TOPOLOGIES FOR EQUATIONS OF NEUTRAL TYPE AND CLASSICAL ELECTRODYNAMICS

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Abstract

This paper is motivated by the two-body problem of classical electrodynamics. The equations for the three-dimensional motion of two charged particles (ignoring radiation reaction) can be written as a delay differential system of neutral type, with delays depending on the unknown trajectories.

The formulation of a well-posed problem for a system of neutral type is directly connected with the choice of topology for the space of solutions. Indeed, the topology selected determines which neutral equations can be considered, and what type of existence, continuous dependence, and stability theorems one might hope to prove. By consideration of examples, the paper discusses the implications of using each of the following norms on solutions:

$$||x||_{C^1} = \sup_{a \leq t \leq b} |x(t)| + \sup_{a \leq t \leq b} |x'(t)| ,$$

$$||x||_{AC} = |x(a)| + \int_a^b |x'(t)| \, dt ,$$

$$||x||_{C^0} = \sup_{a \leq t \leq b} |x(t)| .$$

After concluding that neither the C^1 norm nor the C^0 norm is appropriate for the equations of the two-body problem, we prove an existence and uniqueness theorem for this system with consistent with use of the AC norm.

TOPOLOGIES FOR EQUATIONS OF NEUTRAL TYPE AND CLASSICAL ELECTRODYNAMICS

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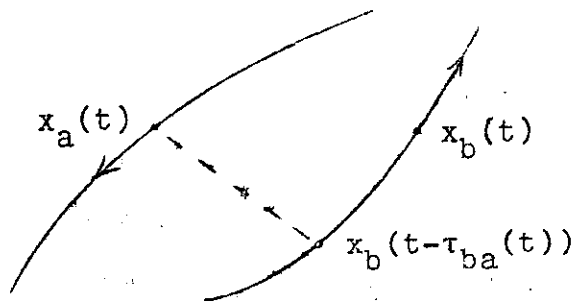
1. Motivation

The interaction of two charged particles under the laws of classical electrodynamics can be described as follows.

At time t let the two particles, a and b, be at positions $x_a(t)$ and $x_b(t)$ in R^3. Here time and position are measured in some given inertial reference frame. For $t \geq 0$, each particle is assumed to be subject only to the influence of the other.

Since electromagnetic interactions propagate at the speed of light, c, the influence on particle a at time $t \geq 0$ must have been "emitted" by particle b at an earlier instant, $t-\tau_{ba}(t)$, where



$$\tau_{ba}(t) = |x_a(t) - x_b(t-\tau_{ba}(t))|/c, \qquad (1_a)$$

with $|\cdot|$ being the Euclidean norm in R^3. Similarly, the influence on particle b at time t was emitted by particle a at time $t-\tau_{ab}(t)$, where

$$\tau_{ab}(t) = |x_b(t) - x_a(t-\tau_{ab}(t))|/c. \qquad (1_b)$$

Let m_a and m_b be the rest masses of the two particles, and let q_a and q_b be the magnitudes of their charges. Now assume for the moment that Eqs. (1) have unique solutions for the two delays, $\tau_{ba}(t)$ and $\tau_{ab}(t)$, and that $|x_a'(t)| < c$ and $|x_b'(t)| < c$. Then the equations of motion can be written in the form

$$x_a'(t) = v_a(t), \tag{2_a}$$

$$x_b'(t) = v_b(t), \tag{2_b}$$

$$\begin{aligned} v_a'(t) = \frac{q_a q_b}{m_a}[f^{(1)}(x_a(t)-x_b(t-\tau_{ba}(t)),v_a(t),v_b(t-\tau_{ba}(t))) \\ + f^{(2)}(x_a(t)-x_b(t-\tau_{ba}(t)),v_a(t),v_b(t-\tau_{ba}(t)))v_b'(t-\tau_{ba}(t))], \end{aligned} \tag{3_a}$$

and a similar equation (3_b) with a and b interchanged. The exact form of the functions $f^{(1)}$ and $f^{(2)}$, which can be found in [2], will not concern us here. It suffices to note that both $f^{(1)}(\xi,\eta_a,\eta_b)$ and $f^{(2)}(\xi,\eta_a,\eta_b)$ are continuously differentiable (in fact analytic) as long as

$$\xi \neq 0, \quad |\eta_a| < c \quad \text{and} \quad |\eta_b| < c.$$

Some authors would include additional "radiation reaction" terms in Eqs. (3), representing the force of each particle upon itself [3]. Such terms are omitted here because they lead to paradoxical behavior of solutions [12].

