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Radiation Reaction for a Rotating Sphere with Rigid Surface Charge

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Radiation reaction in classical electrodynamics is studied using as a simple model a rotating sphere with rigid surface charge. Because of the transparent geometric nature of the problem, we obtain an exact integral expression for the "self-torque", which depends causally and linearly on the angular speed. This leads to a linear equation of motion, which does not have any runaway solutions, as long as no negative mechanical moment-of-inertia is added to the system. The results are illustrated with a few examples corresponding to predetermined motions and external torques.

1. Introduction

The reaction of a radiation field on the charges which sustain it, i.e. radiation reaction [1, 2], has been studied for a rigid spherical charge distribution of radius R undergoing a translatory motion (Lorentz Model) and also for its limiting case ($R \to 0$) of a point charge. For the extended charge distribution, the self-force is an infinite sum over higher derivatives of the velocity [2], which is not easy to work with. By taking the $R \to 0$ limit, all terms involving $\dddot{\boldsymbol{v}}$ and higher derivatives of $\boldsymbol{v}$ disappear, but the electromagnetic mass becomes infinite. When this infinite mass is replaced by the finite physical mass, using either intuitive physical arguments or Dirac's method, based on radiation (retarded minus advanced) fields [3], one obtains an equation of motion, which has unphysical "runaway" solutions (the particle accelerates itself). And when one avoids these solutions by imposing asymptotic conditions [3, 4], one ends up with pre-accelerated solutions [1] (acceleration starts before the force is applied).

It seems therefore worthwhile to have a simple model of a radiating system, which is mathematically manageable, so that one can study radiation reaction for non-trivial motions, and, in particular, to show explicitly that the usual retarded fields lead to consistent and causal solutions. In this article we discuss such a simple model: It is a sphere which is rigid in a geometric sense, which also carries a *rigid* surface charge and rotates around a fixed axis*. A valid objection is that the assumption of rigidity violates relativity, since it implies that a force acting on any part of the sphere will cause all other parts to move instantaneously. But our assumption is not inconsistent with electrodynamics and it makes the problem manageable. The present model does not suffer from infinities, and turns out to be a linear system and therefore relatively simple to work with. We use retarded fields throughout and obtain physically reasonable solutions, which conserve energy and momentum. If a mechanical moment-of-inertia I_{mech} is added to the pure electromagnetic system, the equation of motion will have no runaway solutions as long as $I_{mech} \geqq 0$, and develops runaway solutions only for $I_{mech} < 0$, as shown in III [6].

In Sect. 2 the retarded electromagnetic fields are calculated, first for oscillations, then for constant rotation and finally for arbitrary rotations. The torque of these fields on the sphere, the self-torque, is calculated in Sect. 3: It is a convolution integral that depends linearly and causally on the angular acceleration. In Sect. 4 this torque is used to set up the equation of motion for the sphere, which has, in addition to the charge, a mechanical moment of inertia I_{mech} and also a restoring force $-\kappa\phi$, where ϕ is the angular displacement. One obtains an integrodifferential equation in $\phi(t)$. This equation has an infinite number of homogeneous solutions, whose interesting properties are discussed in Sect. 5. The inhomogeneous solution is given in Sect. 6: It is a convolution integral of the external torque and a formally-known kernel; the dependence on the external torque is linear and causal for $I_{mech} \geqq 0$. In Sect. 7 the solutions for δ-function and step-function external pulses are discussed. In Sect. 8 our techniques are analysed, and it is concluded that all rotationally invariant objects, rotating around their axis of symmetry, will also lead to linear causal systems similar to the specific model in this paper. Finally, a summary is given in Sect. 9.

2. Radiation Fields

a) Oscillations

Let a sphere of radius R carry a rigid homogeneous surface charge σ and oscillate around a fixed axis with the angular speed**

$$u_\omega(t) = U(\omega)\, e^{-i\omega t}\, \hat{n}$$

* The spherical shell model has a long history (see Erber's review article, especially pages 380–381). It turns out for example that already in 1904, Sommerfeld [5] had calculated the oscillatory eigenmodes of the system. (See Sect. 5 of the present article.) See also Schott [5]. But as far as we know, the fields for general rotations have not been calculated explicitly before.

** For convenience, speeds, fields and forces will be written as complex quantities. The physical quantities are equal to the real parts of the complex ones.

("^" will always be used to denote unit vectors $\hat{\boldsymbol{u}} \equiv \boldsymbol{u}/|\boldsymbol{u}|$). The corresponding current*

$$\begin{aligned} \boldsymbol{j}(\boldsymbol{r}, t) &= \rho(\boldsymbol{r})\, \boldsymbol{v}(\boldsymbol{r}, t) \\ &= \sigma\,\delta(r-R)[\boldsymbol{u}(t) \times \boldsymbol{r}] \end{aligned} \tag{2.1}$$

leads to the following *retarded* vector potential (in Gaussian units)

$$\begin{aligned} A_\omega(\boldsymbol{r}, t) &= \frac{1}{c}\int \frac{\boldsymbol{j}(\boldsymbol{r}', t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d^3 r' \\ &= \frac{\sigma R^3}{c}\int \frac{[\boldsymbol{u}_\omega(t-|\boldsymbol{r}-\boldsymbol{r}'|/c) \times \hat{\boldsymbol{r}}']}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d\Omega' \\ &= \frac{eR}{4\pi}\left[\boldsymbol{u}_\omega(t) \times \int \hat{\boldsymbol{r}}'\, \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|/c}}{|\boldsymbol{r}-\boldsymbol{r}'|}\, d\Omega'\right] \\ &= \frac{ekR}{c}\,[\boldsymbol{u}_\omega(t) \times \hat{\boldsymbol{r}}]\, j_1(kr_<)\, h_1(kr_>) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} r_< &\equiv \min(r, R), \qquad r_> \equiv \max(r, R) \\ e &\equiv 4\pi R^2 \sigma, \qquad k \equiv \omega/c \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\ n_1(x) &= -\left(\frac{\cos x}{x^2} + \frac{\sin x}{x}\right) \\ h_1(x) &\equiv i\, h_1^{(1)}(x) = -n_1(x) + i j_1(x) = i\,\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right) \end{aligned} \tag{2.4}$$

are the spherical Bessel, Neumann and Hankel functions.

In deriving (2.2) we made use of the familiar expansion of

$$\mathrm{Exp}\,(ik\,|\boldsymbol{r}-\boldsymbol{r}'|)/|\boldsymbol{r}-\boldsymbol{r}'|$$

in terms of the Legendre polynomials and also of the equality

$$\int \hat{\boldsymbol{r}}'\, P_l(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}')\, d\Omega' = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \delta_{l1}\, \hat{\boldsymbol{r}}. \tag{2.5}$$

From (2.2) we calculate the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Their spherical components, defined by

$$\boldsymbol{E} = E_r \hat{\boldsymbol{r}} + E_\phi \hat{\boldsymbol{\phi}} + E_\theta \hat{\boldsymbol{\theta}}$$

* With this ansatz one assumes that the charges are *distinguishable* and therefore it is meaningful to speak of the rotation of a homogeneously charged rigid shell. A

where

$$\hat{\phi} \equiv \hat{u} \times \hat{r} / |\hat{u} \times \hat{r}| = \hat{u} \times \hat{r} / \sin\theta$$
$$\hat{\theta} = \hat{\phi} \times \hat{r} \tag{2.6}$$

are:

The Fields Inside the Sphere ($r < R$)

$$E_r = E_\theta = B_\phi = 0$$
$$E_\phi = i \frac{e}{c^2} U(\omega)\, \omega X h_1(X) j_1(x) \cdot \sin\theta\, e^{-i\omega t}$$
$$B_r = 2 \frac{e}{c^2} U(\omega)\, \omega X h_1(X) x^{-1} j_1(x) \cdot \cos\theta\, e^{-i\omega t} \tag{2.7a}$$
$$B_\theta = -\frac{e}{c^2} U(\omega)\, \omega X h_1(X) x^{-1} \frac{d}{dx}(x j_1(x)) \cdot \sin\theta\, e^{-i\omega t}.$$

