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Electromagnetic Scattering as a Radiation Reaction Problem

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Electromagnetic scattering is treated as a dynamical problem, by equating the external torque exerted by the incident wave on the sphere to the self torque due to the radiated (scattered) wave. For $I_{\text{mech}}=0$, our scattering amplitude is equal to the usual P-wave amplitude of the electromagnetic scattering on an infinitely conducting sphere. The poles of the scattering amplitude, in particular their dependence on I_{mech}, are studied. For example, a pole on the positive imaginary axis, which usually corresponds to a bound state, corresponds to a runaway solution in our case. Non-decaying resonances are also discussed.

1. Introduction

Electromagnetic scattering is usually calculated by solving the Maxwell Equations with given boundary conditions at the surface of the scatterer. For this, it is not necessary to use mechanical concepts such as the mass or the velocity of the charges nor worry about the forces acting on them; the radiation reaction force is implicitly taken into account, through the boundary conditions.

Here we study the scattering of a plane electromagnetic wave on the sphere to show explicitly how radiation reaction enters into play in scattering phenomena. For $I_{\text{mech}}=0$, we reproduce the usual P-wave phase shift for the scattering on an infinitely conducting sphere.

In Sect. 2 we calculate the phase shift and the scattering cross-section. In Sect. 3 the poles of the scattering amplitude are discussed. In the Appendix we prove that the distant complex poles must lie in the lower half of the k-plane and that their widths Γ_n grow logarithmically with n.

2. Scattering Cross Section

Let the incident electromagnetic plane wave move in the z-direction, and be linearly polarized in x-direction, i.e.

$$\boldsymbol{E}_{\text{inc}}=\hat{\boldsymbol{x}}E_0\,e^{i(kz-\omega t)}. \tag{2.1}$$

* Deceased

We take the center of the sphere as the origin, and its axis of oscillation is constrained to be the y-axis ($\hat{u} \equiv \hat{y}$).

The torque exerted by this wave on the sphere, is due to its magnetic dipole component only:

$$\begin{aligned} d_{\mathrm{ext}}(t) &= \sigma R^2 \int \hat{y} \cdot [\boldsymbol{R} \times \boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{R}, t)]\, d\Omega \\ &= \sigma R^3 E_0 (\int \cos\theta\, e^{ikR\cos\theta}\, d\Omega)\, e^{-i\omega t} \\ &= i e E_0 R j_1(kR)\, e^{-i\omega t} \end{aligned} \tag{2.2}$$

where we used $\hat{y} \cdot [\boldsymbol{R} \times \hat{x}] = R\cos\theta$. The torque (2.2) vanishes for all eigenfrequencies ω_n, which means that the sphere does not "feel" the incident wave at all, and therefore the cross section vanishes at these frequencies, as we shall see below. The external torque (2.2) forces the sphere to oscillate with the frequency of the incident wave, but with a phase shift $\delta(\omega)$. The amplitude and the phase of this oscillation follow by substituting (2.2) in Eq. (4.5) of paper I [1]:

$$\begin{aligned} u(t) &\equiv U(\omega)\, e^{-i\omega t} \equiv U_0(\omega)\, e^{i(\delta - \omega t)} \\ &= \frac{-e E_0 R j_1(kR)}{I_{\mathrm{el}}\, \omega D(k)} \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} D(k) &\equiv |D(k)|\, e^{-i\delta(k)} \equiv \Lambda + \psi(kR) - (\kappa/I_{\mathrm{el}})\,\omega^{-2} \\ &= [I_{\mathrm{mech}}/I_{\mathrm{el}} - 3kR j_1(kR)\, n_1(kR) - \kappa/(I_{\mathrm{el}} c^2 k^2)] \\ &\quad + i 3kR j_1^2(kR). \end{aligned} \tag{2.4}$$

From

$$\sin\delta = -\operatorname{Im} D / |D| \tag{2.5}$$

we get the useful relation

$$U_0(\omega) = \frac{3cE_0}{2eRk^2 j_1(kR)} \sin\delta. \tag{2.6}$$

We also need the expression

$$\begin{aligned} \tan\delta &= \frac{-\operatorname{Im} D}{\operatorname{Re} D} \\ &= \frac{-3kR j_1^2(kR)}{(I_{\mathrm{mech}}/I_{\mathrm{el}}) - ekR j_1(kR)\, n_1(kR) - \kappa/(I_{\mathrm{el}} c^2 k^2)}. \end{aligned} \tag{2.7}$$

Note that δ is independent of I_{el} when $I_{\mathrm{mech}} = \kappa = 0$.