The functional equations (1), for $\tau_{ba}(t)$ and $\tau_{ab}(t)$, can be replaced by delay differential equations which appear more tractable. Specifically, the following has been proved [4], [3].

Lemma 1. Let x_a and x_b be given continuously differentiable functions, with $v_a \equiv x'_a$ and $v_b \equiv x'_b$, satisfying

$$x_a(t) \neq x_b(t), \quad |v_a(t)| < c, \text{ and } |v_b(t)| < c \text{ on } (\alpha,\beta).$$

If Eq. (1_a) has a solution $\tau_{ba}(t_0)$, at some $t_0 \in (\alpha,\beta)$, then Eq. (1_a) has a unique solution, $\tau_{ba}(t)$, for each $t \in [t_0,\beta)$ and

$$\tau'_{ba}(t) = \frac{\langle x_a(t)-x_b(t-\tau_{ba}(t)),\ v_a(t)-v_b(t-\tau_{ba}(t))\rangle}{c^2\tau_{ba}(t) - \langle x_a(t)-x_b(t-\tau_{ba}(t)),\ v_b(t-\tau_{ba}(t))\rangle}, \qquad (4a)$$

where $\langle\cdot,\cdot\rangle$ indicates the usual inner product in R^3.

Conversely, if $\tau_{ba}(t)$ satisfies Eq. (4_a) on $[t_0,\beta)$ and satisfies (1_a) at $t = t_0$, then it satisfies (1_a) on all of $[t_0,\beta)$.

Similar assertions and an Eq. (4_b) are obtained by interchanging a and b.

So we can replace the original system of Eqs. (1), (2), and (3) with Eqs. (2), (3), and (4).

Let us generalize this for the moment, to a system of functional differential equations of neutral type of the form

$$x'(t) = F(t,\ x(t),\ x(t-g(t,x(t))),\ x'(t-h(t,x(t)))), \qquad (5)$$

where x is the unknown function, taking values in R^n, and F, g, and h are given functions. There may be many delays, $g = (g_1,\dots,g_m)$ and $h = (h_1,\dots,h_p)$ in (5), in which case $x(t-g(t,x(t)))$ stands for

$$x(t-g_1(t_1x(t))),\ \dots,\ x(t-g_m(t,x(t)))$$

and $x'(t-h(t,x(t)))$ stands for

Each g_j is assumed to be a non-negative-valued function, and each h_k is assumed to be positive valued.

We should like to be able to solve system (5) for $t \geq t_0$ subject to given initial data

$$x(t) = \phi(t) \quad \text{on} \quad [\alpha, t_0]. \tag{6}$$

It might be remarked here that in the special case of two charged particles moving on the x_1 axis, $f^{(2)}$ in Eqs. (3) vanishes and the system (2), (3), (4) becomes a delay differential system with state-dependent delay. This simpler problem has been treated quite thoroughly in [2], [4], [5], [7], and [15].

2. The Choice of Topology for a Well-Posed Problem

Many of the difficulties encountered in trying to formulate a well-posed problem for Eq. (5) can be illustrated with the much simpler equation of neutral type

$$x'(t) = F(t, x(t), x(t-g(t)), x'(t-h(t))), \tag{7}$$

where the delays are known functions of t. Let the given functions F, g, h, and ϕ be continuous, with ϕ' also continuous, and let $\alpha \leq t - g_j(t) \leq t$ and $\alpha \leq t - h_k(t) < t$ for each $j=1,\cdots, m$ and $k=1,\cdots, p$ when $t \geq t_0$. Then it is easy to prove the existence of a continuous solution, x, of (7) and (6) on $[\alpha, t_1)$ for some sufficiently small $t_1 > t_0$. (Among other things when choosing t_1, we make sure that each $t - h_k(t) < t_0$ for $t_0 \leq t < t_1$.) If, in addition, F satisfies an appropriate Lipschitz condition, then this solution is unique.

