the equation

(4.8) $$D(\omega) = \omega^2-\bar{\omega}^2+\lambda^2\omega^2(\rho|(L-\omega^2)^{-1}\rho) = 0$$

D(ω) is real analytic in the open intervals $(\pm\sqrt{r_i}, \pm\sqrt{r_{i+1}})$ and has poles at $\omega^2 = r_i$. Furthermore one notes that

(4.9) $$\frac{d}{d\omega^2} D(\omega) = 1+\lambda^2(\rho|(L-\omega^2)^{-2}\rho) > 0$$

A simple graphical argument hence shows that there is exactly one zero $\pm\omega_i$ of $D(\omega)$ in each of the intervals $(-\sqrt{r_i}, -\sqrt{r_{i-1}})$ respectively $(\sqrt{r_{i-1}}, \sqrt{r_i})$ with i = 1, 2, ... and r_0 = 0. All of these are simple

zeros.

This result is important because it says that perturbations of the given metric in N such as distortion of symmetry or increase of the volume of N which will increase the density of eigenvalues of the operator L, will do the same to the coupled system. These perturbations will thus, roughly speaking, tend to increase the recurrence time of the system. This is expected on intuitive grounds, since the almost periodic behaviour of, say, the oscillator is due to the fact that waves, after emission, agitate the oscillator again after having "travelled round the universe". Perturbations will, of course, in general decrease the efficiency of this process of "self-interaction via topology" and make it negligible for the actual universe (if it turned out to be closed). From the point of view o physical effects the situation is less ridiculous in the case of, for example, a radiating body enclosed between ideal reflecting walls, a problem which may easily be modelled by making minor changes in the present work.

Despite the fact that some more progress in the study of our system might perhaps be attainable on grounds of general theorems, we now prefer to make life much easier by specifying the manifold N to be S^3 .

V. The Einstein Universe

Now the Lorentz metric is given by

(5.1) $ds^2 = dt^2 - R^2[d\alpha^2 + \sin^2\alpha\, d\Omega^2]$

Here $0 \leq \alpha \leq \pi$, $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the line element on the 2 -sphere. There are coordinate singularities at $\alpha, \vartheta = 0, \pi$.

For ρ we take for simplicity a point distribution at $\alpha = 0$:

(5.2) $$\rho(\vec{x}) = \rho(\alpha) = -\frac{1}{2\pi\,\mathcal{R}^3}\,\frac{\delta'(\alpha)}{\sin\alpha}$$

According to (2.4), the singular nature of ρ demands a "renormalisation" $\omega_0^2 \to \infty$. Therefore, to be completely rigorous, one would have to take an extended $\rho = \rho_\epsilon$ with $\lim_{\epsilon\to 0} \rho_\epsilon = -\frac{1}{2\pi\mathcal{R}^3}\,\frac{\delta'(\alpha)}{\sin\alpha}$ and take the limit $\epsilon \to 0$ in the final results. Such a procedure has been described in detail in [1] for Minkowski space and will be simply omitted here since it does not add to the insight into the problem.

For S^3 the eigenfunctions of L are the well-known hyperspherical harmonics [11] which we denote by $|n\ l\ m)$ with n = 1,2, ...; $0 \le \ell \le n-1$; $-\ell \le m \le \ell$. One has

(5.3) $$-L\,|n\ \ell\ m) = \frac{n^2}{\mathcal{R}^2}\,|n\ \ell\ m)$$

In $\vec{x}$- space we have

(5.4) $$|n\ 0\ 0) = \frac{1}{(2\pi^2\mathcal{R}^3)^{\frac{1}{2}}}\,\frac{\sin n\alpha}{\sin\alpha}$$

which is all of $|n\ \ell\ m)$ which will be needed.

