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Point Data Problems for Functional Differential Equations

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Consider, as a simple prototype, the linear scalar delay differential equation

$$x'(t) = f(t)x(t-1), \tag{1}$$

and let us ask whether it makes any sense to seek solutions on R, or just on $(-\infty, t_0]$, subject to given point data

$$x(t_0) = x_0. \tag{2}$$

At first glance, this problem looks quite hopeless. Indeed, if f is a nonzero constant it is easily seen that there are infinitely many solutions of (1) valid on R that vanish (together with all their derivatives) at t_0 [4].

Doss and Nasr [2] in 1953, and essentially also Fite [5] in 1921, observed that if f is continuous and

$$\int_{-\infty}^{t_0} |f(t)|\, dt < 1, \tag{3}$$

then Eqs. (1) and (2) have a unique *bounded* solution on $(-\infty, t_0]$.

As another example, the functional differential equation

$$x'(t) = f(t)x(g(t)), \tag{4}$$

with nonconstant delay (or advance), has a unique solution satisfying (2) provided f and g are continuous and

$$|g(t) - t_0| \le |t - t_0| \qquad \text{for all} \quad t. \tag{5}$$

This was shown by Fite [5] and was implicit in Polossuchin's dissertation [8] in 1910.

The above authors all considered more general equations than (1) and (4). But their papers are apparently not well known, for minor variants of these

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elementary observations have been rediscovered and republished dozens of times in the past few years.

Here we will show how a natural extension of Fite's and Doss and Nasr's theorems does actually have interesting applications.

Let $J \subset R$ be an interval (open, closed, or half open as may be appropriate). Let D be some subset of R^n and let G be a functional mapping $J \times C(J, D) \to R^n$. We write $|\cdot|$ for any convenient norm on R^n. Then we shall consider the functional differential system

$$x'(t) = G(t, x) \tag{6}$$

on J, with given point data

$$x(t_0) = x_0 \tag{7}$$

for some finite $t_0 \in J$ and $x_0 \in D$.

A *solution* of (6) and (7) will mean a differentiable function $x: J \to D$, which satisfies (6) on J and satisfies (7).

Note that this formulation covers systems involving advanced as well as delayed arguments. Note further that the interval J *must be unbounded* whenever constant delays or advances are involved in G.

In order to get uniqueness of the solution of (6) and (7) we will, in general, have to restrict further the class of functions considered (as Doss and Nasr [2], Myškis [6], and others did). Instead of demanding boundedness of x, let us introduce a function

$$\mu \in C(J, (0, 1]),$$

and seek solutions of (6) and (7) in some set

$$S \subset \{z \in C(J, D) : \mu(t)|z(t)| \text{ is bounded}\}.$$

Theorems 1 and 2 give sufficient conditions for existence, uniqueness, and continuous dependence of solutions *in the class S*.

Theorem 1. Assume that

(i) (S, d) is a complete metric space when

$$d(z, \tilde{z}) \equiv \sup_{t \in J} |z(t) - \tilde{z}(t)|\mu(t).$$

(ii) For $z \in S$, $G(\cdot, z) \in C(J, R^n)$ and $Tz \in S$, where

$$(Tz)(t) \equiv x_0 + \int_{t_0}^{t} G(s, z)\, ds.$$

(iii) There exists a function $K_1 \in C(J, R_+)$ such that whenever $z, \tilde{z} \in S$,

$$|G(t, z) - G(t, \tilde{z})| \leq K_1(t)d(z, \tilde{z}),$$

while $\mu(t)|\int_{t_0}^t K_1(s)\, ds| \leq r < 1$ for each $t \in J$.

Then (6) and (7) have a unique solution in S.

Proof. If $z, \tilde{z} \in S$, then $d(Tz, T\tilde{z}) \leq rd(z, \tilde{z})$, and the assertion follows from the contraction mapping theorem.

To study continuous dependence of solutions, let G also depend on a parameter λ in some metric space (Λ, ρ). Specifically, let $G : J \times C(J, D) \times \Lambda \to R^n$ and consider the system

$$x'(t) = G(t, x; \lambda) \tag{8}$$

on J with point data (7). Then the following theorem asserts continuous dependence on everything in sight (jointly).