However, any attempt to continue the solution leads at once to difficulty because $x'(t)$ is not, in general, defined at $t = t_0$. Thus if $t-h_k(t) \equiv 0$ on some interval to the right of t_0, the right hand side of Eq. (7) becomes undefined there. To avoid this difficulty, we shall assume that each equation

$$t - h_k(t) = \text{a constant}$$

has no more than a finite number of solutions in any interval. Then the solution of (7) can be continued, say by a method of steps, yielding a continuous and piecewise-continuously differentiable x on $[\alpha,\beta)$ which satisfies Eq. (7) for $0 < t < \beta$ except at certain isolated points.

The next question which must be asked concerns the effect on solutions of small changes in F, g, h, t_0, and ϕ. This question forces one to carefully define the solution space and to select a topology for that space.

Consider a function, x, defined on $[a, b] \to R^n$. (The interval $[a, b]$ will actually be some appropriate subinterval of $[\alpha,\beta)$.) Let $|\cdot|$ be any norm in R^n.

Several authors have used the C^1 norm,

$$||x||_{C^1} = \sup |x(t)| + \sup |x'(t)|,$$

where the first supremum is taken over $[a, b]$ and the second is taken over those points in $[a, b]$ where $x'(t)$ is defined. But this norm is quite unsatisfactory because an arbitrarily small change in t_0 or h will, in general, shift the points where $x'(t)$ is undefined. This will cause a large change in the solution as measured by $||\cdot||_{C^1}$.

This difficulty was avoided in [6] by use of the norm

$$||x||_{AC} = |x(a)| + \int_a^b |x'(s)| \, ds$$

instead. If this norm is adopted, then one is naturally led to seek, as a solution of (7) and (6), an absolutely continuous function which satisfies (7) only almost everywhere (when $t > t_0$). This suggests that one should also permit ϕ to be merely absolutely continuous. Four simple examples given in [6] showed that, in order to use the AC norm for solutions of Eqs. (7) and (6), one should have

(i) F linear with respect to $x'(t-h(t))$ (or at least growing no faster than linearly with respect to these terms), and

(ii) h continuously differentiable with each $h_k'(t) < 1$ for $t \geq t_0$.

These restrictions might seem too severe if it were not for the fact (as we shall see in Section 3) that similar conditions are actually satisfied in the electrodynamics equations. Under these hypotheses, and using the AC norm, the existence and uniqueness of solutions and their continuous dependence jointly on t_0, ϕ, F, g, and h was proved in [6].

The assumption of linearity of F with respect to $x'(t-h(t))$ could be avoided if we assumed ϕ' to be essentially bounded. But this appears to be an unnecessary and unnatural assumption for the equations encountered thus far in applications. Moreover, if one then used the norm

$$||x|| = \sup |x(t)| + \text{ess sup } |x'(t)|,$$

the same difficulties would be encountered as with the C^1 norm.

Hale and others (see [1], [10], [11], [13], and further references given there) have been motivated by some systems of neutral type arising in transmission line problems. These systems have the form

$$x'(t) = \sum_{j=1}^{m} q_j x'(t-\tau_j) + f(t, x(t), x(t-\tau)), \tag{8}$$

where each q_j and τ_j is constant, $0 < \tau_1 < \tau_2 < \cdots < \tau_m$, and $x(t-\tau)$ represents

$$x(t-\tau_1), \cdots , x(t-\tau_m).$$

Such a system is considered for $t \geq t_0$ with initial condition (6) on $[\alpha, t_0] = [t_0-\tau_m, t_0]$. Equation (8) then yields the integral delay system,

$$x(t) = \sum_{j=1}^{m} q_j x(t-\tau_j) + \phi(t_0) - \sum_{j=1}^{m} q_k \phi(t_0-\tau_j) \tag{9}$$

$$+ \int_{t_0}^{t} f(s,x(s), x(s-\tau))\, ds \quad \text{for } t \geq t_0.$$

The special form of Eq. (9) now makes it possible to consider ϕ merely continuous, and to seek a merely continuous solution, x, of (9) and (6). Then one can define $x_t(s) = x(t+s)$ for $-\tau_m \leq s \leq 0$ when $t \geq t_0$, and use the C^0 norm

$$||x_t||_{C^0} = \sup_{-\tau_m \leq s \leq 0} |x_t(s)|.$$

The authors cited above (Hale, Cruz, Henry, and Lopes) and others have developed an extensive theory of existence, uniqueness, continuous dependence, stability, and oscillation of solutions using the C^0 norm. They treat equations more general than (8) or (9). In fact it is not at all obvious that the C^0 norm is unsuitable for Eq. (7), provided the equation is linear with respect to $x'(t-h(t))$ and provided each $h_j'(t) < 1$.