ρ may be expanded in the distributional sense as

(5.5) $$|\rho) = \sum_{n=1}^{\infty} \frac{n}{(2\pi^2\mathcal{R}^3)^{\frac{1}{2}}}\,|n\ 0\ 0)$$

So one is in case B) of Sec. IV. By eigenfunction summation we obtain

(5.6) $$\phi_\omega(\alpha) = (L-\omega^2)^{-1}\lambda|\rho) = \frac{\lambda}{4\pi\mathcal{R}\sin\alpha}\,\frac{\sin\omega\mathcal{R}(\pi-\alpha)}{\sin\omega\mathcal{R}\pi} \qquad \left(\omega \ne \pm\frac{n}{\mathcal{R}}\right)$$

Since attention will be restricted to the motion of the oscillator, we only have to consider the eigenvalues of A which are $\ne \pm\frac{n}{\mathcal{R}}$. These are the solutions $\omega_i\ (\omega_{-i})$ to the equation

(5.7) $$D(\omega)=\omega^2-\bar{\omega}^2-2\Gamma\omega\,\mathrm{ctg}\,\omega\mathcal{R}\pi+\frac{2\Gamma}{\mathcal{R}\pi}=0 \qquad \left(2\Gamma=\frac{\lambda^2}{4\pi}\right)$$

The associated normalized eigenvectors are

(5.8) $$|f_{\omega_i}\rangle = c_i\,|1,\ i\omega_i,\ \phi_{\omega_i},\ i\omega_i\phi_{\omega_i}\rangle$$

where

(5.9) $$c_i^{-2}=\left[\frac{\omega^2}{2}\frac{d}{d\omega^2}D(\omega)\right]_{\omega=\omega_i}=\omega_i^2-\Gamma\omega_i\,\mathrm{ctg}\,\omega_i\mathcal{R}\pi+\Gamma\omega_i^2\frac{\mathcal{R}\pi}{\sin^2\omega_i\mathcal{R}\pi}$$

Using (5.7) this gives

(5.10) $$c_i^{-2}=\frac{1}{2}\left(\omega_i^2+\bar{\omega}^2\right)+\frac{\Gamma}{\mathcal{R}\pi}+\Gamma\mathcal{R}\pi\omega_i^2\left[1+\left(\frac{\omega_i^2-\bar{\omega}^2+\frac{2\Gamma}{\mathcal{R}\pi}}{2\Gamma\omega_i}\right)^2\right]$$

and (5.6) may be written

(5.11) $$\phi_{\omega_i}(\alpha)=\frac{\lambda}{4\pi\mathcal{R}\sin\alpha}\left[\cos\omega_i\mathcal{R}\alpha-\frac{\omega_i^2-\bar{\omega}^2+\frac{2\Gamma}{\mathcal{R}\pi}}{2\Gamma\omega_i}\sin\omega_i\mathcal{R}\alpha\right]$$

Spectrally decomposing $|f(t)\rangle = e^{iAt}\,|f(0)\rangle$ yields

(5.12) $$|f(t)\rangle=\sum_{i=-\infty}^{\infty}e^{i\omega_i t}\langle f_{\omega_i}|f(0)\rangle\,|f_{\omega_i}\rangle$$

This means for Q(t)

(5.13) $$Q(t)=\sum_{i=-\infty}^{\infty}\frac{e^{i\omega_i t}}{2c_i^2}\left[\omega_i^2Q-i\omega_i\dot{Q}+\omega_i^2(\phi_{\omega_i}|\phi)-i\omega_i(\phi_{\omega_i}|\dot{\phi})\right]$$

where $Q, \dot{Q}, \phi, \dot{\phi}$ are the Cauchy data.

Next we ask the question what happens to (5.13) in the limit $\mathcal{R}\to\infty$. Consider for the moment only the $Q,\dot{Q}$ - terms in Eq. (5.13). From the general discussion of the zeros of $D(\omega)$ and (5.10) one observes that the sums get replaced by integrals. The surviving terms are

(5.14) $$Q(t)=\dot{G}(t)Q+G(t)\dot{Q}$$

where

$$(5.15)\quad G(t)=\frac{2\Gamma}{\pi}\int_{-\infty}^{+\infty}d\omega\,\frac{-i\omega e^{i\omega t}}{4\Gamma^2\omega^2+(\omega^2-\bar{\omega}^2)^2}$$

The integrand in (5.15) has a typical Lorentzian shape which is characteristic for resonance scattering of light in the presence of damping (see [12,13]).

Performing the integration by contour-closing gives

$$(5.16)\quad G(t)=e^{-\Gamma|t|}\frac{\sin(\bar{\omega}^2-\Gamma^2)^{\frac{1}{2}}t}{(\bar{\omega}^2-\Gamma^2)^{\frac{1}{2}}}$$

Therefore, if $\Phi(\vec{x},t)=\dot{\Phi}(\vec{x},t)=0$ at t=0, the oscillator suffers exponential damping in both time directions.

