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Mass Renormalization and Runaway Solutions

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Abstract. The relationship between negative mechanical mass and runaway solutions is studied using the model of a charged spherical shell. The model yields unexpected results: for example, for $-I_{el} < I_{mech} < 0$ a runaway solution appears, but it disappears again for $I_{mech} < -I_{el}$, where I_{el} and I_{mech} are the electromagnetic and mechanical moments of inertia of the shell. The runaway solutions of this shell are contrasted with those of the quasi-stationary electron. The standard interpretation proposed by Wildermuth of how the negative mass causes runaway solutions is shown to be incorrect. However, an alternative interpretation, based on regarding mass renormalization as if it were an "external hidden force" is capable of explaining the above runaway solutions.

1. Introduction

For many years no progress was made towards the understanding of the relationship between mass renormalization and runaway solutions in classical field theories. Such solutions appear, for example, for the Lorentz-Dirac equation of motion [1] of the point electron and in its non-relativistic limit, the Abraham-Lorentz equation [2]. In particular, it was not clear whether these solutions are caused by the *infinity* of the mass renormalization; for the *point* electron mass renormalization is necessary to bring the infinitely positive electromagnetic mass down to the observed mass of the electron.

Wildermuth [3] made an important contribution towards understanding the role of negative mechanical mass by illustrating that runaway solutions are not necessarily connected to infinities, and that any negative mass renormalization could lead to such solutions. He considered the quasi-stationary limit of the equation of motion of the Lorentz electron, which is obtained by neglecting the non-linear powers in the velocity and its derivatives:

$$F_{ext}(t) - \frac{e^2}{3R^2 c}[\dot{x}(t) - \dot{x}(t-2R/c)] = 0. \tag{1.1}$$

This equation does not have runaway solutions, but by adding to it a mechanical mass term (renormalizing its mass), one obtains the equation

$$F_{ext} - \frac{e^2}{3R^2 c}[\dot{x}(t) - \dot{x}(t-2R/c)] = m_{mech}\,\ddot{x}(t) \tag{1.2}$$

which has an *infinite* number of runaway solutions for $m_{mech} < 0$. Wildermuth interpreted the effect of the negative mass as follows: "An electromagnetic braking force (Bremskraft), e.g., the self-force, accelerates the electron more and more, since force and acceleration have opposite directions for a negative mechanical mass". Wildermuth's example has been much quoted [4, 5] and his interpretation is apparently accepted in the literature [5].

Now, Wildermuth's example is based on Eq. (1.1), which is an *approximation* of the equation of motion of a physical system and it is very unlikely (see Sect. 4) that Eq. (1.1) can describe *exactly* the motion of any localized charged distribution, which need not be rigid, spherical, or homogeneous. Hence, one can not examine the conservation of energy and momentum in Wildermuth's example. Therefore, it is important to have a similar example which is based on a model which has a simple and *exact* equation of motion and for which energy and momentum conservation is well understood. Such a model was described earlier [6], and is based on a rigid charged spherical shell, which can rotate around a fixed axis.

In this paper, we examine the runaway solutions which appear when a negative mechanical moment of inertia I_{mech} is added to the charged shell. We obtain interesting and rather unexpected results: for $-I_{\text{el}} < I_{\text{mech}} < 0$, where I_{el} is the electromagnetic moment of inertia, the equation of motion has only one runaway solution, which disappears when I_{mech} is made more negative. Thus, in contrast to the quasi-stationary approximation, not only the infinite number of radiationless modes fail to become runaway solutions for $I_{\text{mech}} < 0$, but also the single exponential runaway solution is present only for special negative values of I_{mech}. Moreover, if we attach the sphere to a spring, which provides a restoring torque, we find that the single runaway solution persists for all negative values of I_{mech}. These results could not be understood on the basis of Wildermuth's interpretation. And, in fact, a closer look at this interpretation showed it to be wrong. The above results can, however, be understood in terms of the hidden force interpretation [7].

In the next section the runaway solutions for the spherical shell are discussed, in Section 3 Wildermuth's interpretation is examined and in Section 4 the results are explained by the hidden-force interpretation.