The Fields Outside the Sphere ($r > R$)

$$E_\theta = B_\phi = 0, \quad E_r = \frac{e}{r^2}$$
$$E_\phi = i \frac{e}{c^2} U(\omega)\, \omega X j_1(X) h_1(x) \cdot \sin\theta\, e^{-i\omega t}$$
$$B_r = 2 \frac{e}{c^2} U(\omega)\, \omega X j_1(X) x^{-1} h_1(x) \cdot \cos\theta\, e^{-i\omega t} \tag{2.7b}$$
$$B_\theta = -\frac{e}{c^2} U(\omega)\, \omega X j_1(X) x^{-1} \frac{d}{dx}(x h_1(x)) \cdot \sin\theta\, e^{-i\omega t}$$

where

$$x \equiv kr \quad \text{and} \quad X \equiv kR. \tag{2.8}$$

The fields outside the sphere are pure magnetic dipole radiation fields. Eq. (2.7b) leads immediately to the interesting result that the oscillating fields outside the sphere vanish identically at infinitely many eigenfrequencies $\omega_n \equiv X_n/\tau$ ($\tau \equiv R/c$), where X_n are the zeros of the Bessel function $j_1(X_n) = 0$, or

$$\tan X_n = X_n, \quad n = 1, 2, 3, \ldots. \tag{2.9}$$

This is a famous equation first found by Sommerfeld [5]. For these ω_n the fields inside the sphere need not vanish. Therefore a sphere oscillating with these frequencies could theoretically store electromagnetic energy without any dissipation. Further discussion of these and other modes is given in Sect. 5.

b) Constant Rotation

The fields for a constant (or static) rotation u_{st} follow from Eqs. (2.7) by taking the limit $\omega \to 0$. The electric field will be just the Coulomb field outside the sphere, $\boldsymbol{E} = (e/r^2)\hat{\boldsymbol{r}}$, and the magnetic field will have the form:

$$\boldsymbol{B} = \begin{cases} \dfrac{2\boldsymbol{m}}{R} & \text{for } r < R \quad (2.10\text{a}) \\ \dfrac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} + \boldsymbol{m}}{r^3} & \text{for } r > R \quad (2.10\text{b}) \end{cases}$$

where

$$\boldsymbol{m} \equiv \frac{eR^2}{3c}\,\boldsymbol{u}_{st}. \tag{2.11}$$

The above magnetic field is constant inside the sphere and outside the sphere it corresponds to that of a static magnetic dipole $\boldsymbol{m}$. It is useful for later remarks to give the total magnetic field energy U^B_{st} and the angular momentum $\boldsymbol{L}_{st}$ of these static fields

$$U^B_{st} = \frac{1}{8\pi}\int \boldsymbol{B}^2\, d^3r = \boldsymbol{m}^2/R^3, \tag{2.12}$$

$$\boldsymbol{L}_{st} = \frac{1}{4\pi c}\int_{r>R} [\boldsymbol{r}\times[\boldsymbol{E}\times\boldsymbol{B}]]\, d^3r = \frac{2e\boldsymbol{m}}{3cR}. \tag{2.13}$$

One may check explicitly that 2/3 of the total energy U^B_{st} is contained inside the sphere and only 1/3 of it is outside. Defining an electromagnetic moment-of-inertia I_{el} by the relation

$$\boldsymbol{L}_{st} = I_{el}\,\boldsymbol{u}_{st} \tag{2.14}$$

gives

$$I_{el} = \frac{2e^2}{9c^2}R. \tag{2.15}$$

For constant e, I_{el} vanishes as $R \to 0$, which reflects the fact that a rotating point charge does not radiate. I_{el} will play an important role later, also for oscillating fields. Using (2.12), we can immediately check that

$$U^B_{st} = \tfrac{1}{2} I_{el}\, u^2_{st}. \tag{2.16}$$

c) Arbitrary Rotations

Since the vector potential depends linearly on the current and since our current depends linearly on the angular speed, the vector potential

$\boldsymbol{A}(\boldsymbol{r}, t)$ for an arbitrary motion

$$\boldsymbol{u}(t) = \hat{\boldsymbol{u}}\, u(t) = \hat{\boldsymbol{u}} \int_{-\infty}^{\infty} U(\omega)\, e^{-i\omega t}\, d\omega \tag{2.17}$$

is a superposition of the vector potentials for different oscillatory motions:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} \boldsymbol{A}_\omega(\boldsymbol{r}, t)\, d\omega. \tag{2.18}$$

We shall now calculate $\boldsymbol{A}(\boldsymbol{r}, t)$ explicitly for $r > R$. Substituting (2.2) into (2.18) gives

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}, t) &= \frac{eR}{c^2} [\hat{\boldsymbol{u}} \times \hat{\boldsymbol{r}}] \int_{-\infty}^{\infty} \omega j_1(\omega R/c)\, h_1(\omega r/c)\, U(\omega)\, e^{-i\omega t}\, d\omega \\ &= \hat{\boldsymbol{\phi}} \sin\theta \int_{-\infty}^{\infty} \mathfrak{A}(r, t')\, u(t - t')\, dt' \end{aligned} \tag{2.19}$$

where

$$\mathfrak{A}(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{eR\omega}{c^2} j_1(\omega R/c)\, h_1(\omega r/c)\, e^{-i\omega t}\, d\omega \qquad \text{for } r > R. \tag{2.20}$$

Evaluating this integral using contour integration gives

$$\mathfrak{A}(r, t) = \begin{cases} \dfrac{e}{4R} \left[1 + \dfrac{R^2}{r^2} \left(1 - \dfrac{t^2}{\tau^2}\right)\right] & \text{for } \tau_- < t < \tau_+ \\ 0 & \text{otherwise} \end{cases} \tag{2.21}$$

where

$$\begin{aligned} \tau &\equiv R/c \\ \tau_\pm &= \tau_\pm(r) \equiv (r \pm R)/c. \end{aligned} \tag{2.22}$$

Substituting (2.21) in (2.17) gives

$$A_\phi(\boldsymbol{r}, t) = \sin\theta \int_0^{2\tau} \mathfrak{A}(r, \tau_- + t')\, u(t - \tau_- - t')\, dt'. \tag{2.23}$$

We see that the potential at position $\boldsymbol{r}$ is only affected by the motion of the sphere at the earlier time interval $t - \tau_+ \leqq t' \leqq t - \tau_-$, where the retardation times τ_+ and τ_- are the times required for a signal to reach the observation point $\boldsymbol{r}$ from the farthest and the nearest point on the sphere.

Using (2.23) and $A_\theta = A_r = 0$, we calculate the fields for $r > R$ for an arbitrary rotation $u(t)$:

$$E_r = \frac{e}{r^2}, \quad E_\theta = B_\phi = 0$$

$$E_\phi(r,t) = -\frac{1}{c}\dot{A}_\phi(t) = -\frac{\sin\theta}{c}\int_0^{2\tau} \mathfrak{A}(r, \tau_- + t')\,\dot{u}(t - \tau_- - t')\,dt'$$

$$B_r(r,t) = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} = \frac{2}{r}\cot\theta\, A_\phi \tag{2.24}$$

$$B_\theta(r,t) = -\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r} = -\sin\theta\int_0^{2\tau}\left[\frac{1}{r}\frac{d}{dr}\left(r\mathfrak{A}\left(r, \frac{r}{c} - \frac{R}{c} + t'\right)\right)\right.$$

$$\left.\cdot\, u(t - \tau_- - t') - \frac{1}{c}\mathfrak{A}(r, \tau_- + t')\,\dot{u}(t - \tau_- - t')\right] dt'$$

$$= \sin\theta\int_0^{2\tau}\frac{1}{r^3}(4t' - t'^2/\tau)\,u(t - \tau_- - t')\,dt' - E_\phi$$

where in calculating B_θ, the total derivative with respect to r was taken.