The following two examples will show that the C^0 norm is not suitable for such equations.

<u>Example 1</u> (<u>Linear</u>). For each $\varepsilon \in (0,1)$ there is a continuously-differentiable function h on $[0,1]$ such that

$$-1 \le t-h(t) \le 0, \quad |h(t)-1| \le \varepsilon, \quad \text{and} \quad |h'(t)| \le \varepsilon,$$

and yet the (absolutely continuous) solution of the linear scalar equation

$$x'(t) = x'(t-h(t)) \quad \text{for } 0 \le t < 1 \tag{10}$$

does not depend continuously on the (absolutely continuous) initial function ϕ on $[-1, 0]$ when the C^0 norm is used.

<u>Proof</u>. Let $\varepsilon \in (0, 1)$ be given.
Define $w: [-1, 0] \to [0, 1]$ by

$$w(\sigma) = \begin{cases} 0 & \text{for } \sigma = -1, \\ 1 + \sigma + \frac{\varepsilon}{10}(1+\sigma)^{5/2} \cos \frac{\pi}{1+\sigma} & \text{for } -1 < \sigma \le -\frac{1}{3}, \\ 1 + \sigma + \frac{\varepsilon\pi}{10}(\frac{2}{3})^{1/2}(\sigma+3\sigma^2) & \text{for } -\frac{1}{3} < \sigma \le 0. \end{cases}$$

Then w is continuously differentiable, and

$$w'(\sigma) = \begin{cases} 1 & \text{for } \sigma = -1, \\ 1 + \frac{\varepsilon}{4}(1+\sigma)^{3/2}\cos \frac{\pi}{1+\sigma} + \frac{\varepsilon\pi}{10}(1+\sigma)^{1/2}\sin \frac{\pi}{1+\sigma} & \text{for } -1 < \sigma \le -\frac{1}{3}, \\ 1 + \frac{\varepsilon\pi}{10}(\frac{2}{3})^{1/2}(1+6\sigma) & \text{for } -1/3 < \sigma \le 0. \end{cases}$$

From these we find, for $-1 \le \sigma \le 0$,

$$1 + \sigma - \frac{\varepsilon}{5} < w(\sigma) < 1 + \sigma + \frac{\varepsilon}{5} \quad \text{and} \quad 1 - \frac{\varepsilon}{2} < w'(\sigma) < 1 + \frac{\varepsilon}{2}.$$

Now define $h(t) = t - w^{-1}(t)$ on $[0, 1]$. Then it follows that

$$-1 \leq t-h(t) = w^{-1}(t) \leq 1 \quad \text{for } 0 \leq t \leq 1,$$

and

$$-\varepsilon < 1 - \frac{1}{1-\varepsilon/2} < h'(t) = 1 - \frac{1}{w'(w^{-1}(t))} < 1 - \frac{1}{1+\varepsilon/2} < \frac{\varepsilon}{2} .$$

From the latter and the fact that $h(0) = 1$, we have

$$1 - \varepsilon \leq h(t) \leq 1 + \frac{\varepsilon}{2} \quad \text{for } 0 \leq t \leq 1.$$

Thus all the asserted estimates for h and h' are fulfilled.

Now for each integer $n \geq 2$, define ϕ_n on $[-1, 0]$ by setting $\phi_n(-1) = 0$ and

$$\phi_n'(s) = \begin{cases} 0 & \text{for } 0 < s+1 < \frac{1}{2n^2+1} \text{ and } \frac{1}{2n-1} < s+1 < 1, \\ k^{3/2} & \text{for } \frac{1}{2k+1} < s+1 < \frac{1}{2k} \\ -k^{3/2}\left(\frac{2k-1}{2k+1}\right) & \text{for } \frac{1}{2k} < s+1 < \frac{1}{2k-1} \end{cases} \Bigg\} \; k = n, n+1, \ldots, n^2.$$

Since ϕ_n' is integrable on $[-1, 0]$, it follows that ϕ_n is absolutely continuous.

