Reprinted from THE PHYSICAL REVIEW, Vol. 178, No. 5, 2051–2057, 25 February 1969
Printed in U. S. A.

A "Backwards" Two-Body Problem of Classical Relativistic Electrodynamics*

RODNEY D. DRIVER†
Sandia Laboratory, Albuquerque, New Mexico 87115
(Received 8 October 1968)

The two-body problem of classical electrodynamics can be formulated in terms of action at a distance by using the retarded Liénard-Wiechert potentials (or a combination of retarded and advanced potentials). The resulting equations of motion in the retarded case, for example, form a complicated functional delay-differential system. For such equations in the case of one-dimensional motion, as shown in an earlier paper, one can specify rather arbitrary past histories of the particles, and then solve for the future trajectories. Yet it is often assumed or asserted that unique trajectories would be determined by the specification of Newtonian initial data (the positions and velocities of the particles at some instant). Simple examples of delay-differential equations should lead one to doubt this. For example, even if the values of x and *all* its derivatives are given at t_0, the equation $x'(t)=ax(t)+bx(t-\tau)$, with $\tau>0$ and $b\neq 0$, still has infinitely many solutions valid for all t. Nevertheless, under certain special conditions for the electrodynamic equations it is found that instantaneous values of positions and velocities do indeed determine the solution uniquely. The case treated in this paper involves two charges of like sign moving on the x axis, assumed to have been subject only to their mutual retarded electrodynamic interaction for all time in the past. The similar questions of existence and uniqueness for a more general model, e.g., three-dimensional motion or half-retarded and half-advanced interactions, or even for charges with opposite signs remain open.

1. INTRODUCTION

FOR 60 years the two-body problem of classical electrodynamics remained completely unsolved. The equations of motion were expressible in terms of retarded action at a distance by use of the Liénard-Wiechert expressions. However, this led to a complicated functional delay-differential system which had not been treated mathematically. (Some authors also

* Work supported by the U. S. Atomic Energy Commission.

† Present address: Department of Mathematics, University of Rhode Island, Kingston, R. I. 02881.

introduced half-retarded and half-advanced interactions, and some explicitly included "radiation-reaction" terms in the equations of motion.)

In 1960 it was proved[1] that, in the special case of one-dimensional motion under the influence of retarded interactions (without radiation-reaction terms), future trajectories satisfying the equations of motion are uniquely determined by the specification of rather arbitrary past trajectories.

However, it is frequently conjectured, asserted, or implicitly assumed that, as in Newtonian mechanics, the positions and velocities of the particles *at some instant* will determine the state of the system. More precisely, it has been assumed[2,3] that, if the electrodynamic equations of motion are satisfied for all time, then instantaneous values of positions and velocities of the particles will determine their trajectories.

It is the purpose of this paper to show that, since the equations of motion form a functional-differential system, such a property of the trajectories *should not be expected.* However, in at least one special case it does hold.

2. EQUATIONS OF MOTION

We shall follow Synge[2] in our choice of model. Assume the "fields" produced by a moving charge to be those calculated from the retarded Liénard-Wiechert potentials, and omit radiation-reaction terms.

To simplify the problem we shall restrict our attention to one-dimensional motion, say on the x axis.

Let the world lines of the two particles as functions of their respective "proper times" s_1 and s_2 be $(t_1(s_1),z_1(s_1))$ and $(t_2(s_2),z_2(s_2))$, where

$$\dot{t}_i(s_i)>0 \quad \text{and} \quad c^2[\dot{t}_i(s_i)]^2-[\dot{z}_i(s_i)]^2=1 \quad \text{for} \quad i=1,2, \tag{1}$$

a dot indicating differentiation with respect to the proper time of the particle in question. In case $t_1(s_1)=t_2(s_2)$ let us assume $z_1(s_1)<z_2(s_2)$, i.e., no collisions.

To save writing we shall formulate expressions involving particles i and j with the understanding always that $(i,j)=(1,2)$ or $(2,1)$.

Now if, for a given point $(t_i(s_i),z_i(s_i))$ on the world line of particle i, one can solve for the retarded proper time s_j from the equation

$$c(t_i(s_i)-t_j(s_j))=|z_i(s_i)-z_j(s_j)|, \tag{2}$$

the equation of the motion of particle i, Synge's Eq. (2.9),[2] reduces to

$$\ddot{z}_i(s_i) =\frac{k}{m_i}\frac{[1+\dot{z}_i^2(s_i)]^{1/2}\operatorname{sgn}[z_i(s_i)-z_j(s_j)]}{\{[z_i(s_i)-z_j(s_j)]\dot{z}_j(s_j)-|z_i(s_i)-z_j(s_j)|c\dot{t}_j(s_j)\}^2}. \tag{3}$$

Here $k=e_1e_2/c^2$ and m_i is the rest mass of particle i.

If we had not restricted ourselves to one-dimensional motion, (3) would be replaced by a much more complicated system which would also contain terms involving $\ddot{z}_j(s_j)$. That problem is much more difficult mathematically.