Since

$$\mathfrak{A}(r, \tau_- + t) = \begin{cases} e\left[\dfrac{1}{2r}\left(1 - \dfrac{t}{\tau}\right) + \dfrac{c}{2r^2}\left(t - \dfrac{t^2}{2\tau}\right)\right], & 0 < t < 2\tau \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

the fields emitted at time t fall off radially like $1/r$ and faster, as they propagate in space. Only $E_\phi(r, t + \tau_-)$ and $B_\theta(r, t + \tau_+)$ have terms which fall off like $1/r$, and these depend only on $\dot{u}$ but not on u. These and only these terms are responsible for the irreversible radiation, since their Poynting vector drops off like $1/r^2$, so that *all* the energy entering a spherical shell must leave it again. Since these terms in E_ϕ and B_θ are equal, except for a sign, they contribute a non-negative term to the Poynting vector which is always directed radially outwards. The rest of the Poynting vector has terms that drop like r^{-3} and stronger. These have non-vanishing divergence and thus carry energy that is deposited in space as reversible field energy.

A quantitative account of the above remarks leads to the following expression for the irreversible radiation $\mathfrak{R}(t)$ emitted by the sphere at time t:

$$\mathfrak{R}(t) = \lim_{r\to\infty}\int S_r(r, t + (r - R)/c)\,r^2\,d\Omega$$

$$= \frac{c}{4\pi}\lim_{r\to\infty}\int(\mathrm{Re}\,E_\phi(r, t + \tau_-))^2\,r^2\,d\Omega \tag{2.26}$$

$$= \frac{e^2}{6c}\left[\int_0^{2\tau}\left(1 + \frac{t'}{\tau}\right)\mathrm{Re}(\dot{u}(t - t'))\,dt'\right]^2$$

where $S_r(r, t+\tau_-)$ is the radial component of the Poynting vector at position r and time $t+\tau_-$.

Since

$$\int_0^{2\tau} (1-t'/\tau)\,dt'=0. \tag{2.27}$$

Eq. (2.26) leads to the following interesting result: *the sphere does not emit irreversible radiation if it rotates with constant angular acceleration.*

With the help of the relation

$$\int_0^{2\tau} \left(1-\frac{t'}{2\tau}\right) e^{i\omega t'}\,dt' = -2\tau i\, e^{i\tau\omega} j_1(\tau\omega) \tag{2.28}$$

we can easily calculate the irreversible radiation for an oscillatory motion $u(t)=u_0\, e^{-i\omega t}$:

$$\Re(t)=\left(\frac{2e^2\tau^2}{3c}\right) u_0^2\,\omega^2 j_1^2(\tau\omega)\cos^2[(t-\tau)\omega]. \tag{2.29}$$

Note that $\Re(t)\sim\omega^4$ for small ω and oscillates like $\sin^2\tau\omega$ for $\omega\to\infty$. $\Re(t)$ vanishes indentically for the eigenfrequencies ω_n given by Eq. (2.9), as expected.

3. Self-Torque

The electromagnetic field radiated by the sphere acts back on the sphere with the usual Lorentz force. It can be shown that for a surface charge distribution such as ours, this force depends on the *average* of the fields on both sides of, and immediately outside, the charged surface $\left(\text{for example, } \boldsymbol{E}_{\text{av}}(\boldsymbol{R})=\frac{1}{2}\frac{e}{R^2}\hat{\boldsymbol{r}}+E_\phi(\boldsymbol{R})\hat{\boldsymbol{\phi}}\right)$:

$$\boldsymbol{F}_{\text{self}}=\sigma\boldsymbol{E}_{\text{av}}(\boldsymbol{R})+\frac{\sigma}{c}[[\boldsymbol{u}(t)\times\boldsymbol{R}]\times\boldsymbol{B}_{\text{av}}(\boldsymbol{R})]. \tag{3.1}$$

This force exerts a total torque on the sphere, which, because of the symmetry of the problem, must be in the direction of the rotation axis:

$$\boldsymbol{d}_{\text{self}}(t)=d_{\text{self}}(t)\,\hat{\boldsymbol{u}}$$

with

$$\begin{aligned} d_{\text{self}}(t) &= \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{d}}_{\text{self}}=\int\hat{\boldsymbol{u}}\cdot[\boldsymbol{R}\times\boldsymbol{F}_{\text{self}}]\,R^2\,d\Omega \\ &= \sigma R^3\int[\hat{\boldsymbol{u}}\times\hat{\boldsymbol{R}}]\cdot\boldsymbol{E}_{\text{av}}\,d\Omega \\ &= -\frac{eR}{4\pi c}\int[\hat{\boldsymbol{u}}\times\hat{\boldsymbol{R}}]^2\,d\Omega\cdot\int_0^{2\tau}\mathfrak{A}(R,t)\,\dot{u}(t-t')\,dt' \\ &= -I_{\text{el}}\int_0^{2\tau}D(t')\,\dot{u}(t-t')\,dt' \end{aligned} \tag{3.2}$$

where

$$D(t)=\frac{3e}{c}\,\mathfrak{A}(R,t)=\begin{cases}\frac{3}{2\tau}\left(1-t^2/(2\tau^2)\right) & \text{for } 0<t<2\tau \\ 0 & \text{otherwise.}\end{cases} \tag{3.3}$$

The contribution of the magnetic field vanishes, since

$$[\boldsymbol{u}\times\boldsymbol{R}]\cdot[[\boldsymbol{u}\times\boldsymbol{R}]\times\boldsymbol{R}]=0.$$

This insures that the self-torque is linear in u, since E is linear in u. A detailed discussion of linearity is given in Sect. 8.

Eq. (3.3) is an important result. Unlike in the Lorentz model of an extended electron, the self-torque here is (exactly!) linear in u and it is given by a simple integral expression instead of an infinite series. The integral also has a simple intuitive interpretation: the self-torque at time t depends only on the angular *acceleration* of the sphere during the time interval 2τ prior to t; 2τ is the maximum time required for a signal to travel between any two points on the sphere.

d_{self} can be written as a sum over higher derivatives of u, by expanding $u(t-t')$ in powers of t' under the integral (3.2):

$$\begin{aligned} d_{\text{self}}(t)&=-I_{\text{el}}\sum_{n=1}^{\infty}\alpha_n\frac{d^n u(t)}{dt^n} \\ &=-I_{\text{el}}\left(\dot{u}-\frac{2}{5}\tau^2\dddot{u}+\frac{\tau^3}{9}\ddddot{u}+\cdots\right)\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\alpha_n&=\frac{3}{2\tau}\frac{(-1)^n}{n!}\int_0^{2\tau}(1-t^2/2\tau^2)\,t^n\,dt \\ &=(-1)^{n+1}\frac{3(n-1)(2\tau)^n}{(n+1)!\,(n+3)}.\end{aligned} \tag{3.5}$$

It is important to note however that the above series expansion is not equivalent to the original integral expression, unless the Taylor expansion of $u(t-t')$ has a radius of convergence not smaller than 2τ. Therefore this expansion can not be used for suddenly changing (discontinuous) motions or external torques (see Sect. 7a).

Note that the $\ddot{u}$ term vanishes in our case. In contrast, the corresponding Schott term $\frac{2e^2}{3c^2}\ddot{\boldsymbol{v}}$ is the *only* term responsible for radiation in the non-relativistic equation of motion of the *point* electron.

We need for later use the Fourier transform of d_{self}:

$$D_{\text{self}}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d_{\text{self}}(t)\,e^{i\omega t}\,dt=-I_{\text{el}}\,\psi(\tau\omega)\left(-i\omega\,U(\omega)\right) \tag{3.6}$$

where

$$\psi(X) \equiv 3X j_1(X)\, h_1(X) = \frac{3i}{2X^3}\left(e^{2iX}(X+i)^2 + (X^2+1)\right) \tag{3.7}$$

has the property that $\psi(X) \to 1$ as $X \to 0$.

