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Existence and Continuous Dependence of Solutions of a Neutral Functional-Differential Equation

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Introduction

Differential equations which express $x'(t)$ as a function of present and past values of x and past values of x' have been called functional-differential equations of neutral type or differential equations with deviating argument of neutral type. For brevity we shall refer to them simply as neutral-differential equations.

The equation, or system of equations, which we shall consider is

$$x'(t)=f\big(t,\,x(t),\,x(g_1(t)),\,\ldots,\,x(g_m(t)),\,x'(h_1(t)),\,\ldots,\,x'(h_p(t))\big)\quad\text{for}\quad t>t_0,$$

where $x=(x_1,\ldots,x_n)$ is the unknown function, $f=(f_1,\ldots,f_n)$; $g_1,\ldots,g_m$; and $h_1,\ldots,h_p$ are given functions such that, for some $\alpha<t_0$, $\alpha\leqq g_j(t)\leqq t$ $(j=1,\ldots,m)$ and $\alpha\leqq h_k(t)<t$ $(k=1,\ldots,p)$ for all $t\geqq t_0$, and where $x'\big(h_k(t)\big)$ represents $dx(s)/ds$ evaluated at $s=h_k(t)$. In case $\alpha=-\infty$ all intervals of the form $[\alpha,\ldots$ should be interpreted as $(-\infty,\ldots$.

The problem, briefly stated, is as follows: Let $x(t)=\varphi(t)$, a given function, on $[\alpha,t_0]$, then seek a continuous extension of $x(t)$ which satisfies the differential equation for $t>t_0$.

To shorten the notation further, we introduce $g(t)=\big(g_1(t),\ldots,g_m(t)\big)$ and $h(t)=\big(h_1(t),\ldots,h_p(t)\big)$ and represent the differential equation by

$$(1)\quad x'(t)=f\big(t,\,x(t),\,x(g(t)),\,x'(h(t))\big)\quad\text{for}\quad t>t_0\quad\text{with}\quad x(t)=\varphi(t)\quad\text{on}\quad[\alpha,t_0].$$

Neutral-differential equations have been discussed by many authors. Among the more recent have been E.M. WRIGHT [23, 24, 25] and BELLMAN & COOKE [1] in the case of constant delays (i.e., $g_j(t)=t-\tau_j$ and $h_k(t)=t-\sigma_k$) and EL'SGOL'TS [6, 7], KAMENSKII [11, 12], and DRIVER [3, 4] in the case of variable delays. References to other works can be found in [1], [6], [23], [24], [25], in MYSHKIS' classical paper [15], and in the comprehensive surveys by HAHN [8] and by ZVERKIN, KAMENSKII, NORKIN, and EL'SGOL'TS [26].

In this introduction we shall review the simplest method of attack on these problems and attempt to justify the more sophisticated approach which will be used in this paper.

Let us assume that f, g, and h are continuous functions of their arguments in appropriate domains, each $g_j(t)<t$ (a strict inequality), φ is continuous on $[\alpha,t_0]$, and $\varphi'(t)$ is continuous on $[\alpha,t_0)$ — a one-sided derivative being implied at the

closed end of the interval. Then a *local* existence theorem is readily found as follows (*cf.* EL'SGOL'TS [*6*], Chapter 5, §12, or KAMENSKII [*11*]).

Let $t_1 > t_0$ be the largest number such that $\alpha \leqq g_j(s) \leqq t_0$ $(j=1, \ldots, m)$ and $\alpha \leqq h_k(s) < t_0$ $(k=1, \ldots, p)$ for all $s \in [t_0, t_1)$ (possibly $t_1 = \infty$). On the interval $[t_0, t_1)$ the problem reduces to the solution of the ordinary differential equation

$$x'(t) = f\big(t, x(t), \varphi(g(t)), \varphi'(h(t))\big) \quad \text{with} \quad x(t_0) = \varphi(t_0).$$

The continuity of f assures the existence of a solution in some sufficiently small interval. If, in addition, we assume that f satisfies a Lipschitz condition with respect to x then this solution is unique.

In spite of all these continuity assumptions, we observe that, in general, $x'(t)$ is not defined (let alone continuous) at t_0. This results from the fact that, in general, $\varphi'(t_0^-) \neq f\big(t_0, \varphi(t_0), \varphi(g(t_0)), \varphi'(h(t_0))\big)$.

Now if the procedure described above happened to provide a solution on the entire interval $[t_0, t_1)$, we would naturally hope to extend the solution further by repeating the procedure, with $x(t)$ on $[\alpha, t_1)$ regarded as the initial function. However, difficulty would certainly arise if it happened, for example, that $h_k(t) \equiv t_0$ for some $k = 1, \ldots,$ or p on some interval to the right of t_1, because for such t the right hand side of equation (1) would be undefined. We rule out this possibility by requiring that each equation of the form $h_k(t) = c$, a constant, have at most a finite number of solutions on any finite sub-interval of $[t_0, \infty)$.

We can now take $t_2 > t_1$ to be the largest number such that

$$g_j(s) \leqq t_1 \quad \text{for all} \quad s \in [t_1, t_2], \qquad j = 1, \ldots, m,$$

and

$$h_k(s) \neq t_0 \quad \text{or} \quad t_1 \quad \text{for} \quad s \in (t_1, t_2), \qquad k = 1, \ldots, p.$$

Then (1) is again equivalent to a well-behaved ordinary differential equation on $[t_1, t_2)$ with initial condition at t_1.

Proceeding in this manner, we can extend the solution of (1) as far as the successive ordinary differential equations can be solved. The solution will be unique if, say, f satisfies a Lipschitz condition with respect to $x(t)$.

In general, $x'(t)$ will be only piecewise continuous (discontinuities occurring at $t_0, t_1, t_2, \ldots$). Thus we might as well have generalized the problem slightly by allowing $\varphi'(t)$ to be piecewise continuous. By this we mean that $\varphi'(t)$ is continuous on any compact interval except at a finite number of points at which $\varphi'(t)$ may have discontinuities of the first kind.

A procedure similar to the above can even be performed in the more complicated case in which g and h are functions of both t and $x(t)$ (see DRIVER [*4*]).

If we were to use the existence and uniqueness theorems for delay-differential equations (see MYSHKIS [*15*], EL'SGOL'TS [*6*], or DRIVER [*5*]) instead of those for ordinary differential equations, we could relax the condition $\alpha \leqq g_j(t) < t$ to $\alpha \leqq g_j(t) \leqq t$ for $j = 1, \ldots, m$.

Now, in order to discuss continuous dependence or stability of the solutions of (1) with respect to changes in any of the given functions, it is of primary importance to select an appropriate metric. One quickly discovers that if a "small" change in φ is to produce only a "small" change in the solution, x, then the measure of smallness must involve the derivatives as well as the values of the functions.

Recent authors have used metrics which compared two solutions x and $\tilde{x}$ by measuring pointwise the magnitude of their difference and the difference of their derivatives, *e.g.* on an interval $[\alpha, \beta]$

$$\|x-\tilde{x}\| = \sup_{\alpha \leqq s \leqq \beta} |x(s)-\tilde{x}(s)| + \sup_{\alpha \leqq s \leqq \beta} |x'(s)-\tilde{x}'(s)|, \quad \text{where} \quad |x| = \max_{i=1,\ldots,n} |x_i|.$$

See for example KAMENSKII [*12*], EL'SGOL'TS [*7*], and BELLMAN & COOKE [*1*].

This gives results in certain cases when one varies $\varphi \in C^1$, *provided* one does not allow t_0 to change. However, as soon as one permits pertubations of t_0 or of $h(t)$, it becomes impossible, due to the unavoidable discontinuities of $x'(t)$, to obtain a reasonable regularity theorem using such a metric.

To overcome this difficulty, in this paper we shall use a metric of the form

$$\|x-\tilde{x}\| = \sup_{\alpha \leqq s \leqq \beta} |x(s)-\tilde{x}(s)| + \int_{\alpha}^{\beta} |x'(s)-\tilde{x}'(s)|\, ds,$$

or the simpler equivalent metric,

$$\|x-\tilde{x}\|^{[\alpha,\beta)} = |x(\alpha)-\tilde{x}(\alpha)| + \int_{\alpha}^{\beta} |x'(s)-\tilde{x}'(s)|\, ds.$$

(*cf.* DRIVER [*3*].) This choice of metric is, perhaps, the most crucial point of the paper. This metric permits two solutions to be considered close even though their derivatives may differ by large amounts, provided these large differences occur only for brief time intervals.