Since the sphere is forced to oscillate, it therefore radiates a magnetic dipole wave, which *is* the scattered wave. And since the sphere does not absorb nor create energy, the average energy radiated by it must be equal to the average work $\bar{A}$ done on it by the incident plane wave.

$$\begin{aligned}\bar{A}&=\frac{1}{T}\int_0^T \mathrm{Re}\,[d_{\mathrm{ext}}(t)]\,\mathrm{Re}\,[u(t)]\,dt\\&=eE_0 R j_1(kR)\,U_0(\omega)\left(\frac{1}{T}\int_0^T \sin\omega t\cos(\delta-\omega t)\,dt\right)\\&=eE_0 R j_1(kR)\,U_0(\omega)(\tfrac{1}{2}\sin\delta)\\&=\frac{3cE_0^2}{4k^2}\sin^2\delta\end{aligned}\tag{2.8}$$

where T is the period ($T\equiv 2\pi/\omega$). In obtaining (2.8) we made use of the relation (2.6).

The total cross section is equal to the average scattered energy per unit time, $\bar{A}$, divided by the average incident energy per unit time and unit area, $(c/8\pi)E_0^2$:

$$\sigma_{\mathrm{tot}}=\frac{\bar{A}}{(c/8\pi)E_0^2}=\frac{6}{k^2}\sin^2\delta(\omega). \tag{2.9}$$

This expression is consistent with the general one for the total cross section of electromagnetic scattering [2]:

$$\sigma_{\mathrm{tot}}=\frac{2\pi}{k^2}\sum_{l=1}^{\infty}(2l+1)[\sin^2\delta_l+\sin^2\delta_l'] \tag{2.10}$$

when all the electric and magnetic phase shifts, δ_l and δ_l', are set to zero except for the magnetic dipole phase shift δ_1', which corresponds to our δ.

Moreover, in the limit $I_{\mathrm{mech}}=\kappa\doteq 0$, we have

$$\tan\delta\to j_1(kR)/n_1(kR) \tag{2.11}$$

so that our phase* δ is exactly equal to the magnetic dipole phase shift of the familiar scattering of an electromagnetic wave on a perfectly conducting sphere [2]. This is expected, since in both cases there is no mechanical inertia to overcome, and hence the only force which counteracts the external force of the incident wave is the radiation reaction force. However, whereas for a perfectly conducting sphere currents corresponding to all partial waves are created by the incident wave, our assumption of a rigid homogeneous charge distribution and a fixed axis of rotation restricts our current to that of a magnetic dipole radiation and thus serves as an effective projection operator of magnetic P-waves only.

The angular distribution of the scattered wave can be obtained by using the Poynting vector for the radiation. One can show easily that the angular distribution is proportional to $\sin^2\beta$, where β is the angle between the rotation axis and the direction of the scattered wave $\hat{k}$

* This phase also appears in non-relativistic quantum mechanics, in the scattering on a rigid sphere.

($\cos\beta = \hat{y}\cdot\hat{k}$). This leads to the differential cross section

$$\frac{d\sigma}{d\Omega} = \sigma_{tot}\frac{3}{8\pi}\sin^2\beta = \sigma_{tot}\cdot\frac{3}{8\pi}(1-\sin^2\theta\sin^2\phi) \tag{2.12}$$

where θ is the scattering angle and ϕ is the azimuthal angle

$$(\hat{k} \equiv (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)).$$

3. Singularities of the Scattering Amplitude

Since there is no absorption, the partial wave scattering amplitude $A(k)$ is determined on the real axis by the known phase shift $\delta(k)$:

$$\begin{aligned} A(k) &= k^{-1}\sin\delta(k)\,e^{i\delta(k)} \\ &= -\operatorname{Im} D(k)/(kD(k)) \end{aligned} \tag{3.1}$$

where we used Eqs. (2.4) and (2.5). Since $D(-k^*) = D^*(k)$, we can replace $(-\operatorname{Im} D(k))/k$ in (3.1) by the analytic function

$$N(k) = [D(-k) - D(k)]/2ik = -3R\,j_1^2(kR) \tag{3.2}$$

and obtain an analytic expression for $A(k)$ for the whole k-plane:

$$\begin{aligned} A(k) &= N(k)/D(k) \\ &= -3Rj_1^2(kR)/[\Lambda + \psi(kR) - \kappa/(I_{el}c^2k^2)] \\ &\xrightarrow[\Lambda=\kappa=0]{} -j_1(kR)/(kh_1(kR)). \end{aligned} \tag{3.3}$$