Note that

$$D_{\text{self}}(\omega)\, e^{-i\omega t} = -[I_{\text{el}}\, \psi(\tau\omega)]\, \dot{u}(t) \tag{3.8}$$

is the self-torque for the oscillatory motion $u(t) = U(\omega)\, e^{-i\omega t}$. Thus $I_{\text{el}}\psi(\tau\omega)$ is the effective moment of inertia for this motion, which for $\omega \to 0$ goes to I_{el}. $D_{\text{self}}(\omega)$ can be calculated directly using the fields (2.7) for the oscillatory motions. And since $d_{\text{self}}(t)$ is linear in $u(t)$, one could have obtained $d_{\text{self}}(t)$ by integrating $D_{\text{self}}(\omega)\, e^{-i\omega t}$ over all ω, without first calculating the general fields (2.25) for arbitrary motions $u(t)$.

4. Equation of Motion

We now write down and solve formally the equation of motion for the *general case*, where the charged sphere has a mechanical moment of inertia I_{mech} in addition to I_{el}, and also a mechanical restoring force $-\kappa\phi$, where ϕ is the angular displacement. The special cases $I_{\text{mech}} = 0$ and/or $\kappa = 0$ are then obtained by considering the limits $I_{\text{mech}} \to 0$ and/or $\kappa \to 0$.

The equation of motion for our general system is

$$I_{\text{mech}}\, \ddot{\phi}(t) - d_{\text{self}}(t) + \kappa\,\phi(t) = d_{\text{ext}}(t) \tag{4.1}$$

where d_{ext} is the external torque. Eq. (4.1) becomes an integro-differential equation in ϕ on substituting for $d_{\text{self}}(t)$ its integral expression (3.2):

$$I_{\text{mech}}\, \ddot{\phi}(t) + I_{\text{el}} \frac{3}{2\tau} \int_0^{2\tau} \left(1 - \frac{(t')^2}{2\tau^2}\right) \ddot{\phi}(t-t')\, dt' + \kappa\,\phi(t) = d_{\text{ext}}(t). \tag{4.2}$$

We solve this equation by taking the Fourier transform of both sides:

$$f(\omega)\, \phi(\omega) = D_{\text{ext}}(\omega) \tag{4.3}$$

with

$$f(\omega) \equiv -\left(I_{\text{mech}} + I_{\text{el}}\, \psi(\tau\omega)\right) \omega^2 + \kappa \tag{4.4}$$

where we used expression (3.6) for $D_{\text{self}}(\omega)$. Dividing formally by $f(\omega)$ gives $\phi(\omega)$ and hence the formal solution $\phi(t)$ of (4.2), by taking the inverse Fourier transform

$$\phi(t) = \frac{-1}{2\pi} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{D_{\text{ext}}(\omega)\, e^{-i\omega t}\, d\omega}{\left(I_{\text{mech}} + I_{\text{el}}\, \psi(\tau\omega)\right)\omega^2 - \kappa} + \text{“h.s.”}. \tag{4.5}$$

The $i\delta$ indicates that the integration path should be *above* and parallel to the real axis. This insures causality, since then all the zeros of $f(\omega)$ (for $I_{\text{mech}} \geqq 0$) will be below the integration path.

"h.s." stands for homogeneous solutions, i.e. solutions of (4.2) for the $d_{\text{ext}}(t) \equiv 0$. We shall see in Sect. 5 that (4.2) has infinite number of homogeneous solutions, and we can take in (4.5) any linear combination of them, as needed for describing the motion of the sphere before the external torque starts to act. The necessity of including homogeneous solutions in (4.5) is clearly seen by noting that $D_{\text{ext}}(\omega) \equiv 0$ for $d_{\text{ext}}(t) \equiv 0$. Hence the integral in (4.5) vanishes indentically for zero external torque, so that in this case $\phi(t)$ can be equal to any linear combination of the homogeneous solutions, as expected. This freedom of adding homogeneous solutions follows, since in obtaining $\phi(\omega)$ we devided by the function $f(\omega)$, which has infinite number of zeros. Each of these zeros corresponds to a homogeneous solution, as we shall see immediately in Sect. 5.

In Sect. 6 the integral in (4.5) will be transformed into a convolution integral involving $d_{\text{ext}}(t)$ and the Green function $G(t)$ of Eq. (4.2).

5. Homogeneous Solutions or Forceless Modes

The homogeneous solutions correspond to possible motions (modes) of the sphere when there is no external torque acting on it. Consequently, they are called *forceless* or *torqueless modes*. In looking for these solutions, the restoring force $-\kappa\phi$ will be treated as part of the system. Besides the forceless modes, there are also "*radiationless modes*", for which the system does not radiate energy. These two types of modes are not always identical: there are forceless modes which radiate energy, and radiationless modes which require an external force to maintain it (see Erber [1])★. Our system provides interesting examples of these modes. We shall first derive the formal expressions for the general homogeneous solutions, and then discuss in details the three cases defined by different values of the parameters I_{mech} and κ.

If $\phi_h(t)$ is a homogeneous solution of (4.2), then its Fourier transform $\phi_h(\omega)$ must satisfy Eq. (4.3) with $D_{\text{ext}}(\omega) = 0$, i.e.

$$f(\omega)\,\phi_h(\omega) \equiv \left[-\left(I_{\text{mech}} + I_{\text{el}}\, 3\tau\omega\, h_1(\tau\omega)\, j_1(\tau\omega)\right)\omega^2 + \kappa\right]\phi_h(\omega) = 0 \quad (5.1)$$

where we have written out $\psi(\tau\omega)$ explicitely. Therefore $\phi_h(\omega)$ must vanish identically, except at the isolated (complex) zeros $\omega = \gamma_n$ of $f(\omega)$, where we may formally write

$$\phi_n(\omega) = \begin{cases} c_n\,\delta(\omega - \gamma_n) & \text{if } \gamma_n \text{ is a simple zero} \\ c_n\,\delta(\omega - \gamma_n) - i\,c_n'\,\delta'(\omega - \gamma_n) & \text{if } \gamma_n \text{ is a double zero} \end{cases} \quad (5.2)$$

★ [illegible] and the work-free modes see Erber and Prastein [8] where c_n and c_n' are constants – and so on, if γ_n is a zero of higher order. Taking formally the Fourier transform, one gets the homogeneous solutions

$$\phi_n(t)=\begin{cases} c_n e^{-i\gamma_n t} & \text{if } \gamma_n \text{ is a simple zero} \\ (c_n+c_n' t)\, e^{-i\gamma_n t} & \text{if } \gamma_n \text{ is a double zero.} \end{cases} \tag{5.3}$$

This result can be directly verified and intuitively understood by noting the physical meaning of $f(\omega)$: substituting $\phi^e(t)\equiv e^{-i\omega t}$ (ω may be complex) in Eq. (4.2) gives the external torque necessary for maintaining the exponential motion $\phi^e(t)$:

$$d^e_{\text{ext}}(t)=f(\omega)\, e^{-i\omega t}$$

which shows that the external torque vanishes for $\omega=\gamma_n$. The external torque corresponding to $\phi^{te}(t)\equiv t e^{-i\omega t}=i\dfrac{d}{d\omega} e^{-i\omega t}$ can be obtained directly from $d^e_{\text{ext}}(t)$ by noting that d_{ext}, given in (4.2), is equal to a linear operator applied onto $\phi(t)$. This operator commutes with the differential operator $i\dfrac{d}{d\omega}$, where ω acts here just as a parameter. Hence

$$d^{te}_{\text{ext}}(\omega, t)=i\frac{d}{d\omega} \quad d^e_{\text{ext}}(\omega, t)=[f(\omega)\, t+i f'(\omega)]\, e^{-i\omega t}. \tag{5.5}$$

This equation shows that $\phi(t)=t e^{-i\gamma_n t}$ can not be a solution, unless $\phi(t)=e^{-i\gamma_n t}$ is also a solution, since in order for d^{te}_{ext} to vanish identically, both $f(\gamma_n)$ and $f'(\gamma_n)$ must vanish, and consequently d^e_{ext} must also vanish.