To perform the limit in the $\Phi,\dot{\Phi}$ -terms in (5.13), one has to combine $R\longrightarrow\infty$ with $\alpha\rightarrow 0$ such that $\lim_{R\to\infty}R\sin\alpha=\lim_{R\to\infty}R\alpha=r$. Hereby the metric (5.1) approaches the Minkowski-metric in polar coordinates (t,r,ϑ,φ). Defining, as in [1],

$$(5.17)\quad \psi(r)=r\int\frac{d\Omega}{4\pi}\Phi(\vec{x})\;,\quad \chi(r)=r\int\frac{d\Omega}{4\pi}\ddot{\Phi}(\vec{x})$$

one gets for the remaining terms

$$(5.18)\quad \lambda\int_0^t dr\left[\dot{G}(t-r)\psi(r)+G(t-r)\chi(r)\right]$$

The Eqs. (5.14,15,18) are in full accordance with [1]. An analogous statement may be shown to be true for $\Phi(\vec{x},t)$ in the limit $R\longrightarrow\infty$.

The result just obtained is, however, just a special case of the next one. (We treat them separately, because the methods of proof are different and interesting in themselves).

For reasons described in the Introduction, there is to be expected a close similarity of the motion of the oscillator for $t<R\pi$ with the one in flat space ($R=\infty$). More

specifically, this will - for causality reasons - hold for the $Q, \dot{Q}$ - terms in (5.13) for $t < 2\mathcal{R}\pi$ and for the $\phi, \dot{\phi}$ - terms only for $t < \mathcal{R}\pi$.

In order to check this, it is convenient to work in the representation (3.6). Using (3.4, 6) and (5.7) we get for the $Q, \dot{Q}$ - term

(5.19) $$Q(t) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{st}}{s^2 + 2\Gamma s \coth \mathcal{R}\pi s + \bar{\omega}^2 - \frac{2\Gamma}{\mathcal{R}\pi}}\left[sQ + \dot{Q}\right]$$

Now the denominator in the integrand in (5.19) may be split up

(5.20) $$\frac{1}{s^2 + 2\Gamma s \coth \mathcal{R}\pi s + \bar{\omega}^2 - \frac{2\Gamma}{\mathcal{R}\pi}} = \frac{1}{s^2 + 2\Gamma s + \bar{\omega}^2 - \frac{2\Gamma}{\mathcal{R}\pi}} + \text{rest}$$

where the rest consists of $e^{-2\mathcal{R}\pi s}, e^{-4\mathcal{R}\pi s}$ multiplied by terms which are regular and vanish with $|s| \to \infty$ in the half plane $\mathrm{Re}\, s \geq 0$. Therefore, for $0 < t < 2\mathcal{R}\pi$ we are left with

(5.21) $$Q(t) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{st}}{s^2 + 2\Gamma s + \bar{\omega}^2 - \frac{2\Gamma}{\mathcal{R}\pi}}\left[sQ + \dot{Q}\right]$$

Here $\mu^2 = \bar{\omega}^2 - \frac{2\Gamma}{\mathcal{R}\pi}$ plays the role of $\bar{\omega}^2$ in the flat case. We assume that $\mu^2 > 0$. Then (5.21) may be integrated to give

(5.22) $$Q(t) = \dot{H}(t)Q + H(t)\dot{Q}$$

where

(5.23) $$H(t) = e^{-\Gamma t}\frac{\sin(\mu^2 - \Gamma^2)^{\frac{1}{2}}t}{(\mu^2 - \Gamma^2)^{\frac{1}{2}}}$$

Defining

(5.24) $$\psi(\alpha) = \mathcal{R}\sin\alpha\int\frac{d\Omega}{4\pi}\,\phi(\vec{x})\ ,\ \chi(\alpha) = \mathcal{R}\sin\alpha\int\frac{d\Omega}{4\pi}\,\dot{\phi}(\vec{x})$$

the $\phi, \dot{\phi}$ -terms become for $0 < t < \mathcal{R}\pi$

$$(5.25)\quad \lambda \int_0^{t/\mathcal{R}} d\alpha \left[\dot{H}(t-\alpha)\psi(\alpha) + H(t-\alpha)\chi(\alpha) \right]$$

It should be noted that the process just described may be used to calculate the explicit solution in every interval $n\mathcal{R}\pi < t < (n+1)\mathcal{R}\pi$. Essentially the expression corresponding to H(t) will pick up terms t^k ($k \leq n$).