Note that $|x(s)-\tilde{x}(s)| \leqq \|x-\tilde{x}\|^{[\alpha,\beta)}$ for all $s \in [\alpha, \beta]$.

Now if this is the correct type of metric for neutral-differential equations, then it appears that instead of requiring x' to be piecewise continuous we should only require that it be Lebesgue integrable. We shall, therefore, use the Carathéodory existence theory (*cf.* CODDINGTON & LEVINSON [*2*], Chapter 2) and require a solution, x, to be an absolutely continuous function which satisfies (1) almost everywhere.

It should be noted that earlier authors, H.R. PITT [*18, 19*] and E.M. WRIGHT [*23, 24*], considering linear integro-differential and difference-differential equations, allowed $x'(t)$ to be integrable and independently obtained estimates for $\int_{\alpha}^{t} |x'(s)|\, ds$ ([*19*], Theorem 8, and [*23*], Theorem 3). Thus the metric chosen here is not without precedent.

Note. If, for some i, the arguments of f_i do not involve any of the terms $x'(h_k(t))$, then $x_i'(t)$ will be continuous for $t > t_0$. This situation occurs, for example, when the first order system (1) is obtained from a higher order equation in the natural manner. Such cases may suggest simple modifications of the metric.

I. Extended Existence and Uniqueness

In stating the problem precisely we shall use the following.

Notation. Let

$$(t, x, X, Z) = (t; x_1, \ldots, x_n; x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn}; z_{11}, \ldots, z_{1n}, \ldots, z_{p1}, \ldots, z_{pn})$$

denote a typical vector in $E^{1+n+nm+np}$, Euclidean space of $1+n+nm+np$ dimensions. Let $D=D^{1+n+nm}$ be a domain (a connected open set) in E^{1+n+nm}, the t, x, X-space, and let $f(t, x, X, Z)=\big(f_1(t, x, X, Z), \ldots, f_n(t, x, X, Z)\big)$ be a real, n-vector-valued function over $D\times E^{np}$. Thus f is defined over a domain in $E^{1+n+nm+np}$ whose boundary puts no restriction on the values of the z_{ki} $(i=1, \ldots, n;\ k=1, \ldots, p)$.

Let D^1 be the projection of D onto the t-axis. This will be an open interval (possibly the entire t-axis or half of it). Let $g(t)=\big(g_1(t), \ldots, g_m(t)\big)$ and $h(t)=\big(h_1(t), \ldots, h_p(t)\big)$ be defined over D^1. Let $\gamma=\sup\{t\in D^1\}$.

In terms of this notation we can now state:

Problem *P*. Let $\varphi(t)$ be an absolutely continuous n-vector-valued function on $[\alpha, t_0]$ where $\alpha<t_0$. We seek an absolutely continuous extension, $x(t)$, of $\varphi(t)$ to $[\alpha, \beta)$ where $\beta>t_0$ such that

(a) $\big(t, x(t), x(g(t))\big)\in D$ for $t_0\leqq t<\beta$,

and

(b) $x'(t)=f\big(t, x(t), x(g(t)), x'(h(t))\big)$ for almost all $t\in(t_0, \beta)$.

In the existence theorems we shall make use of some hypotheses which have not yet been mentioned. To justify these, we consider the following examples.

Example 1. Let $x'(t)=[x'(t-1)]^2$ on $(0, 1)$. Then the absolutely continuous initial function $\varphi(t)=\sqrt{1+t}$ on $[-1, 0]$ gives rise to a problem which has no solution for any $t>0$.

We can eliminate this example in either of two ways. In Theorem 1 we shall require that $\varphi'(t)$ be essentially bounded, as it would be if it were piecewise continuous. In Theorem 2 we shall require that $x'\big(h(t)\big)$ enter the equation linearly. The required linearity actually occurs in the more complicated equations of a two-body problem of classical electrodynamics [*4*].

Example 2. Let $\varphi'(t)$ be a bounded measurable function on $[-1, 0]$ and $h(t)$ a continuous strictly increasing function mapping $[0, 1]$ into $[-1, 0]$ with the property that $\varphi'\big(h(t)\big)$ is not measurable (see HALMOS [*9*], p. 83). Then the equation $x'(t)=x'\big(h(t)\big)$ on $(0, 1)$ with $x(t)=\varphi(t)$ on $[-1, 0]$ has no absolutely continuous solution.

We shall eliminate this example by requiring that each $h_k'(t)$ be continuous and positive on $[t_0, \gamma)$. In particular this makes $h_k^{-1}(s)$ absolutely continuous on $\big[h_k(t_0), h_k(\gamma)\big)$ so that a measurable function of $h_k(t)$ will be measurable (*cf.* VON NEUMANN [*16*], p. 81). An analog of the condition $h_k'(t)>0$ is actually satisfied in the electrodynamics problem [*4*].

One might object, at this point, that the assumptions that f be linear in $x'\big(h(t)\big)$ and that $h_k'(t)>0$ are not necessary when one only admits piecewise continuously differentiable solutions. By way of further justification, therefore, we present two more examples. These show that, if one agrees on the choice of metric given in the introduction, then without these additional assumptions the solution will not depend continuously on the initial data, even in the case of piecewise continuously differentiable solutions.

Example 3. Let $x'(t)=[x'(t-1)]^2$ on $(0, 1)$. Then $\varphi(t)\equiv 0$ on $[-1, 0]$ gives the solution $x(t)\equiv 0$ on $[-1, 1]$. However, for any $\delta>0$, the initial function $\tilde{\varphi}$

such that $\tilde{\varphi}(-1)=0$ and

$$\tilde{\varphi}'(t)=\begin{cases}\frac{1}{\delta^2} & \text{on} \quad [-1,-1+\delta^3) \\ 0 & \text{on} \quad [-1+\delta^3, 0),\end{cases}$$

gives

$$\tilde{x}'(t)=\begin{cases}\frac{1}{\delta^4} & \text{on} \quad [0,\delta^3) \\ 0 & \text{on} \quad [\delta^3, 1).\end{cases}$$

Thus, while $\|\varphi-\tilde{\varphi}\|^{[-1,0)}=\delta$ is being made small, $\|x-\tilde{x}\|^{[-1,1)}=\delta+\frac{1}{\delta}$ becomes large.

Example 4. Let $x'(t)=x'(t^2-1)$ on $(0,1)$. Again $x(t)=0$ on $[-1,1]$ is a solution. However, with the initial function $\tilde{\varphi}$ such that $\tilde{\varphi}(-1)=0$ and

$$\tilde{\varphi}'(t)=\begin{cases}\frac{1}{\delta^3} & \text{on} \quad [-1,-1+\delta^4) \\ 0 & \text{on} \quad [-1+\delta^4, 0),\end{cases}$$

we again find that $\|\varphi-\tilde{\varphi}\|^{[-1,0)}=\delta$ while $\|x-\tilde{x}\|^{[-1,1)}=\delta+\frac{1}{\delta}$. In this example $h'(t_0)=0$.

Theorem 1 (Extended Existence for System (1)). *Let $f(t,x,X,Z)$, $g(t)$, and $h(t)$ be continuous on $D\times E^{np}$, $[t_0,\gamma)$, and $[t_0,\gamma)$ respectively, and let $\varphi(t)$ be absolutely continuous on $[\alpha,t_0]$. Let $\alpha\leqq g_j(t)\leqq t$ $(j=1,\ldots,m)$ and $\alpha\leqq h_k(t)<t$ $(k=1,\ldots,p)$ for all $t\in[t_0,\gamma)$ and let $h_k'(t)$ be continuous and positive on $[t_0,\gamma)$ for $k=1,\ldots,p$. Let $\varphi'(t)$ be bounded for almost all $t\in[\alpha,t_0]$, and let $(t_0,\varphi(t_0),\varphi(g(t_0)))\in D$.*

Then there exists a solution, $x(t)$, of problem P on $[\alpha,\beta)$, where $t_0<\beta\leqq\gamma$ and for each compact set $F\subset D$ there is a sequence of numbers $t_0<\xi_1<\xi_2<\cdots\to\beta$ such that

$$(\xi_i, x(\xi_i), x(g(\xi_i)))\in D-F \quad \textit{for} \quad i=1,2,\ldots.$$

Remark. Actually, instead of the condition that each $h_k'(t)>0$, all we use for this existence theorem is that each $h_k(t)$ is increasing and each $h_k^{-1}(s)$ is absolutely continuous on $[h_k(t_0),h_k(\gamma))$. Note however that example 4 shows that continuous dependence can fail even though $h_k^{-1}(t)$ is absolutely continuous.