Except for some peculiarities, $A(k)$ has the usual properties of a partial scattering amplitude: It satisfies unitarity, $\operatorname{Im} A(k) = k|A(k)|^2$, by construction. It has the usual threshold behavior $A_l(k) \sim k^{2l}$ for $k \simeq 0$, but goes as k^4 instead of k^2 for $\kappa \neq 0$. It satisfies the relation $A(-k^*) = A^*(k)$, so that $A(k^2)$ is real analytic (as a function of k^2). It can be written in the usual N over D form (3.3), with D satisfying the dispersion relation

$$D(k) = \Lambda - \kappa/(I_{el}c^2k^2) + \frac{6R}{\pi}\int_0^\infty \frac{k'^2 j_1(k'R)\,dk'}{k'^2 - k^2} \quad \text{for } \operatorname{Im} k > 0, \tag{3.4}$$

but $N(k^2)$ having no left-hand cut.

D was chosen such that $\lim D(k) \to$ const for $k \to \infty$ and $\Lambda \neq 0$. Therefore, $D(k)$ has a double pole at $k=0$ for $\kappa \neq 0$.

Finally, $D(k)$ may be identified with the Jost function $J(k)$ by the relation $J(-k) = D(k)$. In the following we discuss and interpret the zeros of $D(k)$ which are the poles of $A(k)$.

Radiationless Modes as Non-Decaying Resonances

For $I_{\text{mech}} \neq 0$, $A(k)$ has infinite number of poles $\gamma_n^{\pm} = \pm\Omega_n + i\frac{1}{2}\Gamma_n$ ($n = 1, 2, \ldots$), which lie *below* the real axis, as shown in an Appendix of paper III. Such poles usually correspond to resonances. This is true also in our case, where these poles correspond for $I_{\text{mech}} \neq 0$ to oscillating decaying modes, which radiate energy to infinity.

It is interesting to see what happens to the above decaying resonances as $|I_{\text{mech}}| \to 0$. As $|I_{\text{mech}}|$ decreases, the zeros of $D(k)$ approach the real axis and they all lie exactly on it for $I_{\text{mech}} = 0$. The residues of the corresponding poles of $A(k)$ become smaller and vanish exactly in this limit. Therefore, looking at $A(k)$ for $I_{\text{mech}} = 0$, one would observe no poles on the real axis. This is expected, since otherwise unitarity would be violated. Here we have a rare example of resonances that can not be detected as poles of the scattering amplitude, nor by the $\delta = (n + \frac{1}{2})\pi$ criterion, as discussed further below.

It seems physically reasonable to interpret such real zeros as corresponding to *non-decaying resonances*. In our case they correspond to radiationless modes, for which the sphere can store energy inside it for indefinite time without any dissipation. However, the amount of the stored energy depends on the amplitude of the oscillation, which is arbitrary in our case, since our system is classical and not quantized. Therefore, one can not associate a physical mass to such resonances, as one does to resonances in quantum mechanics.

However, we could associate masses to our resonances, if we "quantize" the sphere as follows: If the n-th eigenmode decays, it is expected to emit radiation with frequencies concentrated around ν_n. If one assumes that when such an excited resonance decays, it emits all its stored energy by a single photon of frequency ν_n, then the excitation energies $\hbar\omega_n$ are completely determined, if the radius R is given.