Equations (1), (2), and (3) for $(i,j)=(1,2)$ and $(2,1)$ represent a system of functional and functional-differential equations. The first mathematical inconvenience arises from the presence of the two trajectory parameters s_1 and s_2. However, it is quite simple to convert to a system in which the only independent variable is the "observer time," t.

Let us introduce the symbol $x_i(t)$ to denote the position of particle i as a function of t, i.e., $x_i(t)=z_i(t_i^{-1}(t))$, for $i=1,2$. Let primes indicate differentiation with respect to t, e.g., $x_i'(t)=dx_i(t)/dt=\dot{z}_i(t_i^{-1}(t))/\dot{t}_i(t_i^{-1}(t))$. Thus $|x_i'(t)|<c$.

We shall use (1) to obtain, for $s_i=t_i^{-1}(t)$,

$$\dot{t}_i(s_i)=\frac{1}{c[1-(x_i'(t))^2/c^2]^{1/2}},$$

and

$$\dot{z}_i(s_i)=\frac{x_i'(t)}{c[1-(x_i'(t))^2/c^2]^{1/2}},$$

and we define two delays, $\tau_{ji}(t)$ for $(i,j)=(1,2)$ and $(2,1)$, as functions of t as follows. Given any t, let $s_i=t_i^{-1}(t)$ and let s_j be the retarded proper time obtained from (2) (if a solution exists). Then define

$$\tau_{ji}(t)=t-t_j(s_j).$$

Equation (2) now becomes

$$c\tau_{ji}(t)=|x_i(t)-x_j(t-\tau_{ji}(t))|.$$

Since we have assumed $x_1(t)<x_2(t)$ for all t under consideration, this can be reduced to

$$c\tau_{ji}(t)=(-1)^i[x_i(t)-x_j(t-\tau_{ji}(t))]. \tag{4}$$

For, assuming $(-1)^ix_i(t)<(-1)^ix_j(t-\tau_{ji}(t))$ leads to the contradiction

$$\begin{aligned}c\tau_{ji}(t)&=(-1)^ix_j(t-\tau_{ji}(t))-(-1)^ix_i(t)\\&<(-1)^ix_j(t-\tau_{ji}(t))-(-1)^ix_j(t)<c\tau_{ji}(t).\end{aligned}$$

Now, introducing

$$x_i'(t)=v_i(t), \tag{5}$$

(3) becomes

$$\frac{v_i'(t)}{[1-v_i^2(t)/c^2]^{3/2}}=\frac{(-1)^ik}{m_i\tau_{ji}^2(t)}\frac{c-(-1)^iv_j(t-\tau_{ji}(t))}{c+(-1)^iv_j(t-\tau_{ji}(t))}. \tag{6}$$

---

[1] R. D. Driver, Am. Math. Soc. Notices 7, 522 (1960); Ann. Phys. (N. Y.) 21, 122 (1963); and R. D. Driver and M. J. Norris, *ibid.* 42, 347 (1967).

[2] J. L. Synge, Proc. Roy. Soc. (London) A177, 118 (1940).

[3] H. Van Dam and E. P. Wigner, Phys. Rev. 138, B1576 (1965); 142, 838 (1966).

The four equations represented by (5) and (6) form a system of delay-differential equations in which the "delays," $\tau_{ji}(t)$, are defined by the two functional equations (4).

Had we considered three-dimensional motion, the equations would have become a functional-differential system of neutral type.[4] And, had we considered a theory involving both retarded and advanced effects, the equations would have involved both retarded arguments (delays) and advanced arguments.

3. DELAY-DIFFERENTIAL EQUATIONS

A great many papers have been written on delay-differential equations in the past 200 years, and especially during the past 30 years. Most of them, however, have dealt with delays which were either constants or specified functions of t.

Consider, for example, the simplest case—a scalar linear equation with constant coefficients and a single constant delay:

$$x'(t)=ax(t)+bx(t-\tau), \quad \text{with} \quad \tau>0,\ b\neq 0. \tag{7}$$

The usual problem posed for such an equation is an "initial-function" problem. The unknown function is set equal to some arbitrarily specified continuous function, $x(t)=\varphi(t)$, on an "initial set" $[t_0-\tau, t_0]$. Then one seeks a continuous extension, $x(t)$, which satisfies the delay-differential equation for $t>t_0$. Indeed this is the only problem formulation for which any well-developed theory exists.

It has also been proved[1] that for the equations of motion of two charged particles in one dimension, (4), (5), and (6), as described in Sec. 2, such an "initial-function" problem is well posed. That is, one can specify rather arbitrary continuously differentiable trajectories for $t\leq t_0$, and the future trajectories, for $t>t_0$, will be determined. Various properties of these solutions have also been found.[1]

Another approach would be to specify positions and velocities of the two particles at some instant and seek trajectories which satisfy the equations of motion for all past (and future) time—the "backwards" problem.

It was assumed by Synge[2] that this would be a well-posed problem, at least if $m_1\neq m_2$.