Proof of Theorem 1. We shall carry out the proof in detail for the case when $\alpha\leqq g_j(t)<t$ for each j, and then point out the modifications necessary in case of the weak inequalities.

Even though we are now allowing an integrable derivative rather than requiring piecewise continuity, it will still be convenient to give a step-by-step existence proof similar to that outlined in the introduction. The proof is broken down into three parts.

1. Construction and properties of a sequence $t_0<t_1<t_2\cdots\to\gamma$

By induction, we construct a sequence $t_0<t_1<t_2<\cdots$ (possibly finite) as follows. For $\nu=1,2,\ldots$, let

$$\begin{aligned} t_\nu=\sup\{t\in(t_0,\gamma): g_j(s)\leqq t_{\nu-1} \quad &\text{for all} \quad s\in[t_0,t], \qquad j=1,\ldots,m, \quad \text{and} \\ h_k(s)\neq t_0,t_1,\ldots, \quad \text{or} \quad t_{\nu-1} \quad &\text{for} \quad s\in(t_{\nu-1},t), \qquad k=1,\ldots,p\}; \end{aligned}$$

and if ever, for some ν, $t_\nu=\gamma$, then terminate the sequence at that point.

Since each $g_j(t) < t$, each $h_k(t) < t$, and each $h_k(t)$ is strictly increasing on $[t_0, \gamma)$, it follows that this sequence is well defined and $t_0 < t_1 < t_2 < \cdots < \gamma$. Whenever $t \in (t_{\nu-1}, t_\nu)$, each $g_j(t) \in [\alpha, t_{\nu-1}]$ and each $h_k(t) \in (\alpha, t_0)$, (t_0, t_ν), ..., or $(t_{\nu-2}, t_{\nu-1})$.

Furthermore we shall now show that for any $\beta \in (t_0, \gamma)$ there can be at most a finite number of the points $t_\nu \in [t_0, \beta]$. Since each $t - g_j(t) > 0$ and each $t - h_k(t) > 0$ on the compact set $[t_0, \beta]$, there exists a constant $\tau = \tau(\beta) > 0$ such that

$$t \geqq g_j(t) + \tau \quad \text{for} \quad t_0 \leqq t \leqq \beta, \qquad j = 1, \ldots, m$$

and

$$t \geqq h_k(t) + \tau \quad \text{for} \quad t_0 \leqq t \leqq \beta, \qquad k = 1, \ldots, p.$$

Now for any $\mu = 0, 1, 2, \ldots$ consider all the t_ν's such that $t_\mu < t_\nu < t_\mu + \tau$. At each of these instants we have $g_j(t_\nu) \leqq t_\nu - \tau < t_\mu$ for each j and $h_k(t_\nu) \leqq t_\nu - \tau < t_\mu$ for each k. Therefore it follows, from the definition of t_ν, that for each such ν

$$h_k(t_\nu) = t_0, t_1, \ldots, \quad \text{or} \quad t_{\mu-1} \quad \text{for some} \quad k = 1, \ldots, \quad \text{or} \quad p.$$

But, since $h_k(t)$ can take on a given value at most once, this condition can occur at most $p\mu$ times. Thus

$$t_\nu \geqq t_\mu + \tau \quad \text{whenever} \quad \nu \geqq \mu + p\,\mu + 1 = 1 + (p+1)\,\mu.$$

By applying this estimate inductively for $\mu = 0, 1, 1 + (p+1), 1 + (p+1) + (p+1)^2, \ldots$, one finds that for any integer $\sigma \geqq 0$

$$t_\nu \geqq t_0 + \sigma\,\tau \quad \text{whenever} \quad \nu \geqq 1 + (p+1) + (p+1)^2 + \cdots + (p+1)^{\sigma-1}.$$

From the last condition it follows that for any $\beta \in (t_0, \gamma)$ there can be at most a finite number of points $t_\nu \in [t_0, \beta]$. Consequently the sequence $\{t_\nu\}$ converges to γ.

2. *Reduction to an ordinary differential equation*

Let us assume now that a solution $x(t) = x_{(\nu)}(t)$ has been found on $[\alpha, t_\nu)$ where $t_\nu < \gamma$, and $x'_{(\nu)}(t)$ is essentially bounded on $[\alpha, t_\nu)$. These conditions are certainly satisfied for $\nu = 0$. By the essential boundedness of $x'_{(\nu)}(t)$ we extend $x_{(\nu)}(t)$ continuously to $[\alpha, t_\nu]$.

Now consider equation (1) on $(t_\nu, t_{\nu+1})$. Without loss of generality we can assume $t_{\nu+1} < \gamma$, for in the case $t_{\nu+1} = \gamma$ we could replace $t_{\nu+1}$ by a number less than γ but arbitrarily close to it (or arbitrarily large if $\gamma = \infty$). Also we can assume $\alpha > -\infty$, since each of the continuous functions $g_j(t)$ and $h_k(t)$ must remain bounded in the compact set $[t_0, t_{\nu+1}]$.

Since each $h'_k(t)$ is continuous and positive on $[t_0, t_{\nu+1}]$, it follows that there is a number $\eta = \eta(t_{\nu+1})$ such that

$$h'_k(t) \geqq \eta > 0 \quad \text{on} \quad [t_0, t_{\nu+1}] \quad \text{for} \quad k = 1, \ldots, p.$$

This means that each function $h_k^{-1}(s)$ is absolutely continuous, and hence (*cf.* VON NEUMANN [*16*], p. 81) each $x'_{(\nu)}(h_k(t))$ is measurable and essentially bounded on $[t_\nu, t_{\nu+1}]$. Now, from the continuity of f, it follows that, in any compact subset of the set $\{(t, x) : t_\nu \leqq t < t_{\nu+1}, (t, x, x_{(\nu)}(g(t))) \in D\}$,

$$f\big(t, x, x_{(\nu)}(g(t)), x'_{(\nu)}(h(t))\big)$$

is a measurable function of t for each fixed x (*cf.* MCSHANE [*14*], p. 123), a continuous function of x for almost every fixed t, and is an essentially bounded function of t and x. We can assume without loss of generality that the function is continuous in x for *every* fixed t, since setting the function equal to zero on a set of t of measure zero will only affect it on a set of measure zero in the t, x-space. It then follows, from Carathéodory's existence theorem (*cf.* CODDINGTON & LEVINSON [2], p. 43) applied to the ordinary differential equation

$$x'(t) = f\big(t, x(t), x_{(\nu)}(g(t)), x'_{(\nu)}(h(t))\big) \quad \text{on} \quad [t_\nu, t_{\nu+1}),$$

that $x_{(\nu)}(t)$ can be extended absolutely continuously to a solution $x(t)$ of (1) on $[\alpha, \beta_\nu)$ where $t_\nu < \beta_\nu \leqq t_{\nu+1}$.

Suppose that β_ν is maximal, *i.e.* the solution of the ordinary differential equation is extended as far as possible on $[t_\nu, t_{\nu+1})$. If β_ν fulfills the requirements for β in the statement of the theorem, then the proof is complete. Otherwise there exists a compact set $F \subset D$ such that $\big(t, x(t), x_{(\nu)}(g(t))\big) \in F$ for all $t \in [t_0, \beta_\nu)$ and thence for all $t \in [t_\nu, \beta_\nu]$. Now if β_ν were less than $t_{\nu+1}$, it would follow that the function $f\big(t, x, x_{(\nu)}(g(t)), x'_{(\nu)}(h(t))\big)$ would satisfy the Carathéodory conditions on a rectangle $\{(t, x): \beta_\nu \leqq t \leqq \beta_\nu + a, |x - x(\beta_\nu)| \leqq b\}$. Thus the solution could be extended to the right of β_ν. This shows that if β_ν is maximal but is not the β of the theorem, then $\beta_\nu = t_{\nu+1}$ and $t_{\nu+1} < \gamma$. Moreover $\big(t, x(t), x(g(t))\big) \in F \subset D$ for $t_0 \leqq t \leqq t_{\nu+1}$, and the procedure can be repeated on the interval $[t_{\nu+1}, t_{\nu+2})$.