2. Runaway Solutions of the Spherical Shell

The equation of motion for the spherical shell of radius R is given by Eq. (4.2) of paper I [6]:

$$d_{\text{ext}}(t) = I_{\text{mech}}\,\ddot{\phi}(t) + I_{\text{el}}\frac{3}{2\tau}\int_0^{2\tau}\left(1-\frac{(t')^2}{2\tau^2}\right)\ddot{\phi}(t-t')\,dt' + \kappa\,\phi(t) \tag{2.1}$$

where $\phi(t)$ is the angular displacement and $\tau \equiv R/c$, and κ is the spring constant. In this equation, the external torque d_{ext} is balanced by the mechanical inertia term, by the electromagnetic inertia term due to radiation reaction, and finally by the restoring torque of the spring.

The runaway solution of (2.1) are the solutions for $d_{\text{ext}}(t) \equiv 0$, which become unbounded as the time t goes to infinity. In paper I we showed that any solution of the homogeneous equation must have the form:

$$\phi(t) = (\phi_0 + \phi_1 t + \phi_2 t^2 + \cdots + \phi_n t^n)\, e^{-i\omega t} \tag{2.2}$$

or a linear combination of such exponential forms, where ω is a complex number. We also showed that for any ω the solutions which involve powers of t cannot exist unless the pure exponential solution

$$\phi(t) = \phi_0\, e^{-i\omega t} \tag{2.3}$$

exists. We shall therefore first investigate these latter solutions and then show that the others do not exist in our case.

Substituting (2.3) into (2.1) and setting $d_{\text{ext}} \equiv 0$ gives [6]

$$f(\omega) \equiv -[I_{\text{mech}} + I_{\text{el}}\,\psi(\tau\omega)]\,\omega^2 + \kappa = 0 \tag{2.4}$$

where

$$\psi(X) \equiv 3X\, j_1(X)\, h_1(X) = \frac{3i}{2X^3}\left(e^{2iX}(X+i)^2 + X^2 + 1\right)$$

where $j_1(X)$ and $h_1(X) \equiv i h_1^{[1]}(X)$ are the spherical Bessel and Hankel functions for $l=1$. Note that $I_{\text{el}}\,\psi(\tau\omega)$ can be interpreted as the effective electromagnetic moment of inertia for the oscillatory motion (2.3) [6].

The solutions of the homogenous equation correspond therefore to the zeros of Eq. (2.4). In particular, the exponential solutions (2.3) correspond to simple zeros of (2.4) while solutions of the type $\phi_n t^n e^{-i\omega t}$ correspond to zeros of order n. Since we are only interested in solutions of the form (2.2) which become unbounded for $t \to \infty$, we need only consider the zeros of (2.4) for $\operatorname{Im}\omega > 0$. I shall first consider the zeros which lie on the positive imaginary axis, and then prove that no other zeros exist in the upper half-plane.

These pure imaginary zeros are the zeros of the following relation, which follows from (2.4) by substituting $\tau\omega = iy$:

$$I_{\text{el}}\,\psi(iy) = -I_{\text{mech}} - (\kappa/\tau^2)\, y^{-2}. \tag{2.5}$$

One can show that $\psi(iy)$ has its maximum value, $+1$, for $y=0$ and decreases monotonically as $y \to \infty$, so that

$$0 < \psi(iy) < 1 \quad \text{for } y > 0. \tag{2.6}$$

For the case $\kappa = 0$, Eqs. (2.5) and (2.6) immediately give us a condition for the existence of runaway solutions, namely

$$-I_{\text{mech}} = |I_{\text{mech}}| = I_{\text{el}}\,\psi(iy) < I_{\text{el}} \quad \text{for } y > 0. \tag{2.7}$$

For the case $\kappa > 0$, Eq. (2.5) for $y > 0$ has a single simple zero for any $I_{\text{mech}} < 0$, and hence the system will always have a runaway solution when there is a restoring spring.

It is interesting to follow the pure imaginary zero as I_{mech} decreases from 0 to $-\infty$. As soon as I_{mech} becomes negative, a new zero appears in the complex plane from $\omega = i\infty$ and moves downwards along the imaginary axis as I_{mech} decreases. For the case $\kappa = 0$, this zero reaches the origin for $I_{\text{mech}} = -I_{\text{el}}$ and there it meets a decaying-mode zero, which for $I_{\mathrm{mech}}=0$ is located at $\omega=-i/\tau$ and which moves upwards along the imaginary axis as I_{mech} decreases from 0 to $-I_{\mathrm{el}}$. For $I_{\mathrm{mech}}<-I_{\mathrm{el}}$ these two zeros become two complex zeros $\omega_{\pm}=\pm\alpha-i\beta$ ($\beta>0$) in the lower half-plane. For $\kappa>0$, the runaway and the decay-mode zeros move towards each other as I_{mech} decreases to $-\infty$, but they never meet.