Van Dam and Wigner[3] considered equations of motion involving both retarded and advanced fields and they asserted (again without proof) that instantaneous positions and velocities would determine unique trajectories.

Schild and Schlosser[5] have sought exact solutions of two-body problems in special cases involving circular motion, but have not concerned themselves about uniqueness.

Rzewuski[6] asserted that the solution of a certain generalized electrodynamic system would be uniquely determined by instantaneous initial data. He referred to the proof in a paper[7] which did consider a system somewhat similar to (4), (5), and (6). However, among other things, the authors of the latter imposed conditions which would correspond in our case to assuming $a\leq t-\tau_{ji}(t)\leq t_0$ for $a\leq t\leq t_0$ for some *finite* a and $\tau_{ji}(t)\geq\sigma>0$. Clearly, these are mutually exclusive conditions, thus making the results inapplicable to a system like (4), (5), and (6).

Staruszkiewicz[8] considered a one-dimensional, two-body problem in which (2) was replaced by

$$c|t_i(s_i)-t_j(s_j)|=|z_i(s_i)-z_j(s_j)|, \tag{2'}$$

allowing both retarded and advanced arguments, and (3) was replaced by the more artificial equation

$$\ddot{z}_i(s_i)=-\lambda n[1+\dot{z}_i^2(s_i)]^{1/2}\operatorname{sgn}[z_i(s_i)-z_j(s_j)], \tag{3'}$$

where λ is a constant and $n=0$, 1, or 2 is the number of solutions of (2′). He found special solutions of these equations of motion in certain special cases. However, in a footnote reporting observations by Havas, he notes that, rather than being unique, in some cases the solutions obtained actually yield an infinite family, all having the specified positions and velocities at $t=0$.

Other authors[9] have attempted to overcome the embarrassment of retarded and/or advanced interactions, in both electrodynamic and gravitational theories, by the use of an approximation method. They formally expand objectionable retarded expressions such as (4) in a Taylor series in powers of $x(t)/c\equiv|x_i(t)-x_j(t)|/c$ to obtain

$$c\tau_{ji}(t)=x(t)-(-1)^i x_j'(t)\frac{x(t)}{c}+\frac{(x'(t))^2}{c}\frac{x(t)}{c}+(-1)^i x_j''(t)\frac{x^2(t)}{c^2}+\cdots.$$

Anticipating a "quasistatic" solution, one might arbitrarily terminate this series with the $[x(t)/c]^2$ or

[4] R. D. Driver, in *International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics*, edited by J. P. LaSalle and S. Lefshetz (Academic Press Inc., New York, 1963), p. 474.

[5] A. Schild, Phys. Rev. 131, 2762 (1963); A. Schild and J. A. Schlosser, J. Math. Phys. 9, 913 (1968).

[6] J. Rzewuski, *Field Theory* (Państwowe Wydawnictwo Naukowe, Warszawa, 1958), Part I, p. 239.

[7] A. Krzywicki, J. Rzewuski, J. Zamorski, and A. Zięba, Ann. Polon. Math. 2, 77 (1955).

[8] A. Staruszkiewicz, Am. J. Phys. 35, 437 (1967).

[9] See, for example, L. Page, Phys. Rev. 12, 371 (1918); 24, 296 (1924); C. G. Darwin, Phil. Mag. 39, 537 (1920); G. Breit, Phys. Rev. 34, 533 (1929); A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. 39, 65 (1938); H. P. Robertson, *ibid.* 39, 101 (1938); L. Infeld, Phys. Rev. 53, 836 (1938); A. Eddington and G. L. Clark, Proc. Roy. Soc. (London) A166, 465 (1938); A. Einstein and L. Infeld, Can. J. Math. 1, 209 (1949); G. L. Clark, Proc. Roy. Soc. (Edinburgh) A64, 49 (1954); B. Bertotti, Nuovo Cimento 12, 226 (1954). More examples are listed and discussed by P. Havas, in *Statistical Mechanics of Equilibrium and Non-Equilibrium*, edited by J. Meixmer (North-Holland Publishing Co., Amsterdam, 1965), p. 1.

lower-order terms, and similarly for other retarded or advanced expressions. Then (5) and (6) are replaced by a system of ordinary differential equations requiring only instantaneous values of x_i and v_i to determine a solution.

This way, one might fool himself into thinking that only instantaneous values of x_i and v_i are needed in the original equations. The potential mathematical pitfalls in this "method" are abundant.

In fact, it is a simple matter to show that, *in general*, for delay-differential equations, or for equations involving both retarded and advanced arguments such as

$$x'(t)=ax(t)+bx(t-\tau)+cx(t+\tau),\quad \text{with } \tau>0,\ b\neq0,\ c\neq0, \tag{8}$$

instantaneous initial data will not determine a unique solution.