3. *Continuation to* β

All that remains is to show that if $x(t)$ is a solution for $\alpha \leqq t < \beta \leqq \gamma$ and if $\big(t, x(t), x(g(t))\big) \in F$, some compact subset of D, for $t_0 \leqq t < \beta$, then the solution can be continued beyond β. The compactness of F again implies that $\beta < \gamma$. Thus $[t_0, \beta]$ contains only a finite number of points of the sequence defined in part 1 of the proof, say

$$t_0 < t_1 < t_2 < \cdots < t_q < \beta$$

(β may be t_{q+1}). The same argument as was applied to the interval $[t_\nu, \beta_\nu]$ in part 2 of the proof can now be applied to the interval $[t_q, \beta]$ to show that the solution can be extended beyond β.

In order to replace the condition that each $g_j(t) < t$, on $[t_0, \gamma)$ with the weaker condition $g_j(t) \leqq t$, one requires a more general basic existence theorem than Carathéodory's. One needs an existence theorem for *delay*-differential equations in which the function f is allowed to be merely measurable with respect to t. As has been pointed out by SANSONE [21], such a theorem can be proved in a manner exactly analogous to the proof of Carathéodory's theorem. With this available one need consider only the effect of the h_k's when defining the sequence $t_0 < t_1 < t_2 < \cdots \to \gamma$ in part 1 of the proof of the present theorem. Q.E.D.

Remarks. This existence proof is easily modified to cover the case of piecewise continuously differentiable functions. In that case one can replace the hypothesis that each $h'_k(t) > 0$ on $[t_0, \gamma)$ by the weaker requirement that each equation of the form $h_k(t) = c$, a constant, has at most a finite number of solutions on any compact subinterval of $[t_0, \gamma)$. This then necessitates a slight modification in the estimates in part 1 of the proof.

The hypothesis that $\varphi'(t)$ be essentially bounded on $[\alpha, t_0]$ seems unnatural in the definition and study of continuous dependence of a solution on the initial conditions. It turns out that we can avoid this difficulty by restricting ourselves to an equation in which $x'(h(t))$ enters linearly,

$$(2) \qquad x'(t) = f^{(1)}(t, x(t), x(g(t))) + f^{(2)}(t, x(t), x(g(t)))\, x'(h(t)) \quad \text{for} \quad t > t_0$$
$$\text{with} \quad x(t) = \varphi(t) \quad \text{on} \quad [\alpha, t_0].$$

Here $f^{(1)}$ is an n-vector-valued function over $D \subset E^{1+n+nm}$, and $f^{(2)}$ is an $n \times np$-matrix-valued function over D. We shall still sometimes use the notation $f(t, x, X, Z)$ to stand for $f^{(1)}(t, x, X) + f^{(2)}(t, x, X)\, Z$. Here, $x'(h(t))$ and Z are regarded as np-dimensional column vectors.

For a matrix, A, the norm $|A|$ will be the maximum of the absolute values of the elements of A.

Theorem 2 (Extended Existence for System (2)). *For system* (2), *Theorem 1 holds without the condition that* $\varphi'(t)$ *be essentially bounded, provided* $f^{(1)}(t, x, X)$ *and* $f^{(2)}(t, x, X)$ *are continuous on* D.

Proof. The proof is the same as that of Theorem 1 except for the application of the local Carathéodory-type theorem in part 2.

In the proof of Theorem 1, we concluded that $f(t, x, x_{(\nu)}(g(t)), x'_{(\nu)}(h(t)))$ was essentially bounded from the fact that $x'_{(\nu)}(h(t))$ was essentially bounded. We now let $M_\nu^{(1)}$ be the maximum of $|f^{(1)}(t, x, x_{(\nu)}(g(t)))|$ and $M_\nu^{(2)}$ be the maximum of $|f^{(2)}(t, x, x_{(\nu)}(g(t)))|$ on a compact subset of the set $\{(t, x) : t_\nu \leqq t < t_{\nu+1},\ (t, x, x_{(\nu)}(g(t))) \in D\}$. Then, on that subset,

$$|f^{(1)}(t, x, x_{(\nu)}(g(t))) + f^{(2)}(t, x, x_{(\nu)}(g(t)))\, x'_{(\nu)}(h(t))|$$
$$\leqq M_\nu^{(1)} + np\, M_\nu^{(2)}\, |x'_{(\nu)}(h(t))| = m(t).$$

We now use the fact that each $h'_k(t) \geqq \eta > 0$ on $[t_\nu, t_{\nu+1}]$ (rather than the mere absolute continuity of h_k^{-1}) to assert that each $|x'_{(\nu)i}(h_k(t))|$ is integrable there. This is verified as follows:

$$\int_{t_\nu}^{t_{\nu+1}} |x'_{(\nu)i}(h_k(t))|\, dt \leqq \frac{1}{\eta} \int_{t_\nu}^{t_{\nu+1}} |x'_{(\nu)i}(h_k(t))\, h'_k(t)\, dt$$
$$= \frac{1}{\eta} \int_{h_k(t_\nu)}^{h_k(t_{\nu+1})} |x'_{(\nu)i}(\sigma)|\, d\sigma \leqq \frac{1}{\eta} \int_{\alpha}^{t_\nu} |x'_{(\nu)i}\ \sigma)|\, d\sigma$$

(*cf.* McShane [*14*], p. 214). We shall use an estimate of this type, where $x'_{(\nu)i}$ is replaced by some other integrable function, in the proof of Lemma 2.

Thus, in a suitable rectangle containing the point $(t_\nu, x_{(\nu)}(t_\nu))$, $m(t)$ serves as the integrable function required in Carathéodory's theorem. Q.E.D.

Remarks. Other conditions, weaker than linearity in $x'(h(t))$, could of course be imposed upon the function f of equation (1) to assure the integrability of the associated Carathéodory equations. However, the linearity indicated in equation (2) seems the most natural, and, as mentioned earlier, it actually occurs in the more complicated equation arising in electrodynamics.

We could allow $f^{(1)}$ and $f^{(2)}$ to be measurable with respect to t provided $|f^{(1)}|$ is majorized by an integrable function of t and $|f^{(2)}|$ is bounded.

For ordinary differential equations, a well-known condition due to WINTNER [22] assures the existence of the solution on the entire t-interval under consideration. The following example is due to a suggestion by T. F. BRIDGLAND that such a result should also be possible for neutral-differential equations.

Example 5 (Global Existence). Let the conditions of Theorem 2 be satisfied with

$$D=\{(t, x, X): t_0-a<t<\gamma,\ |x|<\varrho, |X|<\varrho\},$$

with constants $a\in(0,\infty]$, $\gamma\in(t_0,\infty]$, and $\varrho\in(0,\infty]$. Suppose also that there exists a continuous positive function $L(r)$ on $0\leqq r<\varrho$ such that, in D,

$$|f^{(1)}(t, x, X)|\leqq L\left(\max\{|x|, |X|\}\right) \quad \text{and} \quad |f^{(2)}(t, x, X)|\leqq L\left(\max\{|x|, |X|\}\right)$$

and $\int_0^\varrho dr/L(r)=\infty$. Then every solution of (2) exists on the entire interval $\alpha\leqq t<\gamma$.

Proof. Suppose (for contradiction) that $\beta<\gamma$, where β is as described in Theorem 1. Then it must follow that $\limsup_{t\to\beta-}|x(t)|=\varrho$. This implies that for some $\tilde{t}\in(t_0,\beta]$,

$$\|x\|^{[t_0,\tilde{t}]}=|\varphi(t_0)|+\int_{t_0}^{\tilde{t}}|x'(s)|\,ds=\varrho,$$

while $\|x\|^{[t_0,t]}<\varrho$ for $t\in[t_0,\tilde{t})$. Now, almost everywhere on the interval $(t_0,\tilde{t})$,

$$|x'(t)|\leqq L\left(\|x\|^{[t_0,t]}\right)+n\,p\,L\left(\|x\|^{[t_0,t]}\right)|x'(h(t))|,$$

or

$$\int_{t_0}^{\tilde{t}}\frac{|x'(t)|\,dt}{L(\|x\|^{[t_0,t]})}\leqq\tilde{t}-t_0+n\,p\int_{t_0}^{\tilde{t}}|x'(h(t))|\,dt.$$

Changing variables in the integrals and letting $\tau>0$ and $\eta>0$ be constants such that each $h_k(t)\leqq t-\tau$ and each $h_k'(t)\geqq\eta$ on $[t_0,\beta]$, one obtains, for arbitrary $\varrho^*\in(|\varphi(t_0)|,\varrho)$,

$$\int_{|\varphi(t_0)|}^{\varrho^*}dr/L(r)\leqq\tilde{t}-t_0+\frac{n\,p}{\eta}\int_{\alpha}^{\tilde{t}-\tau}|x'(\sigma)|\,d\sigma<\beta-t_0+\frac{n\,p}{\eta}\left(\|\varphi\|^{[\alpha,t_0]}+\varrho\right).$$

This contradicts the divergence of $\int_0^\varrho dr/L(r)$ since $L(r)$ is bounded away from zero on $[0,|\varphi(t_0)|]$.