We shall now discuss the following three cases separately:

a) The Pure Electromagnetic System ($I_{\text{mech}}=\kappa=0$)

For this case, we seek the zeros of

$$f_0(\omega)\equiv -I_{\text{el}}\,\psi(\tau\omega)\,\omega^2=-I_{\text{el}}\, 3\tau\omega^3\, h_1(\tau\omega)\, j_1(\tau\omega)=0. \tag{5.6}$$

We see immediately that $\omega=0$ is a double zero, and (5.3) gives the expected homogeneous solution

$$\phi_0(t)=\phi_0+u_0 t \tag{5.7}$$

where ϕ_0 and u_0 are constant angular displacement and speed. The constant rotation $u(t)=u_0$ is a trivial example of a forceless and radiationless mode.

Next we find that

$$f_0(\omega_n)=0 \qquad n=1, 2, \ldots, \infty \tag{5.8}$$

where ω_n are the zeros of the Bessel function $j_1(\tau\omega)$. They correspond to the non-decaying oscillatory modes,

$$\phi_n(t)=c_n\, e^{-i\omega_n t} \tag{5.9}$$

which were already mentioned in Sect. 2. They are forceless and radiationless modes, since the oscillating fields vanish identically outside the sphere for these eigenfrequencies.

Finally, the most interesting mode corresponds to

$$f_0(-i/\tau)=0 \tag{5.10}$$

where $\omega=-i/\tau$ is the zero of the Hankel function $h_1(\tau\omega)$. It is a *decaying non-oscillatory* mode:

$$\phi_{-1}(t)=c_{-1}\, e^{-t/\tau}. \tag{5.11}$$

Although this mode is torqueless, it is nevertheless a radiating mode. Where does the energy come from? Since for the distant past $t\to-\infty$ the angular speed is infinite, the initial conditions at any negative and large t correspond to non-zero fields and consequently to an electromagnetic field energy stored initially in the space surrounding the sphere. In fact the total energy over all space is infinite at any time t, since the fields at far-away regions $r \gg R$ are determined by the retarded angular speed at $t_{\text{ret}} \simeq t-\dfrac{r}{c}$, and since this speed increases exponentially as t_{ret} decreases. Because of radiation, the energy density at any fixed position r decreases exponentially and with it the speed decreases too. But one wonders how the sphere "*knows*" that it has to slow down, even though it does not "*feel*" any torque? *In this mode the sphere acts, through its motion, simply as a transducer or "an expedition agent" for transferring locally-stored energy to infinity.* This transducer effect does not contradict any generally accepted principles (compare Synge [9]).

b) Sphere with Mechanical Mass $I_{\text{mech}} \neq 0$, but $\kappa=0$

Here $f(\omega)$ reduces to

$$f_1(\omega)\equiv-[I_{\text{mech}}+I_{\text{el}}\,3\tau\omega\, h_1(\tau\omega)\, j_1(\tau\omega)]\,\omega^2=0. \tag{5.12}$$

Clearly (5.7) remains a homogeneous solution, also in this case. However, the non-decaying oscillatory modes become decaying modes, as the zeros ω_n of $f_0(\omega)$ move in the lower half of the ω-plane and become complex zeros

$$\pm\omega_n\to\gamma_n^{\pm}=\pm\Omega_n-i\tfrac{1}{2}\Gamma_n \tag{5.13}$$

so that

$$\phi_n^{\pm}(t)=c_n\, e^{\mp i\Omega_n t}\, e^{-\frac{1}{2}\Gamma_n t} \tag{5.14}$$

where the damping factor or width Γ_n is positive, as shown in the Appendix of paper II. These modes are still torqueless, by definition, but they radiate energy to infinity. The radiated energy are again taken from the energy stored at $t=-\infty$.

As $I_{\rm mech}>0$ increases, the zero $\gamma_{-1}=-i/\tau$ moves downwards along the negative imaginary axis, giving rise to a larger decay constant of original non-oscillatory decaying mode.

c) Harmonic Oscillator ($\kappa \neq 0$)

In this case we look for the zeros of the general $f(\omega)$, which is given in Eq. (5.1). Clearly, for the harmonic oscillator, the solution $\phi(t)=\phi_0+u_0 t$ is no longer possible. The double zero at $\omega=0$, corresponding to this solution for $\kappa=0$, now becomes two complex zeros $\gamma_0^\pm=\pm\Omega_0-i\frac{1}{2}\Gamma_0$.

Let us estimate Ω_0 and Γ_0 for $\Omega_0\tau\ll 1$, i.e. for a small spring constant κ. Near the origin $f(\omega)$ can be approximated by

$$\begin{aligned} f(\omega) &\simeq -\left[I_{\rm mech}+I_{\rm el}\left(1+i(\tau\omega)^3/3\right)\right]\omega^2+\kappa \\ &= -I_{\rm tot}\left(\omega^2-\kappa/I_{\rm tot}+i(I_{\rm el}/I_{\rm tot})\,\tau^3\,\omega^5/3\right) \end{aligned} \tag{5.15}$$

where $I_{\rm tot}\equiv I_{\rm mech}+I_{\rm el}$.

We used $\psi(x)\simeq 1+ix^2/3$, taking only the first leading term of the real and the imaginary parts of ψ into account. Near the zeros $\gamma_0^\pm$, $f(\omega)$ must have the form

$$\begin{aligned} f(\omega) &\simeq \mathrm{const}\left[\omega-(\Omega_0-i\tfrac{1}{2}\Gamma_0)\right]\left[\omega-(-\Omega_0-i\tfrac{1}{2}\Gamma_0)\right] \\ &= \mathrm{const}\left((\omega^2-\Omega_0^2-\tfrac{1}{4}\Gamma_0^2)+i\Gamma_0\,\omega\right). \end{aligned} \tag{5.16}$$

Comparing (5.15) and (5.16) and using that $\Gamma_0\ll\Omega_0$, we get

$$\Omega_0\simeq\left(\frac{\kappa}{I_{\rm mech}+I_{\rm el}}\right)^{\frac{1}{2}}=\begin{cases}(\kappa/I_{\rm el})^{\frac{1}{2}} & \text{for } I_{\rm mech}=0\\ \omega_0-\frac{1}{2}(I_{\rm el}/I_{\rm mech})\,\omega_0 & \text{for } I_{\rm mech}\gg I_{\rm el},\end{cases} \tag{5.17}$$

$$\Gamma_0\simeq\frac{I_{\rm el}}{I_{\rm tot}}\,\frac{\tau^3}{3}\,\Omega_0^4\simeq\begin{cases}\dfrac{\tau^3}{3}\left(\dfrac{\kappa}{I_{\rm el}}\right)^2 & \text{for } I_{\rm mech}=0\\[2ex] \dfrac{I_{\rm el}}{I_{\rm mech}}\,\dfrac{\tau^3}{3}\,\omega_0^4 & \text{for } I_{\rm mech}\gg I_{\rm el}\end{cases} \tag{5.18}$$

where

$$\omega_0\equiv(\kappa/I_{\rm mech})^{\frac{1}{2}} \tag{5.19}$$

is the eigenfrequency of the pure mechanical oscillator. Because of radiation, the system will behave for $I_{\rm el}\ll I_{\rm mech}$ like a damped harmonic oscillator with a damping coefficient Γ_0 which is proportional to ω_0^4 and have a negative level shift $\Delta\omega\equiv\Omega_0-\omega_0<0$