From the Eqs. (5.22, 23, 25) the statement made at the end of Sec. III may easily be inferred.

Appendix

The formalism described in this work to deal with the Cauchy problem for a specific field-particle system relies heavily upon energy conservation and the related self-adjointness of the operator which generates the time evolution. This means, for example, that the presence of absorbing matter is discarded. This will not be very relevant for the asymptotic motion of the oscillator in the case of open space-sections, but will be critical in the closed case. To see this we use the following heuristic argument:

Consider, again, the wave equation

$$(\text{A.1})\quad \left(\Box + \frac{R}{6}\right)\Phi(\vec{x},t) = -\frac{\lambda}{2\pi\mathcal{R}^3}\frac{\delta'(\alpha)}{\sin\alpha}\,Q(t)$$

in the Einstein universe. A "retarded" solution of (A.1) is given by

$$(\text{A.2})\quad \Phi(\vec{x},t) = \frac{\lambda}{4\pi\mathcal{R}\sin\alpha}\left\{ Q(t-\mathcal{R}\alpha) + \sum_{\nu=1}^{\infty}\left[Q(t-\mathcal{R}\alpha-2\nu\mathcal{R}\pi) - Q(t+\mathcal{R}\alpha-2\nu\mathcal{R}\pi)\right]\right\}$$

It is retarded in the sense that $\Phi(\vec{x},t)=0$ for $t < \mathcal{R}\alpha$ if Q = 0 for t $<$ 0. (The sum in (A.2) will not in general

converge. This does not matter for the present purpose: we just sum up the occurring series' formally. For the pair (Q, Φ) one arrives at finally one can always check that it is in fact a solution to (A.1) together with (2.2a)). Inserting (A.2) into (2.2a) leads, after renormalisation, to

$$\ddot{Q}(t)+2\Gamma\dot{Q}(t)+4\Gamma\sum_{\nu=1}^{\infty}\dot{Q}(t-2\nu\mathcal{R}\pi)+\mu^{2}Q(t)=0 \tag{A.3}$$

which may be viewed as an analogue to the Lorentz-Dirac equation of electrodynamics in the present circumstances. It looks time-asymmetric, but it is not. Using (A.3) at time $t-4\pi\mathcal{R}$ and subtracting from (A.3) yields an equation which is manifestly time-symmetric with respect to $t-2\pi\mathcal{R}$. Hence, for any Q(t), which solves (A.3) for all times, Q(-t) is also a solution. For this to be the case it is crucial that the factor in front of the infinite sum in (A.3) is twice the coefficient of $\dot{Q}(t)$. Inserting the ansatz $Q(t) = e^{i\omega t}$ into (A.3) and summing up the series formally, one arrives at the "characteristic equation" $D(\omega)=0$, which is familiar from Secs. III, V. Its having only real solutions is, of course, guaranteed in the preceding treatment by self-adjointness of the operator A.

Imagine now an absorbing medium uniformly distributed in the universe. We model the presence of absorption primitively by replacing (A.3) by

$$\ddot{Q}(t)+2\Gamma\dot{Q}(t)+4\Gamma\sum_{\nu=1}^{\infty}a^{-\nu}\dot{Q}(t-2\nu\mathcal{R}\pi)+\mu^{2}Q(t)=0 \qquad (a>1) \tag{A.4}$$

It is not difficult to convince oneself that the zeros of the corresponding characteristic equation now move to the upper half of the complex ω -plane.

One should, of course, also try to incorporate the expansion

of the universe $\mathcal{R} = \mathcal{R}(t)$ into the model. Then Eq. (A.3) (absorption is discarded now) becomes

$$\text{(A.5)} \quad \ddot{Q}(t) + 2\Gamma\dot{Q}(t) + 4\Gamma\sum_{r=1}^{\infty}\dot{Q}\{f^{-1}[f(t) - 2r\pi]\} + \mu^2 Q(t) = 0$$

where $f(t) = \int^{t} \frac{dt'}{\mathcal{R}(t')}$

In view of the results obtained in [4] it seems likely that the cosmic expansion ($\ddot{f}(t) < 0$) also turns the periodic solutions of (A.3) into damped ones. However, we have been unable to prove this.

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