For example, let $\varphi(t)$ be any (nontrivial) infinitely differentiable function on $[t_0-\tau,\ t_0+\tau]$ with the property that φ and all its derivatives vanish at $t_0-\tau$, t_0, and $t_0+\tau$. Then, solving (7) or (8) for $x(t-\tau)$, one can compute $x(t)$ successively on the intervals $[t_0-2\tau,\ t_0-\tau]$, $[t_0-3\tau,\ t_0-2\tau]$, $\cdots$. Or with (7) as written or (8) solved for $x(t+\tau)$, one can compute $x(t)$ on $[t_0+\tau,\ t_0+2\tau]$, $[t_0+2\tau,\ t_0+3\tau]$, $\cdots$.

Thus it is clear that, in either case, infinitely many solutions exist even if one specifies x and *all* its derivatives at t_0.

Similar arguments can be given for linear equations with coefficients which are C^∞ functions of t.

While a well-posed problem for (7) is known to exist when an initial function is specified on $[t_0-\tau,\ t_0]$, to my knowledge, *no* useful, well-posed problem is known for (8). (It has been suggested by Wheeler and Feynman,[10] without proof, that for the particular case of two charged particles under the influence of half-retarded and half-advanced interactions, initial data on segments analogous to $[t_0-\tau,\ t_0+\tau]$ would be natural. This is quite contrary to Van Dam and Wigner's assumption and would certainly not work for (8) unless the initial data were very special.)

In spite of the difficulties exhibited by the linear examples cited above, there are certain particular functional-differential equations for which instantaneous data *do* suffice to determine a unique solution. Consider, for example, the equation

$$x'(t)=f\big(t,x(t),x(g(t))\big), \tag{9}$$

where $f(t,x,X)$ and $g(t)$ are given continuous functions for $t\in[a,b]$ and $x,X\in R$, and $a\leq g(t)\leq b$ for $a\leq t\leq b$. Assume also that $f(t,x,X)$ satisfies a Lipschitz condition of the form

$$|f(t,x,X)-f(t,\bar{x},\bar{X})|\leq L\max(|x-\bar{x}|,\ |X-\bar{X}|).$$

This equation is a slight variation of one considered by Fite[11] who showed that instantaneous data, $x(t_0)=x_0$ for some $t_0\in[a,b]$, determine a unique solution if either

$$|g(t)-t_0|\leq|t-t_0|\quad \text{for}\quad a\leq t\leq b \tag{10}$$

or

$$L(b-t_0)<1\quad \text{and}\quad L(t_0-a)<1. \tag{11}$$

Fite's paper is apparently not well known, for at least ten authors have since published minor variations or generalizations of his results, most of them considering an infinite interval.[12]

Other special cases—some of them very complicated—were considered long before Fite. See, for example Babbage,[13] Polossuchin,[14] and others listed in Myškis's bibliography.[15]

Thus the above-cited examples of linear equations do not prove that in more complicated, nonlinear equations of motion, such as (4), (5), and (6), a unique solution will *not* be determined by specifying instantaneous positions and velocities. However, this discussion should suggest that it is unsafe to assume so until it has been mathematically proved for the particular model under consideration.

Section 4 contains a proof that in one special case, system (4), (5), and (6), or its equivalent (1), (2), and (3), assumed valid back to $-\infty$, together with instantaneous position and velocity data, does determine a unique solution.

4. EXISTENCE AND UNIQUENESS FOR A SPECIAL CASE OF (4), (5), AND (6)

We shall actually consider the case of two charged particles of like sign, i.e., $k>0$, with the position and velocity of each charge specified at some point on its world line, the two points having a spacelike relation. If, at these points, the particles are approaching each other but are not too close together their past histories are uniquely determined. It follows that their futures are also determined.[1]

We first prove a lemma about solutions of (4) (without regard to the sign of k).

Lemma. Let trajectories be given: $x_i(t)\in C^1$ with $|x_i'(t)|<c$ for $t<t_0$, $i=1, 2$. Let t_{10}, $t_{20}\in(-\infty, t_0]$ be given, and assume $c|t_{20}-t_{10}|\leq x_2(t_{20})-x_1(t_{10})\equiv x_0$.

[10] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 21, 425 (1949).

[11] W. B. Fite, Trans. Am. Math. Soc. 22, 311 (1921).

[12] E.g., L. Silberstein, Phil. Mag. 30, 185 (1940); S. Doss and S. K. Nasr, Am. J. Math. 75, 713 (1953); P. I. Romanovskii, Trudy Moskov Aviation. Inst. 24, 1 (1953); V. P. Skripnik, Matematičeskii Sbornik (N. S.) 62, 385 (1963); S. Sugiyama, Kōdai Math. Seminar Rep. 15, 67 (1963); J. Błaż, Annales Polon. Math. 15, 1 (1964); T. Dlotko and M. Kuczma, Colloq. Math. 12, 107 (1964); D. R. Anderson, SIAM Rev. 8, 359 (1966). Interesting variations of (11) are given by A. B. Nersesjan, Dokl. Akad. Nauk. Arm. SSR 36, 193 (1963).

[13] C. W. Babbage, Phil. Trans. Roy. Soc. (London) 106, 179 (1816).

[14] O. Polossuchin, *Über eine besondere Klasse von differentialen Funktionalgleichungen*, (Universität Zürich, Zürich, 1910).