In potential scattering one encounters a similar behavior to that for $\kappa=0$, where a bound-state zero meets an anti-bound-state zero to become two complex resonance zeros as the potential becomes less attractive. In fact, it turns out that $f(\omega)$ for $\kappa=0$ is proportional to the Jost function [8] of the P partial wave for the scattering off a δ-function potential (2.9): The Jost function $D_l(k)$ is given in general by

$$D_l(k)=1+\frac{2\mu}{\hbar^2}k^{l+1}\int_0^\infty h_l(kr)\,V(r)\,\phi_l(k,r)\,r\,dr. \tag{2.8}$$

Substituting for the potential

$$V(r)=\gamma\,\delta(r-R) \tag{2.9}$$

and noting that the regular solution $\phi_l(k,r)$ for this potential is given by

$$\phi_l(k,r)=r\,j_l(kr)/k^l\sim\frac{r^{l+1}}{(2l+1)!!}\quad \text{for } r\leqq R \tag{2.10}$$

we get for the δ-potential

$$D_l(k)=1+\frac{2\mu}{\hbar^2}\gamma R^2 k\,j_l(kR)\,h_l(kR) \tag{2.11}$$

which for $l=1$ and the following choice of

$$\gamma=\frac{3\hbar^2}{2\mu R}(I_{\mathrm{el}}/I_{\mathrm{mech}})<0\quad \text{for } I_{\mathrm{mech}}<0 \tag{2.12}$$

becomes proportional to our function $f(\omega)$ defined in (2.4). Note that the coupling constant γ becomes less attractive as I_{mech} becomes more negative. The Jost function for a Hermitian potential does not have zeros in the upper-half of the k-plane, except for bound-state zeros which lie on the imaginary axis. This fact is a proof that for $\kappa=0$ our system does not have any oscillatory runaway solutions for $I_{\mathrm{mech}}<0$.

For $\kappa>0$, it is possible to prove the same result by using a generalized version of Levinson theorem which is applicable to functions $f(\omega)$ which do not have constant asymptotic behavior for $|\omega|\to\infty$. In fact, I originally used Levinson theorem to prove the non-existence of oscillatory runaway solutions for $\kappa=0$ before discovering the analogy to the scattering from a δ-potential. I shall not give more mathematical details about the case $\kappa>0$, since I shall show in Section 4 that such oscillatory runaway solutions can not appear on physical grounds.

3. Wildermuth's Interpretation

As mentioned earlier, Wildermuth's interpretation, quoted in the introduction, of *how* the negative mechanical mass causes runaway solutions, has been the standard interpretation for many years. However, this interpretation cannot explain, e.g., why the runaway solution disappears for $I_{\mathrm{mech}}<-I_{\mathrm{el}}$, nor why there are oscillatory runaway solutions for the quasi-stationary electron and none for the sphere.

Although the above interpretation sounds intuitively reasonable, since as shown below, a braking force in the usual sense would in fact accelerate an object instead of decelerating it, if the mechanical mass were negative, it contains two errors: For a braking force the conclusion would hold, but not for the reason given. However, the self-force is not a braking force, and hence also the conclusion is not always true:

1. In general, whether force and acceleration have parallel or opposite directions is irrelevant, since whether there is acceleration or deceleration ($d|\mathbf{v}|/dt>0$ or <0) does not depend on $\hat{\mathbf{a}}$, but only on $\hat{\mathbf{v}}\cdot\hat{\mathbf{a}}$:

$$\frac{d|\mathbf{v}|}{dt}=\frac{1}{2|\mathbf{v}|}\frac{d(\mathbf{v}\cdot\mathbf{v})}{dt}=\hat{\mathbf{v}}\cdot\frac{d\mathbf{v}}{dt}=|\mathbf{a}|\,\hat{\mathbf{v}}\cdot\hat{\mathbf{a}}. \tag{3.1}$$

However, in the specific case of the usual braking free, $\mathbf{F}_{\mathrm{brak}}=-b\mathbf{v}$, $\hat{\mathbf{F}}_{\mathrm{brak}}$ is determined by $\hat{\mathbf{v}}$, and we would indeed have an acceleration,

$$\frac{d|\mathbf{v}|}{dt}=\hat{\mathbf{v}}\cdot\mathbf{a}=\hat{\mathbf{v}}\cdot\left(\frac{-b\mathbf{v}}{m_{\mathrm{mech}}}\right)>0\quad \text{for } m_{\mathrm{mech}}<0, \tag{3.2}$$

as intuitively expected.