Theorem 2. Let $N \subset J$ and $H \subset D$, and assume that hypotheses (i) and (ii) of Theorem 1 hold for each $(t_0, x_0, \lambda) \in N \times H \times \Lambda$. Also assume that

(iii′) There exist functions K_1 and $K_2 \in C(J, R_+)$ such that whenever $z, \tilde{z} \in S$, $\lambda, \tilde{\lambda} \in \Lambda$, and $t \in J$,

$$|G(t, z; \lambda) - G(t, \tilde{z}; \tilde{\lambda})| \leq K_1(t)d(z, \tilde{z}) + K_2(t)\rho(\lambda, \tilde{\lambda}),$$

while, for each $t_0 \in N$ and $t \in J$,

$$\mu(t)\left|\int_{t_0}^t K_1(s)\, ds\right| \leq r < 1, \qquad \text{and} \qquad \mu(t)\left|\int_{t_0}^t K_2(s)\, ds\right| \leq B.$$

Then, given $(t_0, x_0, \lambda) \in N \times H \times \Lambda$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that if $(\tilde{t}_0, \tilde{x}_0, \tilde{\lambda}) \in N \times H \times \Lambda$ with

$$|t_0 - \tilde{t}_0| < \delta, \qquad |x_0 - \tilde{x}_0| < \delta, \qquad \text{and} \qquad \rho(\lambda, \tilde{\lambda}) < \delta,$$

the corresponding unique solutions in S satisfy $d(x, \tilde{x}) < \varepsilon$.

Proof. Consider (t_0, x_0, λ) a given (fixed) point and $(\tilde{t}_0, \tilde{x}_0, \tilde{\lambda})$ a nearby point, both in $N \times H \times \Lambda$. Then, for each $t \in J$,

$$|x(t) - \tilde{x}(t)| \leq |x_0 - \tilde{x}_0| + \left|\int_{\tilde{t}_0}^{t_0} G(s, x; \lambda)\, ds\right|$$

$$+ \left|\int_{t_0}^t K_1(s)\, ds\right| d(x, \tilde{x}) + \left|\int_{t_0}^t K_2(s)\, ds\right| \rho(\lambda, \tilde{\lambda}).$$

Multiply through by $\mu(t)$, recalling that $0 < \mu(t) \leq 1$. Then take the supremum over $t \in J$ to find

$$(1-r)d(x, \tilde{x}) \leq |x_0 - \tilde{x}_0| + \left| \int_{t_0}^{t_0} G(s, x; \lambda)\,ds \right| + B\rho(\lambda, \tilde{\lambda}),$$

from which the asserted continuous dependence follows.

The above proofs were very simple because of the nature of the assumptions on G. The theorems provide a format for unifying some special cases, rather than particularly interesting results themselves.

For brevity, only Theorem 1 will be illustrated with examples.

The first two examples, from Nersesjan [7], show the need for the main hypotheses in Theorem 1. Both of these examples use the linear scalar Eq. (1).

Example 1. Consider Eq. (1) with $J = (-\infty, 0]$, $t_0 = 0$, and $f(t) = 2te^{2t-1}$. Apply Theorem 1 with $\mu(t) \equiv 1$, $S = \{z \in C(J, R)\colon z \text{ bounded}\}$, and $K_1(t) = |f(t)|$. It follows that for each $x_0 \in R$ there is one and only one bounded solution of Eqs. (1) and (2) on J. However, it so happens that there is also an unbounded solution given by $x(t) = x_0 e^{t^2}$. See also de Bruijn [1].

Example 2. Consider Eq. (1) with $J = (-\infty, t_0]$, where $t_0 \leq 0$, and

$$f(t) = \begin{cases} 0 & \text{for} \quad t < -1, \\ -2 - 2t & \text{for} \quad -1 \leq t \leq 0. \end{cases}$$

Again take $\mu(t) \equiv 1$, $S = \{z \in C(J, R): z \text{ bounded}\}$, and $K_1(t) = |f(t)|$. Then for $t_0 < 0$, Theorem 1 guarantees the existence of a unique bounded solution of Eqs. (1) and (2) on J. Moreover, since $f(t) \equiv 0$ for $t < -1$, there can be no unbounded solutions. Thus Eqs. (1) and (2) have a unique solution on J, period!

However, if $t_0 = 0$ condition (iii) of Theorem 1 fails (since $r = 1$). And indeed, it is easily verified that each solution of (1) on $(-\infty, 0]$ has the form

$$x(t) = \begin{cases} c & \text{for} \quad t < -1, \\ -c(2t + t^2) & \text{for} \quad -1 \leq t \leq 0, \end{cases}$$

where c is a constant. Thus there are no solutions of (1) and (2) if $x_0 \neq 0$ and infinitely many if $x_0 = 0$.

Example 3 (Small delays). Consider the vector equation

$$x'(t) = F(t, x_t) \qquad \text{on} \quad J = (-\infty, t_0], \tag{9}$$

where F is a continuous functional mapping $J \times \mathscr{C} = J \times C([-\tau, 0], R^n) \to R^n$. We adopt the usual conventions $x_t(s) = x(t+s)$ for $-\tau \le s \le 0$ and $\|\psi\| = \sup_{-\tau \le s \le 0} |\psi(s)|$ for $\psi \in \mathscr{C}$.