[15] A. D. Myškis, Usp. Mat. Nauk (N. S.) 4, 99 (1949) [English transl.: Am. Math. Soc. Transl., Series 1, 4, 207 (1962)].

Then if (4) has a (positive) solution for $t\leq t_{i0}$, it is unique, $t-\tau_{ji}(t)\leq t_{j0}$, $t-\tau_{ji}(t)$ is a strictly increasing function of t, and for $t<t_{i0}$

$$\tau_{ji}'(t)=\frac{(-1)^i x_i'(t)+(-1)^j x_j'(t-\tau_{ji}(t))}{c+(-1)^j x_j'(t-\tau_{ji}(t))}. \quad (12)$$

If, in addition, there exist constants $u\in(0,1)$, v_{10}, and v_{20} such that $v_0\equiv v_{10}-v_{20}>0$, and $-uc\leq(-1)^i x_i'(t)\leq(-1)^i v_{i0}\leq uc$ for $t\leq t_{i0}$, $i=1$, 2, then (4) does have a solution, $\tau_{ji}(t)$, for $t\leq t_{i0}$ and

$$\frac{x_0(1-u)+v_0(t_{i0}-t)}{c(1+u)}\leq\tau_{ji}(t)\leq\frac{x_0+uc(t_{i0}+t_{j0}-2t)}{c(1-u)}. \quad (13)$$

Proof. Rewrite (4) as

$$ct-(-1)^i x_i(t)=c(t-\tau)+(-1)^j x_j(t-\tau) \quad (4')$$

and observe that both sides are strictly increasing functions of $t\leq t_0$ and $t-\tau\leq t_0$, respectively. Hence if a solution, $\tau=\tau_{ji}(t)>0$, exists it is unique and $t-\tau_{ji}(t)$ is a strictly increasing function of t. Moreover, for $t\leq t_{i0}$, we cannot have $t-\tau_{ji}(t)>t_{j0}$ for this would lead to

$$\begin{aligned}ct_{i0}-(-1)^i x_i(t_{i0})&\geq ct-(-1)^i x_i(t)\\&=c(t-\tau_{ji}(t))+(-1)^j x_j(t-\tau_{ji}(t))\\&>ct_{j0}+(-1)^j x_j(t_{j0})=ct_{j0}-(-1)^i x_i(t_{i0})+x_0\\&\geq ct_{j0}-(-1)^i x_i(t_{i0})+c|t_{20}-t_{10}|\geq ct_{i0}-(-1)^i x_i(t_{i0}),\end{aligned}$$

a contradiction. Equation (12) follows from the implicit-function theorem.

Now add the condition $-uc\leq(-1)^i x_i'(t)\leq(-1)^i v_{i0}\leq uc$ for $t\leq t_{i0}$, $i=1$, 2. Then for $t\leq t_{i0}$

$$\begin{aligned}ct-(-1)^i x_i(t)&\leq ct_{i0}-(-1)^i x_i(t_{i0})\\&=ct_{i0}-x_0+(-1)^j x_j(t_{j0})\leq ct_{j0}+(-1)^j x_j(t_{j0}).\end{aligned}$$

Then, since the right-hand side of (4'), as a function of $t-\tau$, has slope $\geq c-uc>0$, a solution of (4') exists.

For $t\leq t_{i0}$, since $t-\tau_{ji}(t)\leq t_{j0}$ we obtain from (4)

$$\begin{aligned}c\tau_{ji}(t)&=(-1)^i x_i(t_{i0})-(-1)^i\int_t^{t_{i0}}x_i'(s)ds+(-1)^j x_j(t_{j0})\\&\quad-(-1)^j\int_{t-\tau_{ji}(t)}^{t_{j0}}x_j'(s)ds\\&\geq x_0-(-1)^i v_{i0}(t_{i0}-t)-(-1)^j v_{j0}(t_{j0}-t+\tau_{ji}(t))\\&=x_0+v_0(t_{i0}-t)-(-1)^j v_{j0}(t_{j0}-t_{i0}+\tau_{ji}(t))\\&\geq x_0(1-u)+v_0(t_{i0}-t)-uc\tau_{ji}(t).\end{aligned}$$

This gives the first inequality of (13).

For the other half of (13) we merely note that for $t\leq t_{i0}$

$$c\tau_{ji}(t)\leq x_0+uc(t_{i0}-t+t_{j0}-t+\tau_{ji}(t)).$$

We are now ready to prove the main result of this paper—an existence and uniqueness theorem for the backwards problem for system (1), (2), and (3), or (4), (5), and (6) in one special case.

Theorem. Let both charges have the same sign ($k>0$). Let two numbers, v_{10} and v_{20}, be given such that $-c<v_{20}<v_{10}<c$. Let (t_{10},x_{10}) and (t_{20},x_{20}) with $x_{20}>x_{10}$ be two points with spacelike separation, i.e., $c|t_{20}-t_{10}|\leq x_{20}-x_{10}$. Define $v_0=v_{10}-v_{20}$ and $x_0=x_{20}-x_{10}$.