Definition. We shall say that the solution, $x(t)$, of (1) or (2) is *unique* if every solution agrees with $x(t)$ as far as both are defined.

Definition. We shall say that a function $f(t, x, X)$ defined over D satisfies a *local Lipschitz condition* with respect to x (or with respect to x and X) in D if for each compact set $F\subset D$ there exists a (Lipschitz) constant L such that

$$|f(t, x, X)-f(t,\tilde{x}, X)|\leqq L|x-\tilde{x}| \quad \text{whenever} \quad (t, x, X) \quad \text{and} \quad (t,\tilde{x}, X)\in F$$

$$\left(\text{or}\quad |f(t, x, X)-f(t,\tilde{x},\tilde{X})|\leqq L\max\{|x-\tilde{x}|, |X-\tilde{X}|\}\quad \text{whenever}\quad (t, x, X)\quad \text{and}\quad (t,\tilde{x},\tilde{X})\in F\right).$$

Theorem 3 (Uniqueness for System (1)). *Let the conditions of Theorem 1 be satisfied, but with each* $g_j(t) < t$, *and for every* $Z \in E^{np}$ *let* $f(t, x, X, Z)$ *satisfy a local Lipschitz condition with respect to* x *in* D *with Lipschitz constant* $L(Z)$ *where* $L(Z)$ *is a continuous function of* Z *for any given compact subset of* D. *Then the solution of (1) is unique.*

Proof. On each of the intervals $(t_\nu, t_{\nu+1})$, the right hand side of the ordinary differential equation considered in part 2 of the proof of Theorem 1 will (almost everywhere) satisfy a local Lipschitz condition with respect to x. The Lipschitz constant, in a given compact set, can be taken to be

$$L_{(\nu+1)} = \max \{L(Z) : |Z| \leqq \operatorname*{ess\,sup}_{\alpha \leqq t \leqq t_\nu} |x'_{(\nu)}(t)|\}.$$

Uniqueness now follows successively on each of the intervals $[t_\nu, t_{\nu+1})$, $\nu = 0, 1, 2, \ldots$ (*cf.* CODDINGTON & LEVINSON [2], pp. 49—51).

Theorem 4 (Uniqueness for System (2)). *Let the conditions of Theorem 2 be satisfied, but with each* $g_j(t) < t$, *and let* $f^{(1)}(t, x, X)$ *and* $f^{(2)}(t, x, X)$ *satisfy a local Lipschitz condition with respect to* x *in* D. *Then the solution of (2) is unique.*

Proof. In a compact set, as considered for a typical interval $(t_\nu, t_{\nu+1})$ in the proof of Theorem 2, let L_1 and L_2 be Lipschitz constants for $f^{(1)}$ and $f^{(2)}$ respectively. Then

$$|f(t, x, X, Z) - f(t, \tilde{x}, X, Z)| \leqq (L_1 + np\, L_2 |x'_{(\nu)}(h(t))|)\, |x - \tilde{x}|,$$

and the function $L_{(\nu)}(t) = L_1 + np\, L_2 |x'_{(\nu)}(h(t))|$ is integrable on $[t_\nu, t_{\nu+1}]$. The only non-negative solution of the scalar equation $u'(t) = L_{(\nu)}(t)\, u(t)$ on $[t_\nu, t_{\nu+1})$ with $u(t_\nu) = 0$ is $u(t) \equiv 0$. For, if ever $u(t)$ became positive, the divergence of the integral $\int_0^{a>0} du/u$ would contradict the integrability of $L_{(\nu)}(t)$. (This is similar to a condition of Osgood, *cf.* KAMKE [13], p. 100.) Hence again uniqueness follows on the successive intervals $[t_\nu, t_{\nu+1})$, $\nu = 0, 1, 2, \ldots$ (*cf.* CODDINGTON & LEVINSON [2], pp. 49—51).

Remarks. The basic uniqueness theorem for ordinary differential equations, used in the proofs of Theorems 3 and 4, holds for conditions weaker than Lipschitz, hence weaker conditions could have been used here.

If the condition $g_j(t) < t$ is weakened to $g_j(t) \leqq t$ in Theorem 3 or 4, then for uniqueness one needs, in general, some condition on the behavior of $f(t, x, X, Z)$ with respect to x *and* X, e.g. a local Lipschitz condition with respect to x and X. As a matter of fact the resulting uniqueness theorem for system (2) in this case will follow from Theorem 5.

II. Continuous Dependence

In this section we shall show that if, in addition to the hypothesis of Theorem 2, $f^{(1)}(t, x, X)$ and $f^{(2)}(t, x, X)$ satisfy local Lipschitz conditions with respect to x and X, then the solution depends continuously on the functions $f^{(1)}, f^{(2)}, g, h$, and φ. Moreover, we obtain quantitative estimates in the course of the proof.

The proof of Theorem 5 — the main result of the paper — uses two lemmas.

Lemma 1. *Let $r(t)$ be a non-negative continuous function of t for $t_0-\tau\leqq t\leqq\beta<\infty$ such that*

$$r(t)\leqq F(t)+\int_{t_0}^{t}\vartheta(s)\,r(s)\,ds+K(t)\,r(t-\tau)\quad \text{for}\quad t_0\leqq t\leqq\beta,$$

where $F(t)$ and $K(t)$ are non-negative, non-decreasing, continuous functions on $[t_0,\beta)$, $\vartheta(t)$ is a non-negative integrable function on each finite subinterval of $[t_0,\beta)$, and τ is a positive constant. Then, letting $r_0=\max\{r(s):t_0-\tau\leqq s\leqq t_0\}$, we have on $[t_0,\beta)$

$$r(t)\leqq[F(t)+K(t)\,r_0]\,[1+K(t)]^{(t-t_0)/\tau}\,e^{\int_{t_0}^{t}\vartheta(s)\,ds}.$$

Remarks. In case $K(t)\equiv 0$ this reduces to a well-known inequality used in ordinary differential equations, originally due to W. T. Reid [*20*]. Simpler forms of that result date back at least to Peano [*17*]. In case $\vartheta(t)$ is bounded, the present result can be found qualitatively as special cases of theorems given by Pitt ([*19*], Theorem 8) and Wright ([*23*], Theorem 3), in order to justify use of the Laplace transform. In this paper, however, we wish to allow ϑ to be merely integrable.

Proof of Lemma 1. It is sufficient to prove the lemma for the case $F(t)\equiv F$ and $K(t)\equiv K$, constants. Then, to get the result stated for non-decreasing functions, one merely inserts into the estimate the largest values of $F(s)$ and $K(s)$ up to the instant t, namely $F(t)$ and $K(t)$.

We can now assume, without loss of generality, that $r(t)$ is non-decreasing. This follows from the fact that the non-decreasing function $r^*(t)\equiv\sup\{r(s):t_0-\tau\leqq s\leqq t\}$ will also satisfy the assumed inequality, and if the stated result holds for $r^*(t)$ then it certainly holds for $r(t)$.

We shall actually prove that the stated result is a consequence of the following inequality (weaker than the given one):

$$r(t)\leqq F+\int_{t_0}^{t}\vartheta(s)\,r(s)\,ds+K\sum_{\nu=0}^{\infty}u(t-t_0-\nu\tau)\,r(t_0+\nu\tau),$$

where

$$u(s)=\begin{cases}0 & \text{for}\quad s<0\\ 1 & \text{for}\quad s\geqq 0.\end{cases}$$

From this point of view the result is similar to a lemma announced recently by G. S. Jones [*10*]. However, we shall give an independent proof, making repeated use of the same type of calculation which one uses in proving Reid's Lemma (for $K=0$).