Assume there are numbers $A > 0$ and $K > 0$ such that

$$|F(t, 0)| \le Ae^{|t|/\tau} \qquad \text{for} \quad t \le t_0,$$

and whenever (t, ψ) and $(t, \tilde{\psi}) \in J \times \mathscr{C}$,

$$|F(t, \psi) - F(t, \tilde{\psi})| \le K\|\psi - \tilde{\psi}\|.$$

If $K\tau e < 1$, Theorem 1 applies with $\mu(t) = e^{(t-t_0)/\tau}$,

$$S = \{z \in C(J, R^n) : e^{t/\tau}|z(t)| \text{ is bounded}\},$$

and $K_1(t) = Ke^{(t_0 - t)/\tau + 1}$. Thus, for each $x_0 \in R^n$, there is a unique solution of (9) and (7) in S.

Example 3 is of interest because these solutions, extended to R, are actually the "special solutions" that characterize the asymptotic behavior of all solutions of Eq. (9) as $t \to +\infty$. See Rjabov [10], Uvarov [12], and Driver [3].

Corollary (to Example 3). Pointwise degeneracy (see Popov [9]) cannot occur for a system (9) having Lipschitz constant $K < 1/\tau e$ and $F(t, 0)$ bounded.

Example 4 [Generalizing condition (5)]. Consider (6) and (7), with $t_0 = 0$ for convenience. Assume that

(a) $G(\cdot, z) \in C(J, R^n)$ for each $z \in C(J, D)$,

and there exist constants A and K such that

(b) $0 \in D$ and $|G(t, 0)| \le Ae^{K|t|}$ for all $t \in J$,

(c) $|G(t, z) - G(t, \tilde{z})| \le K \sup_{|s| \le |t|, s \in J} |z(s) - \tilde{z}(s)|$, whenever $t \in J$ and $z, \tilde{z} \in C(J, D)$.

Choose any $c > K$ and apply Theorem 1 with $\mu(t) = e^{-c|t|}$,

$$S = \{z \in C(J, D) : e^{-c|t|}|z(t)| \text{ is bounded}\},$$

and $K_1(t) = Ke^{c|t|}$. The conclusion is that Eqs. (6) and (7) have a unique solution in S. Moreover, it can be shown that any solution of (6) and (7) must lie in S. So we actually have unqualified uniqueness.

Example 5. The equations of one-dimensional motion for two classical electrons give rise to a complicated system of functional differential equations with state-dependent delays [4]. However, in several respects, a reasonable "prototype" appears to be the simple equation

$$y''(t) = a/y^2(t-\tau), \tag{10}$$

where a and τ are positive constants. Let us seek a solution of (10) on $J = (-\infty, 0]$ satisfying

$$y(0) = y_0 > 0, \qquad y'(0) = v_0 < 0. \tag{11}$$

It is convenient to put (10) and (11) into the format of (6) and (7) with $n = 1$. This can be done by introducing $v(t) = y'(t)$ and writing

$$v'(t) = a\left[y_0 - \int_{t-\tau}^{0} v(s)\,ds\right]^{-2} \tag{12}$$

on J, with

$$v(0) = v_0 < 0. \tag{13}$$

By some straightforward estimates, one can show that if (10) and (11) have a solution, then

$$y' = v \in S \equiv \{z \in C(J, R) : v_0 + a/y_0 v_0 \le z(t) \le v_0\}.$$

Then, using $\mu(t) \equiv 1$, the above set S, and

$$K_1(t) = -2a\cdot(t-\tau)[y_0 + v_0\cdot(t-\tau)]^{-3},$$

the conditions of Theorem 1 are found to be fulfilled for (12) and (13) *provided* $v_0{}^2 y_0 > a$. Thus it follows that if $v_0{}^2 y_0 > a$, then Eqs. (10) and (11) have a unique solution on $(-\infty, 0]$.

With considerably more work, Theorem 1 can be applied to prove the existence of a unique "backwards" solution of the equations of one-dimensional motion for two electrons treated in [4].

Unsolved Problems. These include Eqs. (10) and (11) on $(-\infty, 0]$ with $y_0 > 0$ and $v_0 = 0$. Existence (but not uniqueness) has recently been proved by Travis [11] for the actual electrodynamics model in [4] for this case (and others).

If this problem has a unique solution, it will then be of interest to study the equations for two charged particles under the influence of both retarded *and advanced* interactions.

Added in Proof: The "unsolved problem" for two electrons described in [4] and mentioned above in connection with Eqs. (10) and (11), with $v_0 = 0$, has now been partially solved. An ingenious proof of existence and uniqueness whenever $v_0^2 + a/y_0$ is sufficiently small and the motion is symmetric was presented by V. I. Zhdanov at the Fourth All-Union Conference on the Theory and Applications of Differential Equations with Deviating Argument, Kiev, USSR, Sept. 1975 (abstracts, pp. 91, 92). In view of the difficulties that Zhdanov had to overcome, Eq. (10) now appears too simple to be a prototype for this problem.

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