2. The electromagnetic self-force cannot be considered a *braking* force: A braking force, such as $\mathbf{F}=-b\mathbf{v}$, tries to slow down the motion (making $|\mathbf{v}|$ smaller), and in the absence of other external forces, it brings the system to a complete standstill, where it no longer has to lose energy into friction. The electromagnetic self-force, on the other hand, tries to bring the system to a constant speed or more generally to radiationless modes, where the system no longer has to lose energy into radiation. In fact, the self-force sometimes increases the speed of the object, in order to bring it to a radiationless mode. We may therefore call the self-force an *inertial* force or some other appropriate name, but it is certainly misleading to call it a braking force.

To conclude, the direction of the self-force $\mathbf{F}_{\mathrm{self}}$ is usually not determined by $\hat{\mathbf{v}}$, and hence no statement can be made on acceleration or deceleration on the basis of the sign of m_{mech} alone.

4. Explaining the Results by the Hidden-Force Interpretation

Elsewhere [7] I show that if we have two equations of motion

$$\mathbf{F}_{\mathrm{ext}}=\mathbf{F}_{1\,\mathrm{reac}}(\mathbf{v}) \tag{4.1}$$

$$\mathbf{F}_{\mathrm{ext}}=\mathbf{F}_{2\,\mathrm{reac}}(\mathbf{v}) \tag{4.2}$$

where $\mathbf{F}_{\mathrm{ext}}$ represent the external force and $\mathbf{F}_{1\,\mathrm{reac}}$ and $\mathbf{F}_{2\,\mathrm{reac}}$ are reaction forces (see an example below), and assuming that we understand how energy and momentum are conserved for the system described by Eq. (4.1), then it is often advantageous to rewrite (4.2) in the form

$$\mathbf{F}_{\mathrm{ext}}^{\mathrm{effective}}\equiv\mathbf{F}_{\mathrm{ext}}+\mathbf{F}_h=\mathbf{F}_{1\,\mathrm{reac}}(\dot{\mathbf{v}}) \tag{4.3}$$

with

$$\mathbf{F}_h\equiv\mathbf{F}_{1\,\mathrm{reac}}-\mathbf{F}_{2\,\mathrm{reac}} \tag{4.4}$$

and look upon system (2) as if it were system (1), but with additional "external hidden force" $\mathbf{F}_h$.

We now apply the above idea to our problem. Rewritting Eq. (2.1) in a form similar to (4.3) gives

$$d_{\mathrm{ext}}(t)+d_h(t)=I_{\mathrm{el}}\frac{3}{2\tau}\int_0^{2\tau}\left(1-\frac{(t')^2}{2\tau^2}\right)\dot{u}(t-t')\,dt+\kappa\,\phi(t) \tag{4.5}$$

where

$$d_h(t)=-I_{\mathrm{mech}}\,\dot{u}(t). \tag{4.6}$$

From now on we shall treat the mechanical mass term as an "external" (hidden) torque, which is acting on the reduced system of the charged sphere with spring but without mechanical mass. This enables us to explain the runaway solutions in terms of energy conservation for the reduced system. We know that the reduced system will not leave its state of rest $\phi(t)\equiv 0$, unless it is acted upon by an external torque which could do positive work on the system, since the *total* energy of the system (the energy of the field plus the energy of the spring) has its lowest possible value for the rest state. But in Eq. (4.5), even for $d_{\mathrm{ext}}\equiv 0$, the effective torque in non zero, being equal to $-I_{\mathrm{mech}}\dot{u}$. This hidden torque could do positive work on the system for a runaway motion

$$\phi(t)=\mathrm{Re}(\phi_0\,e^{-i\omega t})\equiv\mathrm{Re}\left(\frac{i u_0}{\omega}\,e^{-i(\alpha+i\beta)t}\right) \tag{4.7}$$

where we write $\phi_0=i u_0/\omega$ and choose u_0 to be real for convenience:

$$\begin{aligned}W_h(t)&=\int_{-\infty}^{t}\left(-I_{\mathrm{mech}}\,\dot{u}(t')\right)u(t')\,dt'\\&=-\tfrac{1}{2}I_{\mathrm{mech}}\left[u^2(t)-u^2(-\infty)\right]\\&=-\tfrac{1}{2}I_{\mathrm{mech}}\,u_0^2\cos^2\alpha t\,e^{2\beta t}.\end{aligned} \tag{4.8}$$

This equation explains why runaway solutions of the form (4.7) *can* only exist for $I_{\mathrm{mech}}<0$.