For any given $t^*\in[t_0,\beta)$ let n be the integer such that $t_0+n\tau\leqq t^*<t_0+(n+1)\tau$. Then, on each interval $t_0+i\tau\leqq t<t_0+(i+1)\tau$ $(i=0,1,\ldots,n)$, with the understanding that $t_0+(n+1)\tau$ is to be replaced by t^*, we obtain the inequality

$$r(t)-R(t)\leqq F+K\sum_{\nu=0}^{i}r(t_0+\nu\tau),$$

where $R(t)\equiv\int_{t_0}^{t}\vartheta(s)\,r(s)\,ds$. Let us define $\Theta(t)\equiv\exp\left\{-\int_{t_0}^{t}\vartheta(s)\,ds\right\}$ and multiply the last inequality through by $\vartheta(t)\,\Theta(t)$. Integrating the result from $t_0+i\tau$ to

$t_0+(i+1)\tau$ gives

$$R\bigl(t_0+(i+1)\,\tau\bigr)\,\Theta\bigl(t_0+(i+1)\,\tau\bigr)-R(t_0+i\,\tau)\,\Theta(t_0+i\,\tau)$$
$$\leqq F\,[\Theta(t_0+i\,\tau)-\Theta(t_0+(i+1)\,\tau)]+$$
$$+K\sum_{\nu=0}^{i} r(t_0+\nu\,\tau)\,[\Theta(t_0+i\,\tau)-\Theta(t_0+(i+1)\,\tau)].$$

In case $i=1, 2, \ldots, n$ we modify this inequality by adding to it the inequality

$$-K R(t_0+i\,\tau)\,\Theta(t_0+i\,\tau)\leqq K\left[F+K\sum_{\nu=0}^{i-1} r(t_0+\nu\,\tau)\right]\Theta(t_0+i\,\tau)-$$
$$-K\,r(t_0+i\,\tau)\,\Theta(t_0+i\,\tau),$$

which results from the continuity of $r(t)$ and $R(t)$ at $t_0+i\tau$.

Thus, for $i=0$,

$$R(t_0+\tau)\,\Theta(t_0+\tau)\leqq F\,[1-\Theta(t_0+\tau)]+K\,r_0[1-\Theta(t_0+\tau)],$$

and, for $i=1, \ldots, n$,

$$R\bigl(t_0+(i+1)\,\tau\bigr)\,\Theta\bigl(t_0+(i+1)\,\tau\bigr)-(1+K)\,R(t_0+i\,\tau)\,\Theta(t_0+i\,\tau)$$
$$\leqq(1+K)\,F\,\Theta(t_0+i\,\tau)-F\,\Theta\bigl(t_0+(i+1)\,\tau\bigr)+$$
$$+K(1+K)\sum_{\nu=0}^{i-1} r(t_0+\nu\,\tau)\,\Theta(t_0+i\,\tau)-K\sum_{\nu=0}^{i} r(t_0+\nu\,\tau)\,\Theta\bigl(t_0+(i+1)\,\tau\bigr).$$

Multiplying each of these inequalities by $(1+K)^{-i}$ and summing from $i=0$ to n, we obtain

$$\frac{1}{(1+K)^n}\,R(t^*)\,\Theta(t^*)\leqq F-\frac{F}{(1+K)^n}\,\Theta(t^*)+K\,r_0-\frac{K}{(1+K)^n}\sum_{\nu=0}^{n} r(t_0+\nu\,\tau)\,\Theta(t^*).$$

Combining this with the inequality

$$r(t^*)\leqq R(t^*)+F+K\sum_{\nu=0}^{n} r(t_0+\nu\,\tau)$$

gives

$$r(t^*)\leqq(F+K\,r_0)\,(1+K)^n\,\frac{1}{\Theta(t^*)}\leqq(F+K\,r_0)\,(1+K)^{(t^*-t_0)/\tau}\,e^{\int_{t_0}^{t^*}\vartheta(s)\,ds}. \qquad \text{Q.E.D.}$$

The second lemma which we shall use may also be known, but the proof is included here for completeness.

Lemma 2. *Let $\vartheta(t)$ be Lebesgue integrable on an interval $[\alpha, \beta]$. Then for each $\eta>0$ and $\varepsilon>0$ there exists a number $\delta>0$ such that*

$$\int_a^b |\vartheta(h(t))-\vartheta(\tilde h(t))|\,dt<\varepsilon$$

whenever $h(t)$ and $\tilde h(t)$ are absolutely continuous functions from $[a, b]$ into $[\alpha, \beta]$ satisfying the following conditions:

$$h'(t)\geqq\eta \quad \textit{and} \quad \tilde h'(t)\geqq\eta \quad \textit{(almost everywhere) on} \quad [a, b],$$

and

$$|h(t)-\tilde h(t)|<\delta \quad \textit{for all} \quad t\in[a, b].$$

Proof. Choose a continuous function $z(t)$ on $[\alpha, \beta]$ such that

$$\int_\alpha^\beta |\vartheta(t) - z(t)|\, dt < \tfrac{1}{3}\eta\,\varepsilon$$

(*cf.* McShane [*14*], p. 229). Then choose $\delta > 0$ such that

$$|z(t) - z(\tilde{t})| < \frac{\varepsilon}{3(b-a)} \quad \text{whenever} \quad t, \tilde{t} \in [\alpha, \beta] \quad \text{with} \quad |t - \tilde{t}| < \delta.$$

Then, letting $h(t)$ and $\tilde{h}(t)$ satisfy the conditions of the lemma with the chosen δ, we have

$$\int_a^b |\vartheta(h(t)) - \vartheta(\tilde{h}(t))|\, dt \leqq \int_a^b |\vartheta(h(t)) - z(h(t))|\, dt + \\ + \int_a^b |z(h(t)) - z(\tilde{h}(t))|\, dt + \int_a^b |z(\tilde{h}(t)) - \vartheta(\tilde{h}(t))|\, dt.$$

The first and third integrals on the right hand side are each shown to be less than $\frac{1}{3}\varepsilon$ by an estimate like that used at the end of the proof of Theorem 2. The second integral on the right is less than $\frac{1}{3}\varepsilon$ by the uniform smallness of its integrand. Q.E.D.

Remarks. This lemma could easily be generalized to functions $\vartheta \in L_p$ $(p \geqq 1)$ with the L_p metric, instead of L_1. M.L. Slater has pointed out that by making the function $z(t)$ a polynomial and applying the mean value theorem to estimate the second integral, one could reduce the requirement that $|h(t) - \tilde{h}(t)|$ be small to one that $\int_a^b |h(t) - \tilde{h}(t)|\, dt$ be small.

Theorem 5 (Continuous Dependence of Solutions on the Initial Data and on the Right Hand Sides of the Equations). *Let $f^{(1)}(t, x, X)$, $f^{(2)}(t, x, X)$, $g(t)$, $h(t)$, and $\varphi(t)$ satisfy the conditions of Theorem 2 where the interval $[t_0, \gamma)$ is replaced by the larger interval D^1. Let $f^{(1)}$ and $f^{(2)}$ satisfy a local Lipschitz condition with respect to x and X in D. Let $x(t)$ on $\alpha \leqq t < \beta$, where $\beta > t_0$, be a solution (which will turn out to be unique).*

Let any $\bar{\beta} \in (t_0, \beta)$, any $\eta > 0$, and any $\varepsilon > 0$ be given. Then there exist positive numbers $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$, and δ_6 with the following property. If $\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{g}, \tilde{h}$, and $\tilde{\varphi}$ on $[\alpha, \tilde{t}_0]$ are new functions satisfying the conditions of Theorem 2 (with $\tilde{t}_0$ in place of t_0 and D^1 in place of $[t_0, \gamma)$), and if

$$|\tilde{f}^{(1)}(t, x, X) - f^{(1)}(t, x, X)| \leqq \delta_1 \quad in \quad D,$$
$$|\tilde{f}^{(2)}(t, x, X) - f^{(2)}(t, x, X)| \leqq \delta_2 \quad in \quad D,$$
$$|\tilde{g}(t) - g(t)| \leqq \delta_3 \quad on \quad D^1,$$
$$|\tilde{h}(t) - h(t)| \leqq \delta_4 \quad and \quad \tilde{h}_k'(t) \geqq \eta \ (k = 1, \ldots, p) \quad on \quad D^1,$$
$$t_0 - \delta_5 \leqq \tilde{t}_0 < \bar{\beta},$$

and

$$\|\tilde{\varphi} - x\|^{[\alpha, \tilde{t}_0]} \leqq \delta_6,$$

then any solution, $\tilde{x}(t)$, of the new problem, $\tilde{P}$, can be extended to $\alpha \leqq t \leqq \bar{\beta}$ and

$$\|\tilde{x} - x\|^{[\alpha, \bar{\beta}]} < \varepsilon.$$

Remarks. The solution of $\tilde{P}$ may not be unique.