Then if $v_0x_0>0$ is sufficiently large (described in the proof), there exists one and only one solution of (4), (5), and (6), $x_i(t)$ for $t\leq t_{i0}$ and $i=1$, 2, such that $x_i(t_{i0})=x_{i0}$ and $v_i(t_{i0})=x_i'(t_{i0})=v_{i0}$ and $|x_i'(t)|<c$ for $t\leq t_{i0}$, $i=1,2$.

Equivalently there is one and only one solution of (1), (2), and (3) with

$$t_i(0)=t_{i0},\quad z_i(0)=x_{i0},\quad \dot{z}_i(0)=\frac{v_{i0}/c}{(1-v_{i0}^2/c^2)^{1/2}}$$

$$\text{for}\quad i=1,2.$$

Remark. Having found the solution of (4), (5), and (6) into the past, the unique solution into the future follows from previously known theorems.[1]

Proof of Theorem. Let $v_M=\max(|v_{10}|,\ |v_{20}|)$, and $m=\min(m_1,m_2)$. Then choose a constant $u\in(0,1)$ which satisfies the inequality

$$\frac{u}{(1-u^2)^{1/2}}\geq\frac{v_M/c}{(1-v_M^2/c^2)^{1/2}}+\frac{2ck(c+v_M)}{x_0v_0(c-v_M)m}. \quad (14)$$

Since the last term in (14) is positive and since

$$\frac{d}{dy}\frac{y}{(1-y^2)^{1/2}}=\frac{1}{(1-y^2)^{3/2}}>0, \quad (15)$$

one has $uc>v_M$.

Now let us assume that there is a solution of (4), (5), and (6) having the specified values at t_{i0}. Then $(-1)^i v_i'(t)\geq0$ for $t\leq t_{i0}$. Therefore $(-1)^i v_i(t)\leq(-1)^i v_{i0}$ for $t\leq t_{i0}$, $i=1$, 2.

From the lemma, $t-\tau_{ji}(t)\leq t_{j0}$ for $t\leq t_{i0}$ and, similarly to the proof of (13),

$$\begin{aligned}c\tau_{ji}(t_{i0})&=(-1)^i x_i(t_{i0})+(-1)^j x_j(t_{i0}-\tau_{ji}(t_{i0}))\\&=x_0-(-1)^j\int_{t_{i0}-\tau_{ji}(t_{i0})}^{t_{j0}}v_j(s)ds\\&\geq x_0-(-1)^j v_{j0}(t_{j0}-t_{i0}+\tau_{ji}(t_{i0}))\\&\geq x_0-v_M(x_0/c+\tau_{ji}(t_{i0})).\end{aligned}$$

Therefore

$$\tau_{ji}(t_{i0})\geq x_0(c-v_M)/c(c+v_M). \quad (16)$$

Also, for $t<t_{i0}$ we have

$$\tau_{ji}'(t)=\frac{(-1)^i v_i(t)+(-1)^i v_j(t-\tau_{ji}(t))}{c+(-1)^i v_j(t-\tau_{ji}(t))}\leq\frac{-v_0}{c+(-1)^i v_j(t-\tau_{ji}(t))}. \quad (17)$$

Thus, from (6), (15), (16), and (17), for $t<t_{i0}$,

$$\frac{(-1)^i v_i(t)}{[1-v_i^2(t)/c^2]^{1/2}}=\frac{(-1)^i v_{i0}}{(1-v_{i0}^2/c^2)^{1/2}}-\int_t^{t_{i0}}\frac{k}{m_i\tau_{ji}^2(s)}\frac{c-(-1)^i v_j(s-\tau_{ji}(s))}{c+(-1)^i v_j(s-\tau_{ji}(s))}ds\geq\frac{(-1)^i v_{i0}}{(1-v_{i0}^2/c^2)^{1/2}}+\int_t^{t_{i0}}\frac{k}{m_i\tau_{ji}^2(s)}\frac{2c}{v_0}\tau_{ji}'(s)ds$$

$$=\frac{(-1)^i v_{i0}}{(1-v_{i0}^2/c^2)^{1/2}}-\frac{2ck}{m_i v_0}\left[\frac{1}{\tau_{ji}(t_{i0})}-\frac{1}{\tau_{ji}(t)}\right]\geq\frac{-v_M}{(1-v_M^2/c^2)^{1/2}}-\frac{2c^2k(c+v_M)}{mv_0x_0(c-v_M)}\geq\frac{-uc}{(1-u^2)^{1/2}}. \quad (18)$$

Thus $-uc\leq(-1)^i v_i(t)\leq(-1)^i v_{i0}\leq uc$ for $t\leq t_{i0}$.

Hence, as candidates for solutions, we need only consider trajectories such that $-uc\leq(-1)^i v_i(t)\leq(-1)^i v_{i0}\leq uc$ for $t\leq t_{i0}$.

The existence and uniqueness proof will now consist of a Picard iteration simplified by use of the Banach contraction-mapping principle.