The hidden-force interpretation also enables us to understand why the exponential runaway solution disappears for $I_{\mathrm{mech}}<-I_{\mathrm{el}}$ when $\kappa=0$: For an exponential motion

$$u(t)=u_0\,e^{\beta t}\qquad \beta>0, \tag{4.9}$$

the hidden force, with $I_{\mathrm{mech}}<-I_{\mathrm{el}}$, would supply the sphere with more power than it can radiate away into reversible and irreversible radiation:

$$\begin{aligned}P_{\mathrm{hidden}}(t)&=-I_{\mathrm{mech}}\,\dot{u}(t)\,u(t)\\&=u_0^2|I_{\mathrm{mech}}|\,\beta\,e^{2\beta t}>u_0^2 I_{\mathrm{el}}\,\psi(i\tau\beta)\,\beta\,e^{2\beta t}\\&=d_{\mathrm{self}}\,u(t)\equiv P_{\mathrm{radiation}}(t)\end{aligned} \tag{4.10}$$

where we used the inequality (2.6), and that the self-torque d_{self} for the motion (4.9) is equal to

$$I_{\mathrm{el}}\,\psi(i\tau\beta)\,\dot{u}(t)=I_{\mathrm{el}}\,\psi(i\tau\beta)\,u_0\,\beta\,e^{\beta t}.$$

However, for $\kappa>0$, the extra power supplied by the hidden force could be absorbed by the spring and stored as potential energy, and the runaway solution does not disappear for this case.

Finally, we show that the hidden force interpretation can also explain why no *oscillatory* runaway solutions of the form (4.7) *with* $\alpha\neq 0$ appear. For such solutions the work (4.8) vanishes periodically with time. And since the reduced system emits irreversible radiation for such oscillatory modes (see Eq. (2.26) of I), it follows that for every t_0 for which $W_h(t_0)=0$ the irreversible radiation emitted for $-\infty<t<t_0$ would have to come either from the reversible electromagnetic energy of the velocity fields (which for the particle at rest is equal to the electrostatic Coulomb field energy) or from the potential energy of the spring. This is impossible, since for the solution of the form (2.3) the system is initially at rest, $\phi(-\infty)=0$, and has therefore its lowest possible energy. Hence, there cannot be any *oscillatory* runaway solutions. Note that this argument does not apply to a monotonically increasing runaway solution (4.9), since the work $W_h(t)$ in this case never vanishes for a finite t.

It is interesting to note, that since the equation of motion for the electron in the quasi-stationary approximation *does* have *oscillatory* runaway solutions for $m_{\mathrm{mech}}<0$, we have a good indication that this approximate equation cannot describe *exactly* the motion of *any* stable charge distribution, since every charge distribution is expected to radiate irreversibly when it is forced to leave its lowest energy state, the rest state $\mathbf{v}(t)\equiv 0$.

To conclude: By using a model, for which energy and momentum conservation is well understood, I was able to illustrate and confirm Wildermuth's main conclusion that adding any finite negative mechanical mass to a physical system *could* lead to unphysical runaway solutions. I illustrated that such addition is not a sufficient condition for the appearance of runaway solutions, and explained why this is so.

References

1. Dirac, P.A.M.: Proc. Roy. Soc. (London), Ser. A **167**, 148 (1938)
2. See for example: Rohrlich, F.: Classical Charged Particles. Reading, Mass.: Addison-Wesley 1965, and standard textbooks on classical electrodynamics
3. Wildermuth, K.: Z. Naturforschung **10**a, 450 (1955)
4. See e.g. Gromes, D., Petzold, J.: Z. Physik **199**, 299 (1967)
5. Erber, T.: Fortschr. Physik **9**, 343 (1961), especially p. 378
6. Daboul, J., Jensen, J.H.D.: Z. Physik **265**, 479 (1973), hereafter quoted as I
7. Daboul, J.: In preparation
8. Jost, R.: Helv. Phys. Acta **20**, 256 (1947), and standard textbooks on potential scattering

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