6. Inhomogeneous Solutions

The formal solution (4.4) gives $\phi(t)$ as the Fourier transform of a product of two functions $1/f(\omega)$ and $D_{\text{ext}}(\omega)$. Therefore $\phi(t)$ can be written as the convolution of the Fourier transforms of these functions, i.e.

$$\phi(t)=\int_{-\infty}^{\infty} G(t-t')\, d_{\text{ext}}(t')\, dt' + \text{"h.s."} \tag{6.1}$$

where the "Kernel" $G(t)$ is given by

$$G(t)=\frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{e^{-i\omega t}}{f(\omega)}\, d\omega = \frac{-1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta} \frac{e^{-i\omega t}\, d\omega}{(I_{\text{mech}}+I_{\text{el}}\psi(\tau\omega))\omega^2-\kappa}. \tag{6.2}$$

By comparing (4.4) and (6.2) we see that $G(t)$ corresponds to the angular displacement $\phi(t)$ for $D_{\text{ext}}(\omega)\equiv 1$ or to the external torque

$$d_{\text{ext}}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega = \delta(t).$$

Hence, G is the Green function of Eq. (4.2), i.e. G is the solution of the following equation:

$$I_{\text{mech}}\ddot{G}(t)+I_{\text{el}}\int_0^{2\tau} D(t')\, \dot{G}(t-t')\, dt' + \kappa G(t) = \delta(t). \tag{6.3}$$

Since for $G(t)$ the external torque $d_{\text{ext}}(t)$ vanishes everywhere except for $t=0$, we expect $G(t)$ to be a particular linear combination of the homogeneous solutions. The coefficients of this linear combination are determined by the residues of the poles of $1/f(\omega)$, when the integral (6.2) is evaluated by contour integration.

For $t<0$ we close the path $\omega=x+i\delta\ (-\rho<x<\rho)$ by a semi-circle in the upper half-plane and get $G(t)=0$, whereas for $t>0$ we close it by a semi-circle in the lower half plane and get the sum of the residues. One can show that the contribution of these semi-circle vanishes for infinite radii. And although $1/f(\omega)$ has infinite number of poles extending to $|\omega|\to\infty$, the contour integration poses no difficulties, since the sum of the residues for all poles outside the semi-circle approaches zero as its radius ρ goes to infinity. For the three cases, we get

$$G(I_{\text{mech}}=\kappa=0;\, t)=\theta(t)\frac{1}{I_{\text{el}}}\left[\frac{2}{3}\sum_{n=1}^{\infty}\frac{\sin\omega_n t}{\omega_n}+\frac{\tau}{3}e^{-t/\tau}+t\right], \tag{6.4a}$$

$$G(I_{\text{mech}}\neq 0,\, \kappa=0;\, t)$$
$$=\theta(t)\left[\sum^{\infty}\alpha_n\sin(\beta_n+\Omega_n t)\, e^{-\frac{1}{2}\Gamma_n t}+\alpha_{-1}e^{-t/\tau_{-1}}+\frac{t}{[\text{illegible}]}\right] \tag{6.4b}$$

and

$$G(\kappa \neq 0;\, t) = \theta(t) \left[\sum_{n=1}^{\infty} \alpha'_n \sin(\beta'_n + \Omega'_n t)\, e^{-\frac{1}{2}\Gamma'_n t} + \alpha'_{-1}\, e^{-t/\tau'_{-1}} + \alpha_0 \sin(\beta_0 + \Omega_0 t)\, e^{-\frac{1}{2}\Gamma_0 t} \right] \tag{6.4c}$$

where $\theta(t)$ is the usual step function

$$\theta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases} \tag{6.5}$$

The residues for $I_{\text{mech}} = \kappa = 0$ can be evaluated analytically, whereas for $I_{\text{mech}} \neq 0$ only the coefficient of the linear term can be obtained analytically; to calculate the constants α_n and β_n and the poles' positions $(\pm \Omega_n - i\frac{1}{2}\Gamma_n)$, one needs numerical calculation. The same is true for the primed constants.

For $0 < t < 2\tau$ and $I_{\text{mech}} = \kappa = 0$, $G(t)$ has the analytic form

$$\begin{aligned} G(I_{\text{mech}} = \kappa = 0;\, t) &= \theta(t) \frac{\tau}{3I_{\text{el}}} (e^{t/\tau} + e^{-t/\tau}) \\ &= \theta(t) \frac{2\tau}{3I_{\text{el}}} \cosh(t/\tau) \qquad \text{for } 0 < t < 2\tau. \end{aligned} \tag{6.6}$$

This follows from (6.4a) by using the equality

$$\frac{\tau}{3} e^{t/\tau} = 2 \sum_{n=1}^{\infty} \frac{\sin \omega_n t}{\omega_n} + t \qquad \text{for } 0 < t < 2\tau. \tag{6.7}$$

This equality follows by rewriting $G(t)$ as follows:

$$\begin{aligned} G(t) &= \frac{1}{2\pi} \int \frac{e^{-i\omega t}}{f(\omega)}\, d\omega \\ &= \frac{1}{2\pi} \int \frac{e^{-i\omega(2\tau + t)}}{f(\omega)\, e^{-2i\omega\tau}}\, d\omega \\ &\equiv \frac{1}{2\pi} \int \frac{e^{-i\omega t'}}{g(\omega)}\, d\omega \end{aligned} \tag{6.8}$$

and noting that the function $f(\omega)$ has a special structure

$$\begin{aligned} f(\omega) &= -\big(I_{\text{mech}} + I_{\text{el}}\, \psi(\tau\omega)\big)\,\omega^2 + \kappa \\ &= -(I_{\text{mech}} + I_{\text{el}}) \frac{3i}{2(\tau\omega)^3} \big(e^{i2\tau\omega}(\tau\omega + i)^2 + (\tau\omega)^2 + 1\big)\,\omega^2 + \kappa \end{aligned} \tag{6.9}$$

so that $g(\omega)$ and $f(\omega)$ obey similar estimates for $|\omega|\to\infty$. This enables us to close the integration path by a semi-circle in the upper half-plane for $t<0$ and simultaneously in the lower half plane for $t'>0$, that is, for $t>-2\tau$. The first case gives zero and the second gives the r.h.s. of Eqs. (6.4), but without the θ-function. This in turn leads to Eq. (6.7) by substituting $t\to -t$.

For $0<t<\tau$, Eq. (6.7) follows also from the so-called Fourier-Bessel expansion [10] of the function $e^{t/\tau}/\sqrt{t}$. One can check by explicit integration, that the expression (6.6) satisfies the integrodifferential equation (4.2) for $G(t)$ in the whole interval $0<t<2\tau$, but fails to do so for $t>2\tau$, showing that the equality (6.1) is invalid outside the indicated interval.

Because of the $\theta(t)$-function in G, we can rewrite (6.1) in a form that exhibits causality explicitely.

$$\begin{aligned}\phi(t)&=\int_{-\infty}^{t} G(t-t')\,d_{\rm ext}(t')\,dt'+\text{“h. s.”}\\ &=\int_{0}^{\infty} G(t')\,d_{\rm ext}(t-t')\,dt'+\text{“h. s.”}.\end{aligned}\tag{6.10}$$

Differentiating with respect to t gives

$$u(t)=\int_{0}^{\infty} G(t')\,\dot{d}_{\rm ext}(t-t')\,dt'+\text{time derivatives of the homogeneous solutions of Eq. (4.2).}\tag{6.11}$$

7. Motion under Pulse

a) Sharp $\delta(t)$ – Pulse and the Green Function

We saw that the Green function $G(t)$ can be interpreted as the angular displacement $\phi(t)$ for the sharp external pulse $d_{\rm ext}(t)=\delta(t)$. This interpretation allows us to understand some interesting properties of $G(t)$ from general physical arguments.