Consider the complete metric space, S, of ordered pairs of continuous functions (v_1,v_2) such that $(v_1,v_2)\in S$ if $v_i\in C([-\infty,t_{i0}])$ with

$$-uc\leq(-1)^i v_i(t)\leq(-1)^i v_{i0}\quad\text{for}\quad t\leq t_{i0},\quad i=1,2, \quad (19)$$

and metric

$$\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]=\max\left(\sup_{t\leq t_{10}}|v_1(t)-\bar{v}_1(t)|,\ \sup_{t\leq t_{20}}|v_2(t)-\bar{v}_2(t)|\right).$$

Now construct a mapping T of $S\to S$ as follows. $T(v_1,v_2)=(T_1(v_1,v_2),\ T_2(v_1,v_2))$ for arbitrary $(v_1,v_2)\in S$, where we define

$$x_i(t)\equiv x_{i0}+\int_{t_{i0}}^t v_i(s)ds\quad\text{for}\quad t\leq t_{i0},\quad i=1,2, \quad (20)$$

$\tau_{ji}(t)$ is to be the unique solution of

$$\tau=(-1)^i[x_i(t)-x_j(t-\tau)]/c\quad\text{for}\quad t\leq t_{i0}, i=1,2, \quad (21)$$

and finally $T_i(v_1,v_2)(t)$ is determined for $t\leq t_{i0}$, $i=1$, 2 from

$$\frac{T_i(v_1,v_2)(t)}{[1-(T_i(v_1,v_2)(t))^2/c^2]^{1/2}}=\frac{v_{i0}}{(1-v_{i0}^2/c^2)^{1/2}}+\int_{t_{i0}}^t\frac{(-1)^i k}{m_i\tau_{ji}^2(s)}\frac{c-(-1)^i v_j(s-\tau_{ji}(s))}{c+(-1)^i v_j(s-\tau_{ji}(s))}ds. \quad (22)$$

To show that T is a well-defined mapping of S into itself, we make the following observations. Clearly $x_i(t)$ is well defined by (20), and then $\tau_{ji}(t)$ is well defined by (21) since all the conditions of the lemma are fulfilled. Since $x_0>0$ and $v_0>0$ it follows from (13) that $\tau_{ji}(t)>0$ for $t\leq t_{i0}$. Then, performing the same calculations as in (16), (17), and (18) we find that $-uc\leq(-1)^i T_i(v_1,v_2)(t)\leq(-1)^i v_{i0}$ for $t\leq t_{i0}$, $i=1$, 2. So T maps S into itself.

It remains to prove that T is a contraction mapping. Actually we shall only prove this on TS (the image of T) and under an assumption that x_0 is sufficiently large.

Let (v_1,v_2) and $(\bar{v}_1,\bar{v}_2)$ be arbitrary members of TS. Let $\bar{x}_i$ and $\bar{\tau}_{ji}$ be defined from $(\bar{v}_1,\bar{v}_2)$ as are x_i and τ_{ji} from (v_1,v_2) in (20) and (21). Then all the conclusions of the lemma hold for $\tau_{ji}(t)$ and $\bar{\tau}_{ji}(t)$.

Thus for $t\leq t_{i0}$

$$\begin{aligned}c|\tau_{ji}(t)-\bar{\tau}_{ji}(t)|&\leq|x_i(t)-\bar{x}_i(t)|+|x_j(t-\tau_{ji}(t))-x_j(t-\bar{\tau}_{ji}(t))|\\&\qquad+|x_j(t-\bar{\tau}_{ji}(t))-\bar{x}_j(t-\bar{\tau}_{ji}(t))|\\&\leq(t_{i0}-t)\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]+uc|\tau_{ji}(t)-\bar{\tau}_{ji}(t)|\\&\qquad+(t_{j0}-t+\bar{\tau}_{ji}(t))\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)].\end{aligned}$$

Therefore, using (13),

$$\begin{aligned}|\tau_{ji}(t)-\bar{\tau}_{ji}(t)|&\leq\frac{t_{i0}+t_{j0}-2t+\bar{\tau}_{ji}(t)}{c(1-u)}\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]\\&\leq\frac{2x_0+2c(t_{i0}-t)}{c^2(1-u)^2}\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]. \quad (23)\end{aligned}$$

Also, since $(v_1,v_2)\in TS$, i.e., $(v_1,v_2)=T(v_1^{(0)},v_2^{(0)})$, we obtain from (22), (15), and (13)

$$\begin{aligned}|v_i'(t)|&\leq\frac{k}{m\tau_{ji}^2(t)}\frac{1+u}{1-u}\\&\leq\frac{kc^2(1+u)^3}{m(1-u)[x_0(1-u)+v_0(t_{i0}-t)]^2}. \quad (24)\end{aligned}$$

Likewise for $|\bar{v}_i'(t)|$.