The uniqueness of the solution of P — which does not follow from Theorem 4 in the case $g_j(t) \leqq t$ — is an immediate corollary of the present theorem.

Proof of Theorem 5. The proof is broken down into four parts.

1. Preliminary conventions

Let us assume, without loss of generality, that ε is so small that the point (t, x, X) lies in D whenever $|(t, x, X) - (s, x(s), x(g(s)))| < \varepsilon$ for some $s \in [t_0, \bar{\beta}]$.

Then we can also assume, without loss of generality, that $\alpha > -\infty$. This follows from the fact that each of the functions $g_j(t)$ and $h_k(t)$ is continuous, and therefore bounded, on the compact set $[t_0 - \varepsilon, \bar{\beta}]$, while each $\tilde{g}_j(t)$ and $\tilde{h}_k(t)$ will be close to the respective values of $g_j(t)$ and $h_k(t)$.

Since each $h'_k(t)$ is continuous and positive on the compact set $[t_0 - \varepsilon, \bar{\beta}]$, each is bounded away from zero there. Thus we can further assume, without loss of generality, that $\eta > 0$ is so small that each $h'_k(t) \geqq \eta$ as well as $\tilde{h}'_k(t) \geqq \eta$ on $[t_0 - \varepsilon, \bar{\beta}]$.

2. Definition of $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$, and δ_6

Choose any numbers ε_1 and ε_2 such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Then the set

$$D_1 = \{(t, x, X) : |(t, x, X) - (s, x(s), x(g(s)))| \leqq \varepsilon_1 \quad \text{for some} \quad s \in [t_0, \bar{\beta}]\}$$

is a bounded domain, and its closure, $\bar{D}_1$, is a compact subset of D. We shall also use a set D_2 defined in the same way as D_1 except that ε_1 is replaced by ε_2. This gives $\bar{D}_2 \subset D_1$.

By the compactness of $\bar{D}_1$ there exist constants B_1, B_2, L_1, and L_2 such that

$$|f^{(1)}(t, x, X)| \leqq B_1, \qquad |f^{(2)}(t, x, X)| \leqq B_2,$$

$$|f^{(1)}(t, x, X) - f^{(1)}(t, \tilde{x}, \tilde{X})| \leqq L_1 \max\{|x - \tilde{x}|, |X - \tilde{X}|\},$$

and

$$|f^{(2)}(t, x, X) - f^{(2)}(t, \tilde{x}, \tilde{X})| \leqq L_2 \max\{|x - \tilde{x}|, |X - \tilde{X}|\}$$

whenever $(t, x, X), (t, \tilde{x}, \tilde{X}) \in D_1$; and, by the compactness of $[t_0 - \varepsilon_1, \bar{\beta}]$ there exists a constant $\tau > 0$ such that

$$h_k(t) \leqq t - 2\tau \quad \text{for} \quad t_0 - \varepsilon_1 \leqq t \leqq \bar{\beta}, \qquad k = 1, \ldots, p.$$

Now choose positive numbers $\delta_1, \delta_2, \delta_6, \Delta_1, \Delta_2$, and Δ_3 such that

$$S \equiv \left\{ \left[\left(1 + \frac{np}{\eta}(B_2 + \delta_2)\right)(\delta_6 + \Delta_1) + (B_1 + \delta_1)\Delta_3 \right] \left[1 + \frac{np}{\eta}(B_2 + \delta_2)\right] + \right.$$
$$\left. + np(B_2 + \delta_2)\Delta_2 + (\delta_1 + L_1\Delta_1)(\bar{\beta} - t_0) + \frac{np}{\eta}(\delta_2 + L_2\Delta_1)\int_\alpha^{\bar{\beta}} |x'(\sigma)|\, d\sigma \right\} \times$$
$$\times \left[1 + \frac{np}{\eta}(B_2 + \delta_2)\right]^{(\bar{\beta} - t_0)/\tau} \exp\left[L_1(\bar{\beta} - t_0) + \frac{np}{\eta} L_2 \int_\alpha^{\bar{\beta}} |x'(\sigma)|\, d\sigma\right] < \varepsilon_2 - \Delta_1.$$

Then choose δ_3, δ_4, and δ_5 such that

$$\delta_4 \leqq \tau,$$

$$\delta_5 \leqq \min(\Delta_3, \varepsilon_2),$$

$$\min_{k=1,\ldots,p} h_k(t_0 - \delta_5) - \delta_4 \geqq \alpha,$$

$$\int_a^b |\varphi'(\sigma)|\, d\sigma \leqq \Delta_1 \quad \text{whenever} \quad \alpha \leqq a < b \leqq t_0$$

$$\text{with} \quad (b-a) \leqq \max\big(\delta_5, |h(t_0) - h(t_0 - \delta_5)| + 2\delta_4\big),$$

$$|\varphi(b) - \varphi(a)| \leqq \Delta_1 \quad \text{whenever} \quad \alpha \leqq a < b \leqq t_0$$

$$\text{with} \quad (b-a) \leqq \delta_3 + \max_{t_0 - \delta_5 \leqq s \leqq t_0} |g(t_0) - g(s)|,$$

$$|x(b) - x(a)| \leqq \Delta_1 \quad \text{whenever} \quad \alpha \leqq a < b \leqq \bar{\beta}$$

$$\text{with} \quad (b-a) \leqq \delta_3, \quad \text{and}$$

$$\int_{t_0}^{\bar{\beta}} |x'(\tilde{h}(s)) - x'(h(s))|\, ds \leqq \Delta_2 \quad \text{whenever} \quad (\text{for } k = 1, \ldots, p)\ h_k(s)$$

and $\tilde{h}_k(s)$ are absolutely continuous mappings of $[t_0, \bar{\beta}]$ into $[\alpha, \bar{\beta}]$, $h_k'(s) \geqq \eta$, $\tilde{h}_k'(s) \geqq \eta$, and $|\tilde{h}(s) - h(s)| \leqq \delta_4$ for all $s \in [t_0, \bar{\beta}]$.

This last requirement can be achieved by Lemma 2.

3. *Existence of a solution of $\tilde{P}$ in D_1*

Let $\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{g}, \tilde{h}, \tilde{t}_0$, and $\tilde{\varphi}$ satisfy the conditions stated in the theorem using the $\delta_1, \ldots, \delta_6$ defined above. We shall show that $(\tilde{t}_0, \tilde{\varphi}(\tilde{t}_0), \tilde{\varphi}(\tilde{g}(\tilde{t}_0))) \in D_2 \subset D_1$ and thence the existence of a solution, $\tilde{x}$, in D_1.

In case $t_0 \leqq \tilde{t}_0 < \bar{\beta}$,

$$\begin{aligned} &|(\tilde{t}_0, \tilde{\varphi}(\tilde{t}_0), \tilde{\varphi}(\tilde{g}(\tilde{t}_0))) - (\tilde{t}_0, x(\tilde{t}_0), x(g(\tilde{t}_0)))| \\ &\quad \leqq \max\{\delta_6, |\tilde{\varphi}(\tilde{g}(\tilde{t}_0)) - x(\tilde{g}(\tilde{t}_0))| + |x(\tilde{g}(\tilde{t}_0)) - x(g(\tilde{t}_0))|\} \\ &\quad \leqq \delta_6 + \Delta_1 < \varepsilon_2. \end{aligned}$$

In case $t_0 - \delta_5 \leqq \tilde{t}_0 < t_0$,

$$\begin{aligned} &|(\tilde{t}_0, \tilde{\varphi}(\tilde{t}_0), \tilde{\varphi}(\tilde{g}(\tilde{t}_0))) - (t_0, \varphi(t_0), \varphi(g(t_0)))| \\ &\quad \leqq \max\{\delta_5, \delta_6 + \max(|\varphi(\tilde{t}_0) - \varphi(t_0)|, |\varphi(\tilde{g}(\tilde{t}_0)) - \varphi(g(t_0))|)\} \\ &\quad \leqq \max\{\delta_5, \delta_6 + \Delta_1\} < \varepsilon_2. \end{aligned}$$

It follows from Theorem 2, with D_1 playing the role of D, that a solution, $\tilde{x}(t)$, of the new problem exists on $\alpha \leqq t < \tilde{\beta}$ where $\tilde{\beta} > \tilde{t}_0$. Moreover, to each compact set $F \subset D_1$ there corresponds an infinite sequence $\tilde{t}_0 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots \to \tilde{\beta}$ such that $(\tilde{\xi}_i, \tilde{x}(\tilde{\xi}_i), \tilde{x}(\tilde{g}(\tilde{\xi}_i))) \in D_1 - F$ for $i = 1, 2, \ldots$.