Eq. (6.6) shows that $G(t)$ for $I_{\rm mech}=\kappa=0$ jumps from 0 to $2\tau/3I_{\rm el}$ at $t=0$, which means that the angular *displacement* experiences a jump! This seems strange at first: we are familiar with jumps in velocity (e.g. a reflected ball from a wall) for a sharp $\delta(t)$ – type force, but a jump in distance never happens in ordinary mechanical systems. The reason is that for mechanical systems, we always have $d_{\rm ext}=I_{\rm mech}\,\dot{u}$. But in our case things are more complicated: We do have the similar relation ever, *shortly after* a sudden acceleration

$$\dot{u} = \begin{cases} 0 & \text{for } t<0 \\ \neq 0 & \text{for } t>0 \end{cases} \tag{7.1}$$

i.e. for $0<t\ll 2\tau$, we get using (3.2) and (3.3)

$$\begin{aligned} -d_{\text{self}}(t) &= I_{\text{el}} \int_0^{2\tau} D(t')\,\dot{u}(t-t')\,dt' \\ &= I_{\text{el}} \int_0^{t} D(t')\,\dot{u}(t-t')\,dt' \\ &\simeq \frac{8I_{\text{el}}}{2\tau} \int_0^{t} \dot{u}(t-t')\,dt' = \frac{3I_{\text{el}}}{2\tau}\,[u(t)-u(0^-)] \qquad 0 \leqq t \ll 2\tau. \end{aligned} \tag{7.2}$$

So that for $d_{\text{ext}}(t)=\delta(t)$, $u(t)$ has a $\delta(t)$ term and hence $\phi(t)$ has a jump!

One can understand this jump in $\phi(t)$ even more clearly from angular momentum conservation, which follows by integrating the above formulae. A $\delta(t)$ external torque imparts to the system a finite angular momentum, namely $\int \delta(t)\,dt=1$. Since $L_{\text{mech}}=I_{\text{mech}}\,u$ is assumed to hold by definition (rigid body), u can only have a finite jump for mechanical systems, i.e. for $I_{\text{mech}} \neq 0$. On the other hand, (7.1) gives

$$L_{\text{el}} \simeq (3I_{\text{el}}/2\tau)\,\phi(t) \qquad \text{for } 0<t\ll 2\tau \tag{7.3}$$

for $\phi(t)\equiv 0$ for $t<0$. Hence, ϕ must jump with $L_{\text{ext}}(t)$, if $I_{\text{mech}}=0$.

According to the above arguments we expect no jump in $G(t)$ if $I_{\text{mech}} \neq 0$. This is indeed the case: In a way similar to that which led to Eq. (6.7) one can show that $\lim_{t\to 0^+} G(I_{\text{mech}} \neq 0, t)=0$.

By the way, we would not obtain a jump in $\phi(t)$, if instead of the integral expression (3.2) for d_{self}, we would use its series expansion (3.4) and breaks it off after a finite number of terms: For

$$d_{\text{self}} = \sum_{n=1}^{N} \alpha_n \frac{d^n u}{dt^n} \quad \text{and} \quad d_{\text{ext}}=\delta(t),$$

only the highest derivative $d^N u/dt^N$ will be proportional to $\delta(t)$ and the other $\dfrac{d^n u}{dt^n}$ $(n<N)$ will be smoother.

Angular momentum conservation also determines the final *constant* speed u_f. This asymptotic speed appears as the coefficient of the linear term in t in the expressions (6.4a) and (6.4b) for $G(t)$: u_f is equal to $1/I_{\text{el}}$ and $1/(I_{\text{el}}+I_{\text{mech}})$ respectively. This value is expected from angular momentum conservation, since the applied external torque must show up as mechanical and electromagnetic angular momentum, so that

$$L_{ext}=L_{mech}+L_{el}=(I_{mech}+I_{el})\,u_f=I_{tot}\,u_f. \tag{7.4}$$

The relation $L_{el}=I_{el}\,u_f$ follows for two reasons: a) no angular momentum is carried away by the irreversible radiation, and b) although for $I_{mech}=0$ the oscillating modes survive for $t\to\infty$, they nevertheless do not contribute to L_{el}; these modes can not carry electromagnetic angular momentum, since otherwise their angular momentum would also oscillate, which is impossible without an external torque.

Finally, for $\kappa\neq 0$, u_f must be zero, since eventually $\phi(t)$ has to come back to the origin. Therefore, the final mechanical and electromagnetic angular momenta vanish for $t\to\infty$. And since the system does not radiate angular momentum to infinity, it follows that the external angular momentum must go into the "spring" which is responsible for the restoring torque $-\kappa\phi$. This leads to the following "sum rule"

$$\int_0^\infty \kappa\phi(t)\,dt=\kappa\int_0^\infty G(\kappa\neq 0,t)\,dt=1. \tag{7.5}$$

b) Step-Function Pulse

We now apply an external torque of the form

$$d_{ext}(t)=(L/T)[\theta(t)-\theta(t-T)]=\begin{cases}L/T & \text{for } 0<t<T\\ 0 & \text{otherwise}\end{cases} \tag{7.6}$$

to the case $I_{mech}\neq 0$, $\kappa=0$. Substituting

$$\dot{d}_{ext}(t)=(L/T)[\delta(t)-\delta(t-T)] \tag{7.7}$$

in Eq. (6.11), gives

$$u(t)=u_i+(L/T)[G(t)-G(t-T)] \tag{7.8}$$

where the homogeneous solution $u(t)=u_i$ for $t<0$ was chosen. For fixed L and $T\to 0$, Eq. (7.8) reduces to $u(t)=u_i+L\dot{G}(t)$, which is the speed for a δ-function pulse, as expected.

Let us write (7.8) explicitely

$$u(t)=\begin{cases} u_i & \text{for } t<0\\ \left(u_i+\dfrac{L}{I_{tot}}\dfrac{t}{T}\right)+\dfrac{L}{T}\left(\sum\limits_{n=1}^{\infty}\alpha_n\sin(\delta_n+\Omega_n t)\,e^{-\frac{1}{2}\Gamma_n t}+\alpha_{-1}\,e^{-\frac{1}{2}\Gamma_{-1}t}\right) & \text{for } 0<t<T\\ \left(u_i+\dfrac{L}{I_{tot}}\right)+\dfrac{L}{T}\left(\sum\limits_{n=1}^{\infty}\alpha_n[\sin(\delta_n+\Omega_n t)-\sin(\delta_n+\Omega_n(t-T))\right.\\ \quad\left.\cdot\, e^{\frac{1}{2}\Gamma_n T}]\,e^{-\frac{1}{2}\Gamma_n t}+\alpha_{-1}(1-e^{\frac{1}{2}\Gamma_{-1}T})\,e^{-\frac{1}{2}\Gamma_{-1}t}\right) & \text{for } t>T.\end{cases} \tag{7.9}$$

We see that while the constant torque is acting, the speed has a term which increases linearly with t. For $t \to \infty$ the asymptotic speed is $u_f = u_i + (L/I_{tot})$, again in accordance with angular momentum conservation. By the way, this conservation supports the idea that static electric and magnetic fields do carry angular momentum in accordance with the Poynting theory, i.e. there is an energy current flowing all the time around the axis of the sphere, even though the fields are static.