As an illustration, let $R = n r_e \equiv n(e^2/m_e c^2)$, where r_e is the classical electron radius and n some number. Then the excitation energy of the first resonance is

$$\hbar\omega_1 = \frac{\hbar c x_1}{R} = \frac{x_1}{n}\left(\frac{\hbar c}{e^2}\right) m_e c^2 = \frac{4.494}{n\alpha} m_e c^2 \simeq \frac{9}{2n\alpha} m_e c^2 \qquad (3.5)$$

where x_1 is the first zero of $j_1(x)$. Taking $n = 3$(!!) gives

$$\hbar\omega_1 = 205.27\, m_e c^2 \qquad (3.6)$$

which is very close to the excitation energy of the muon relative to "its ground state", the electron.

$$(m_\mu - m_e)c^2 = (205.77 \pm 0.01)\, m_e c^2 \qquad (3.7)$$

This observation has been made by many authors before [3]! Besides all the usual problems associated with a classical model for the electron, a radius of $3r_e \simeq 7.4$ fermis seems too large for what is believed to be a point particle. However, $R=3r_e$ is self-consistent (more so than, say, $R=\frac{1}{3}r_e$), since it leads to positive mechanical mass $m_{\text{mech}}=m_e-\frac{e^2}{R}=\frac{2}{3}m_e>0$ (see paper III).

The $\delta=(n+\frac{1}{2})$ Criterion

Weak resonances, like the Roper resonance in $\pi-N$ scattering, are usually discovered by phase shift analysis, by looking for the phase *increasing* through values of $(n+\frac{1}{2})\pi$. Clearly, the far-away resonances can not be detected by the above $\delta=(n+\frac{1}{2})\pi$ criterion. Our amplitude illustrates this and the less expected fact that non-decaying resonances can not be detected by the above criterion either: For $\delta=(n+\frac{1}{2})\pi$ we have $\tan\delta=\infty$ and this corresponds to solutions of the following equation (see Eq. (2.7)):

$$\begin{aligned} \Lambda &= 3x j_1(x) n_1(x) + \kappa/(I_{\text{el}} x^2) \\ &= \frac{3}{2}\left[\left(\frac{1}{x}-\frac{1}{x^3}\right)\sin 2x - \frac{2}{x^2}\cos 2x\right] + \kappa/(I_{\text{el}} x^2) \\ &\simeq \frac{3}{2x}\sin 2x \qquad \text{for } x \gg 1. \end{aligned} \tag{3.8}$$

For $|\Lambda| \ll 1$ this equation has solutions near the zeros of $j_1(x)$ and $n_1(x)$, but only the zeros of $j_1(x)$ correspond to resonances, as only at these zeros δ *grows* as it passes the values $(n+\frac{1}{2})\pi$. For $\Lambda=0$, these zeros of j_1 are canceled by the double zeros of j_1^2 in the numerator of (2.7) and the δ becomes monotonically *decreasing* function of k, and hence δ can not lead to the detection of the non-decaying resonances. By the way, for $\Lambda=0$ the definition $D=|D|e^{i\delta}$ forces δ to have infinite number of discontinuities (at the zeros of D on the real axis), and should be relaxed to $D=\pm|D|e^{-i\delta_c}$ to get a continuous phase δ_c.

For fixed Λ, Eq. (3.8) has only a finite number of solutions, corresponding to $x<\frac{3}{2}\Lambda$. The larger Λ the smaller is this number, and for large enough Λ there is no solution at all. This reflects the fact that as $|\Lambda|$ increases, the poles move downwards in the complex plane (see Appendix) and the distant poles can no longer be detected by the above $\delta=(n+\frac{1}{2})$ criterion.

Runaway Solutions as Bound States

In III it is shown that $D(k)$ has a zero on the negative imaginary axis for $I_{\text{mech}}>-I_{\text{el}}$ and another zero on the positive imaginary axis for $0>I_{\text{mech}}>-I_{\text{el}}$. (For $I_{\text{mech}}<-I_{\text{el}}$ these two zeros become two complex zeros in the lower half of the k-plane). Such zeros on the imaginary axis are usually interpreted as belonging to an antibound state and to a bound state, respectively. In our case they correspond to non-oscillatory decaying- and growing (runaway) modes, respectively.

The (non-Coulomb) fields corresponding to the runaway solution fall off exponentially with the radius as $r\to\infty$, as can be seen immediately by substituting $\omega=i|\omega|$ in Eqs. (2.7b) of paper I. Therefore, for any fixed time t these *fields are normalizable* and thus have a finite total energy $\frac{1}{8\pi}\int(E^2+B^2)d^3r<\infty$. In this sense, a runaway solution may be regarded as a bound state!