4. *Proof that* $\tilde{\beta} > \bar{\beta}$ *and* $\|\tilde{x} - x\|^{[\alpha, \bar{\beta})} < \varepsilon$

It will suffice to prove that $\|\tilde{x} - x\|^{[\alpha, t)} < \varepsilon_2$ and $(t, \tilde{x}(t), \tilde{x}(\tilde{g}(t))) \in \bar{D}_2$, a compact subset of D_1, for $\tilde{t}_0 \leqq t < \min(\tilde{\beta}, \bar{\beta})$. This will then rule out the possibility that $\tilde{\beta} \leqq \bar{\beta}$ and also lead to $\|\tilde{x} - x\|^{[\alpha, \bar{\beta})} \leqq \varepsilon_2 < \varepsilon$.

Letting $\bar{t}_0 = \max(t_0, \tilde{t}_0)$, we must now consider the two cases $\tilde{t}_0 \leqq t < \min(t_0, \tilde{\beta})$ and $\bar{t}_0 \leqq t < \min(\bar{\beta}, \tilde{\beta})$.

In case $\tilde{t}_0 \leqq t < \min(t_0, \tilde{\beta})$, we find

$$
\begin{aligned}
\|\tilde{x} - x\|^{[\alpha, t)} &= \|\tilde{\varphi} - \varphi\|^{[\alpha, \tilde{t}_0)} + \int_{\tilde{t}_0}^{t} |\tilde{x}'(s) - \varphi'(s)|\, ds \\
&\leqq \delta_6 + \int_{t_0 - \delta_6}^{t_0} [B_1 + \delta_1 + n p (B_2 + \delta_2)\, |\tilde{x}'(h(s))| + |\varphi'(s)|]\, ds \\
&\leqq \delta_6 + (B_1 + \delta_1)\, \delta_5 + \frac{np}{\eta} (B_2 + \delta_2) \max_{k = 1, \ldots, p} \int_{\tilde{h}_k(t_0 - \delta_5)}^{\tilde{h}_k(t_0)} |\tilde{x}'(\sigma)|\, d\sigma + \Delta_1 \\
&\leqq (\delta_6 + \Delta_1) + (B_1 + \delta_1)\, \Delta_3 + \\
&\quad + \frac{np}{\eta} (B_2 + \delta_2) \left[\int_{\alpha}^{t_0} |\tilde{x}'(\sigma) - \varphi'(\sigma)|\, d\sigma + \max_{k} \int_{h_k(t_0 - \delta_5) - \delta_4}^{h_k(t_0) + \delta_4} |\varphi'(\sigma)|\, d\sigma \right] \\
&\leqq \left(1 + \frac{np}{\eta} (B_2 + \delta_2)\right) (\delta_6 + \Delta_1) + (B_1 + \delta_1)\, \Delta_3 \leqq S < \varepsilon_2 .
\end{aligned}
$$

In case $\bar{t}_0 \leqq t < \min(\bar{\beta}, \tilde{\beta})$,

$$
\begin{aligned}
\|\tilde{x} - x\|^{[\alpha, t)} &= \|\tilde{x} - x\|^{[\alpha, \bar{t}_0)} + \int_{\bar{t}_0}^{t} |\tilde{x}'(s) - x'(s)|\, ds \\
&\leqq \|\tilde{x} - x\|^{[\alpha, \bar{t}_0)} + \int_{\bar{t}_0}^{t} |\tilde{f}^{(1)}(s, \tilde{x}(s), \tilde{x}(\tilde{g}(s))) - f^{(1)}(s, x(s), x(g(s)))|\, ds + \\
&\quad + \int_{\bar{t}_0}^{t} |\tilde{f}^{(2)}(s, \tilde{x}(s), \tilde{x}(\tilde{g}(s)))\, \tilde{x}'(\tilde{h}(s)) - f^{(2)}(s, x(s), x(g(s)))\, x'(h(s))|\, ds \\
&\leqq \|\tilde{x} - x\|^{[\alpha, \bar{t}_0)} + \int_{\bar{t}_0}^{t} [\delta_1 + L_1 \max(|\tilde{x}(s) - x(s)|, |\tilde{x}(\tilde{g}(s)) - x(g(s))|)]\, ds + \\
&\quad + np \int_{\bar{t}_0}^{t} [(B_2 + \delta_2)\, |\tilde{x}'(\tilde{h}(s)) - x'(h(s))| + \\
&\quad + (\delta_2 + L_2 \max(|\tilde{x}(s) - x(s)|, |\tilde{x}(\tilde{g}(s)) - x(g(s))|))\, |x'(h(s))|]\, ds \\
&\leqq \|\tilde{x} - x\|^{[\alpha, \bar{t}_0)} + \int_{\bar{t}_0}^{t} [\delta_1 + L_1 \|\tilde{x} - x\|^{[\alpha, s)} + L_1 \Delta_1]\, ds + \\
&\quad + np (B_2 + \delta_2) \int_{\bar{t}_0}^{t} [|\tilde{x}'(\tilde{h}(s)) - x'(\tilde{h}(s))| + |x'(\tilde{h}(s)) - x'(h(s))|]\, ds + \\
&\quad + np \int_{\bar{t}_0}^{t} [\delta_2 + L_2 \|\tilde{x} - x\|^{[\alpha, s)} + L_2 \Delta_1]\, |x'(h(s))|\, ds .
\end{aligned}
$$

Now, using the fact that each $\tilde{h}_k(t) \leqq h_k(t) + \tau \leqq t - \tau$ on $[t_0, \bar{\beta}]$, we obtain

$$\begin{aligned}\|\tilde{x} - x\|^{[\alpha,t)} \leqq & \|\tilde{x} - x\|^{[\alpha,\bar{t}_0)} + (\delta_1 + L_1 \Delta_1)(t - \bar{t}_0) + \\ & + \frac{np}{\eta}(B_2 + \delta_2)\|\tilde{x} - x\|^{[\alpha,t-\tau)} + np(B_2 + \delta_2)\Delta_2 + \\ & + \frac{np}{\eta}(\delta_2 + L_2\Delta_1)\int_\alpha^{t-\tau}\|x'(\sigma)\|\,d\sigma + \int_{\bar{t}_0}^{t}\big(L_1 + npL_2|x'(h(s))|\big)\|\tilde{x} - x\|^{[\alpha,s)}\,ds.\end{aligned}$$

Therefore, by Lemma 1 with $r(t) = \|\tilde{x} - x\|^{[\alpha,t)}$,

$$\begin{aligned}\|\tilde{x} - x\|^{[\alpha,t)} \leqq & \Big\{\|\tilde{x} - x\|^{[\alpha,\bar{t}_0)}\Big(1 + \frac{np}{\eta}(B_2 + \delta_2)\Big) + np(B_2 + \delta_2)\Delta_2 + \\ & + (\delta_1 + L_1\Delta_1)(t - \bar{t}_0) + \frac{np}{\eta}(\delta_2 + L_2\Delta_1)\int_\alpha^{t-\tau}|x'(\sigma)|\,d\sigma\Big\} \times \\ & \times \Big[1 + \frac{np}{\eta}(B_2 + \delta_2)\Big]^{(t-\bar{t}_0)/\tau}\exp\Big[L_1(t - \bar{t}_0) + \frac{np}{\eta}L_2\int_\alpha^{t-\tau}|x'(\sigma)|\,d\sigma\Big].\end{aligned}$$

Applying the estimate for $\|\tilde{x} - x\|^{[\alpha,\bar{t}_0)}$ obtained in the first case and the defining condition for $\delta_1, \delta_2, \delta_6, \Delta_1, \Delta_2$, and Δ_3, we find

$$\|\tilde{x} - x\|^{[\alpha,t)} \leqq S < \varepsilon_2.$$

Now from these two cases it follows that for $\tilde{t}_0 \leqq t < \min(\tilde{\beta}, \bar{\beta})$,

$$|\tilde{x}(\tilde{g}(t)) - x(g(t))| \leqq |\tilde{x}(\tilde{g}(t)) - x(\tilde{g}(t))| + |x(\tilde{g}(t)) - x(g(t))| \leqq S + \Delta_1 < \varepsilon_2.$$

Thus $(t, \tilde{x}(t), \tilde{x}(\tilde{g}(t))) \in D_2$. Q.E.D.

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