We are now ready to compute from (22)

$$|T_i(v_i,v_2)(t)-T_i(\bar{v}_1,\bar{v}_2)(t)|\leq\left|\frac{T_i(v_1,v_2)(t)}{[1-(T_i(v_1,v_2)(t))^2/c^2]^{1/2}}-\frac{T_i(\bar{v}_1,\bar{v}_2)(t)}{[1-(T_i(\bar{v}_1,\bar{v}_2)(t))^2/c^2]^{1/2}}\right|$$

$$\leq\frac{k}{m}\int_t^{t_{i0}}\left[\frac{1}{\tau_{ji}^2(s)}\left|\frac{c-(-1)^iv_j(s-\tau_{ji}(s))}{c+(-1)^iv_j(s-\tau_{ji}(s))}-\frac{c-(-1)^i\bar{v}_j(s-\bar{\tau}_{ji}(s))}{c+(-1)^i\bar{v}_j(s-\bar{\tau}_{ji}(s))}\right|+\frac{c-(-1)^i\bar{v}_j(s-\bar{\tau}_{ji}(s))}{c+(-1)^i\bar{v}_j(s-\bar{\tau}_{ji}(s))}\left|\frac{1}{\tau_{ji}^2(s)}-\frac{1}{\bar{\tau}_{ji}^2(s)}\right|\right]ds. \tag{25}$$

In simplifying this expression we shall make use of two more inequalities: For $y\geq -uc$

$$\left|\frac{d}{dy}\frac{c-y}{c+y}\right|=\frac{2c}{(c+y)^2}\leq\frac{2}{c(1-u)^2} \tag{26}$$

and, for $y\geq[x_0(1-u)+v_0(t_{i0}-t)]/c(1+u)$,

$$\left|\frac{d}{dy}\frac{1}{y^2}\right|=\frac{2}{y^3}\leq\frac{2c^3(1+u)^3}{[x_0(1-u)+v_0(t_{i0}-t)]^3}. \tag{27}$$

Applying these and (24) to (25) gives

$$|T_i(v_1,v_2)(t)-T_i(\bar{v}_1,\bar{v}_2)(t)|\leq\frac{k}{m}\int_t^{t_{i0}}\left\{\frac{2c(1+u)^2/(1-u)^2}{[x_0(1+u)+v_0(t_{i0}-s)]^2}|v_j(s-\tau_{ji}(s))-\bar{v}_j(s-\bar{\tau}_{ji}(s))|\right.$$

$$\left.+\frac{2c^3(1+u)^4}{(1-u)[x_0(1-u)+v_0(t_i-s)]^3}|\tau_{ji}(s)-\bar{\tau}_{ji}(s)|\right\}ds\leq\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]\frac{k}{m}\int_t^{t_{i0}}\left\{\frac{2c(1+u)^2/(1-u)^2}{[x_0(1+u)+v_0(t_{i0}-s)]^2}\right.$$

$$\left.\times\left[1+\frac{kc^2(1+u)^3}{m(1-u)[x_0(1-u)+v_0(t_{i0}-s)]^2}\frac{2x_0+2c(t_{i0}-s)}{c^2(1-u)^2}\right]+\frac{2c^3(1+u)^4}{(1-u)[x_0(1-u)+v_0(t_{i0}-s)]^3}\frac{2x_0+2c(t_{i0}-s)}{c^2(1-u)^2}\right\}ds. \tag{28}$$

Now, noting that

$$\frac{x_0+c(t_{i0}-s)}{x_0(1-u)+v_0(t_{i0}-s)}=\frac{c}{v_0}+\frac{x_0-x_0(1-u)c/v_0}{x_0(1-u)+v_0(t_{i0}-s)},$$

we can write

$$|T_i(v_1,v_2)(t)-T_i(\bar{v}_1,\bar{v}_2)(t)|\leq\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]$$

$$\times\int_{-\infty}^{t_{i0}}\left\{\frac{A}{[x_0(1-u)+v_0(t_{i0}-s)]^2}+\frac{B}{[x_0(1-u)+v_0(t_{i0}-s)]^3}+\frac{C}{[x_0(1-u)+v_0(t_{i0}-s)]^4}\right\}ds, \tag{29}$$

where A is a positive constant depending on u and v_0, and B and C are constants depending on u, v_0, and x_0 but only linearly on x_0.

Integrating (29) and considering it for both $i=1$ and 2, we get

$$\rho[T(v_1,v_2),T(\bar{v}_1,\bar{v}_2)]\leq\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)]\left[\frac{A}{v_0x_0(1-u)}+\frac{B}{2v_0x_0^2(1-u)^2}+\frac{C}{3v_0x_0^3(1-u)^3}\right],$$

so that, if v_0x_0 is sufficiently large,

$$\rho[T(v_1,v_2),T(\bar{v}_1,\bar{v}_2)]\leq\alpha\rho[(v_1,v_2),(\bar{v}_1,\bar{v}_2)],$$

where $\alpha<1$. Note that increasing x_0 does not disturb the choice of u satisfying inequality (14).

The Banach fixed-point theorem now completes the proof.

Remarks. I suspect that the conditions $v_0>0$ and v_0x_0 large may be unnecessary. However, I would not conjecture what the situation will be for $k<0$, or for three-dimensional motion, or for half-retarded and half-advanced interactions.