Let us further compare our *runaway* solution with the solution of an ordinary bound state of massive particle. Mathematically, both are "stationary solutions" of the form

$$\psi(r,t)=e^{-iEt}\psi(r,0) \tag{3.9}$$

with normalizable $\psi(r,0)$, i.e. $\int|\psi(r,0)|^2d^3r<\infty$. The difference lies in the time factor e^{-iEt}, which is usually just a phase and thus harmless whereas it is exponentially increasing in our case, and therefore it causes troubles. The reason for this difference is easy to understand mathematically: For ordinary wave equations that are linear in $\partial/\partial t$, E is an eigenvalue of the equation

$$H\psi(r,0)=E\psi(r,0) \tag{3.10}$$

and hence it must be real, since H is Hermitian and $\psi(r,0)$ normalizable. However, the corresponding equation in our case is the inhomogeneous Maxwell Equation, which, when written in a form similar to (3.10) [4], shows an additional source term, due to the electromagnetic current j_μ:

$$H\psi_i=E\psi_i+4\pi j_i \tag{3.11}$$

where $H\equiv-c\boldsymbol{s}\cdot\boldsymbol{p}=i\hbar c\,\boldsymbol{s}\cdot\boldsymbol{\nabla}$ with s_i being 3×3 Hermitian matrices $(s_i)_{jk}=-i\varepsilon_{ijk}$ and $\psi_j\equiv E_j-iB_j$ ($j=1,2,3$) where $\boldsymbol{E}$ and $\boldsymbol{B}$ are real quantities corresponding to the physical fields. Clearly Eq. (3.11) can have non-real eigenvalues, as the existence of our runaway solution proves.

Appendix

Here we prove that for $I_{\text{mech}}\neq0$ the distant zeros $\gamma_n^\pm\equiv\pm\Omega_n+i\frac{1}{2}\Gamma_n$ of

$$\begin{aligned}D(\omega)&\equiv\Lambda+\psi(\tau\omega)-\kappa/(I_{\text{el}}\omega^2)\\&=\Lambda+\frac{3i}{2z^3}(e^{2iz}(z+i)^2+z^2+1)-\frac{\kappa\omega^{-2}}{I_{\text{el}}}=0,\end{aligned} \tag{A.1}$$

($z\equiv\tau\omega$) are given by

$$\tau\Omega_n\sim[n+\tfrac{1}{2}-\tfrac{1}{2}\operatorname{sign}(\Lambda)]\pi \qquad \text{for large } n \tag{A.2}$$

and

$$\tau\Gamma_n \simeq \ln[\tfrac{2}{3}|\Lambda|\tau\Omega_n] > 0 \quad \text{for large } n. \tag{A.3}$$

Proof. Keeping only the leading terms in Eq. (A.1) for large $z \equiv x+iy$ gives

$$\frac{2}{3i}\Lambda z \simeq -(e^{2iz}+1) \tag{A.4}$$

so that

$$\tfrac{2}{3}|\Lambda z| \simeq |e^{2iz}+1| = |e^{2ix}e^{-2y}+1| \simeq |e^{-2y}| \tag{A.5}$$

and hence

$$y \simeq -\tfrac{1}{2}\ln\tfrac{2}{3}|\Lambda z| < 0. \tag{A.6}$$

Therefore, y is large and negative for large $|z|$. Since $|y| \ll \ln|y|$ for large $|y|$, it follows from (A.6) that $|y| \ll |x|$, so that

$$y \simeq -\tfrac{1}{2}\ln\tfrac{2}{3}|\Lambda x| \tag{A.7}$$

which is (A.7). To determine x, consider the real part of (A.4):

$$\cos 2x \simeq -(\Lambda y+1)e^{2y} \simeq \Lambda|y|e^{-2|y|} \simeq 0. \tag{A.8}$$

Therefore

$$|x| = \text{sign}(\Lambda)\tfrac{1}{4}\pi \mod(n\pi), \tag{A.9}$$

since $x \simeq (n+\frac{1}{2})\pi$ for $\Lambda=0$ and since the poles are expected to move continuously. This leads to (A.2).

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