Let us use the above example to demonstrate how the external work done on the sphere goes partly into irreversible radiation and partly into reversible change in kinetic energy:

$$W = \int_0^T d_{ext}(t)\, u(t)\, dt = W_{rad} + W_{kin} \tag{7.10}$$

where

$$W_{rad} = \left(\frac{L}{T}\right)^2 \int_0^T \left(\sum_n \alpha_n \sin(\delta_n + \Omega_n t)\, e^{-\frac{1}{2}\Gamma_n t} + \alpha_{-1}\, e^{-\frac{1}{2}\Gamma_{-1} t}\right) dt$$

$$W_{kin} = \frac{L}{T} \int_0^T \left(u_i + \frac{L}{I_{tot}} \frac{t}{T}\right) dt = L\left(u_i + \frac{L}{2I_{tot}}\right)$$

$$= \frac{1}{2I_{tot}} (L_f - L_i)(L_f + L_i) = \frac{1}{2I_{tot}} (L_f^2 - L_i^2).$$

The term W_{rad} is quadratic in L and is always positive. It goes into irreversible radiation to infinity, and can not be recovered. In contrast, W_{kin} goes partly in the mechanical kinetic energy and the rest is stored in the form of static magnetic field energy. W_{kin} has a term which is linear in L and u_i, so W_{kin} can be positive or negative. If, for example, we apply a second pulse which is opposite but otherwise identical to the first, at a much later time $t \gg T$ (so that enough time has lapsed for the speed to reach its asymptotic constant value), it will do the work $W_{rad} - W_{kin}$; we have to do the same amount of work for radiation, but we gain back the work spent by the first pulse into kinetic energy, as the static magnetic field collapses back with the angular speed to its original value at $t = 0$.

8. Other Simple Linear Models

Our model has two important properties that simplified setting up and solving the equation of motion:

a) the possibility of expressing the fields, and thus the self-torque as simple convolution integrals with known kernels and

b) The linearity of the system.

We discuss below the origin of these two features to understand our model better and to find out other models, for which the same mathematical techniques are applicable.

a) The Convolution Integral

The main mathematical step, which enabled us to get a simple closed expression for the potential A is essentially the transformation of a retarded integral over 3-*space*.

$$A_i(\boldsymbol{r}, t)=\int \frac{\rho(\boldsymbol{r}')\,\alpha_i(\boldsymbol{r}')\,u(t-|\boldsymbol{r}-\boldsymbol{r}'|/c)}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3 r' \tag{9.1}$$

into a convolution integral over *time*

$$A_i(r, t)=\int_{-\infty}^{t} \mathfrak{A}_i(r, t-t')\,u(t')\,dt'. \tag{9.2}$$

This was done by taking the Fourier-transform of $u(t-|\boldsymbol{r}-\boldsymbol{r}'|/c)$, which allows us to separate out the retardation time $|\boldsymbol{r}-\boldsymbol{r}'|/c$ as an exponential factor $\exp(i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c)$. This separates time and space into two factors, so we can carry out the integration over space and obtain a function of $\boldsymbol{r}$ and ω only*:

$$\begin{aligned} A_i(\boldsymbol{r}, t)&=\int \left(\int \frac{\rho(\boldsymbol{r}')\,\alpha_i(\boldsymbol{r}')\,e^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,d^3 r'\right) U(\omega)\,e^{-i\omega t}\,d\omega \\ &=\int \mathfrak{A}_i(\boldsymbol{r}, \omega)\,U(\omega)\,e^{-i\omega t}\,d\omega. \end{aligned} \tag{9.3}$$

The last form is easily transformed into a convolution integral, since the integral is a product of two functions of ω. Furthermore, the convolution integral must have the causal form (9.2), since

$$\begin{aligned} \mathfrak{A}_i(r, t)&=\int \frac{\rho(\boldsymbol{r}')\,\alpha_i(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,e^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c}\,d^3 r'\,e^{-i\omega t}\,d\omega \\ &=\int \frac{\rho(\boldsymbol{r}')\,\alpha_i(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\delta(t-|\boldsymbol{r}-\boldsymbol{r}'|/c)\,d^3 r \end{aligned} \tag{9.4}$$

vanishes for $t<0$, because of the δ-function.

The above analysis shows that the fields can always be written as convolution integrals, if the current can be decomposed into two factors, one that depends only on space and the other only on time:

$$j_i(\boldsymbol{r}, t)=a_i(\boldsymbol{r})\,b_i(t). \tag{9.5}$$

* This procedure was first utilized by Schott [11].

Such decomposition is possible for all rotationally invariant objects, rotating around their axis of symmetry.

b) Linearity of the System

Our equation of motion in $\phi(t)$ is linear, because of the following three reasons:

1. The charges are *rigid* and rotate around a fixed axis. Because of this, the magnetic field does not contribute to the component of the torque in the direction of the rotation axis:

$$\begin{aligned} \boldsymbol{D}\cdot\hat{\boldsymbol{u}} &= \int \hat{\boldsymbol{u}}\cdot[\boldsymbol{r}\times\boldsymbol{F}(\boldsymbol{r})]\,d^3r \\ &= \int \rho(\boldsymbol{r})[\hat{\boldsymbol{u}}\times\boldsymbol{r}]\cdot(\boldsymbol{E}(\boldsymbol{r})+[[\boldsymbol{u}\times\boldsymbol{r}]\times\boldsymbol{B}(\boldsymbol{r})])\,d^3r \\ &= \int \rho(\boldsymbol{r})[\hat{\boldsymbol{u}}\times\boldsymbol{r}]\cdot\boldsymbol{E}(\boldsymbol{r})\,d^3r. \end{aligned} \tag{9.6}$$

Note that this result is true for *any* charge distribution $\rho(\boldsymbol{r})$ and not just for rotationally invariant $\rho(\boldsymbol{r})$. It is also true for any magnetic field and not just the self-field. In systems where the term $\rho[\boldsymbol{v}\times\boldsymbol{B}_{\text{self}}]$ contributes, the self force must have at least quadratic terms in $\boldsymbol{v}$ and its derivatives, since $\boldsymbol{B}_{\text{self}}$ itself depends on the velocity.

2. The electric field E_{self} is linear in $u(t)$, since $\rho(\boldsymbol{r})$ is time-independent in our model, which makes the current $\boldsymbol{j}$ linear in $u(t)$.

3. By using non-relativistic dynamics, one automatically neglects non-linear effects in v, such as Lorentz contraction or the relativistic change of the moment of inertia due to the increase of the mass by the factor $(1-(v/c)^2)^{-\frac{1}{2}}$, etc.

We conclude that all rotationally invariant objects, rotating around their axis of symmetry, must have the two simplifying properties (a) and (b) of our model, and can therefore serve similarly as simple models for studying radiation reaction or more generally as examples of linear systems. These objects include spheres, cylinders, and toroids with rigid surface- or space-charges.

9. Summary

We have presented a simple model based on assuming a) rigid charge distribution b) the validity of Maxwell's equations c) that the self-force is due to the retarded fields and d) non-relativistic dynamics.

The self-torque is a convolution integral, which depends linearly on the previous motion. It can be written as an infinite series over the acceleration and higher derivatives, but this series expansion is useful for calculations only for smooth motions, as discussed in Sections 3 and 7.

Our model led to a linear system, where the response $\phi(t)$ depends linearly on the external torque. By the way, this means that one can theoretically reach speeds greater than that of light by choosing d_{ext} large enough; this fact reflects our use of non-relativistic dynamics.

Our system is causal for $I_{\text{mech}} \geqq 0$. But for $I_{\text{mech}} < 0$ it has a runaway solution, as discussed in III.

Similar results are expected from all rotationally invariant objects, rotating around their axis of symmetry, as was shown in Sect. 8.

Some of the interesting peculiarities of our system are: a) it has infinite number of homogeneous solutions (Sect. 5), b) it does not radiate irreversibly for constant acceleration (Sect. 2), and c) for $d_{\text{ext}}(t) = \delta(t)$ the angular *displacement* $\phi(t)$ has a jump (Sect. 7).

The present and the following two articles grew out of my Master's Thesis, which I wrote in 1963 under the supervision of Prof. Jensen. I am grateful to Prof. Jensen and Prof. Stech for having invited me to Heidelberg last summer. This fortunate opportunity enabled me to discuss with Prof. Jensen the problems dealt with in the first two articles. He read only the rough draft of this article and the first part of the second article.

I would like to thank Prof. T. Erber, who encouraged me to pursue the problems dealt with in this thesis and was kind enough to read the articles and make useful suggestions.

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