By straightforward calculations, one finds

$$|\phi_n(s)| \leq \frac{1}{4\sqrt{n}} \quad \text{for all } s \in [-1, 0],$$

while for the corresponding absolutely continuous solution, x_n, of $x'(t) = x'(t-h(t))$ a.e.,

$$x_n(\tfrac{2}{3}) = \sum_{k=n}^{n^2} \{k^{3/2}[w(-1+\tfrac{1}{2k}) - w(-1+\tfrac{1}{2k+1})]$$

$$- k^{3/2}(\tfrac{2k-1}{2k+1})[w(-1+\tfrac{1}{2k-1}) - w(-1+\tfrac{1}{2k})]\}$$

$$\geq \frac{\varepsilon}{40\sqrt{2}} \sum_{k=n}^{n^2} \frac{1}{k} > \frac{\varepsilon}{40\sqrt{2}} \log n.$$

<u>Example 2</u> (<u>Autonomous</u>). Let f be analytic on R and let ϕ be absolutely continuous on $[-1, 0]$. Then the (absolutely continuous) solution of the autonomous scalar equation

$$x'(t) = 1 + f(x(t))x'(t-1) \quad \text{for } 0 \leq t < 1, \tag{11}$$

with $x(t) = \phi(t)$ on $[-1, 0]$, does not depend continuously on f and ϕ jointly when the C^0 norm is used for x and ϕ.

<u>Proof</u> (developed with M.L. Slater).

For each positive <u>even</u> integer n, define

$$\omega_n = \frac{n \log (\sqrt{n} + \sqrt{n-1})^2}{\sqrt{n-1}},$$

$f_n(\xi) = n^{-1/4} \sin \omega_n \xi$, and define ϕ_n on $[-1, 0]$ by setting $\phi_n(-1) = 0$ and

$$\phi_n'(t) = \begin{cases} n^{3/4} & \text{for } -1 + \frac{2k}{n} < t < -1 + \frac{2k+1}{n} \\ -n^{3/4} & \text{for } -1 + \frac{2k+1}{n} < t < -1 + \frac{2k+2}{n} \end{cases}$$

for $k = 0, 1, \circ\circ\circ, \frac{n}{2} - 1$.

Then f_n is analytic, ϕ_n is absolutely continuous with $\phi_n(0) = 0$,

$$|f_n(\xi)| \leq n^{-1/4} \text{ on } R \quad \text{and} \quad ||\phi_n||_{C^0} \leq n^{-1/4}.$$

Clearly a unique absolutely continuous x_n exists on $[-1, 1)$ such that

$$x_n(t) = \phi_n(t) \quad \text{on } [-1, 0],$$

and

$$x_n'(t) = 1 + f_n(x_n(t))x_n'(t-1) \quad \text{a.e. on } [0, 1).$$

On the interval $[0, 1/n)$

$$x_n'(t) = 1 + \sqrt{n} \sin \omega_n x_n(t) \quad \text{with } x_n(0) = 0.$$

This yields

$$\log \frac{\tan[\omega_n x_n(t)/2] + \sqrt{n} - \sqrt{n-1}}{\tan[\omega_n x_n(t)/2] + \sqrt{n} + \sqrt{n-1}}$$

$$= \omega_n \sqrt{n-1}\, t - \log(\sqrt{n} + \sqrt{n-1})^2 ,$$

and hence $x_n(t) \to \pi/\omega_n$ as $t \to 1/n$. Thus $x_n(1/n) = \pi/\omega_n$.

Next, considering the interval $[1/n, 2/n]$, one finds $x_n(2/n) = 2\pi/\omega_n$. Similarly $x(k/n) = k\pi/\omega_n$ for $k = 1,2,\cdots, n/2$. Thus

$$x(\tfrac{1}{2}) = \frac{n\pi}{2\omega_n} = \frac{\pi}{4} \frac{\sqrt{n-1}}{\log(\sqrt{n} + \sqrt{n-1})} \to \infty \quad \text{as} \quad n \to \infty.$$

On the other hand, if either $f \equiv 0$ or $\phi \equiv 0$, the solution of (11) yields $x(t) = t$ on $[0, 1)$. So again, continuous dependence fails when using the C^0 norm. ∎

Going beyond continuous dependence to the study of stability, W. R. Melvin has recently noted a problem which even plagues Eq. (8).

Example 3 (Melvin [14]). Choose real numbers a and b such that

$$a > 0, \quad -1 < b < -a^2/4, \quad \text{and} \quad |a| + |b| > 1.$$

Then every solution of the equation

$$x'(t) = ax'(t-1) + bx'(t-2) \tag{12}$$

tends exponentially to a constant as $t \to \infty$.

But, for every positive even integer n, no matter how large, the perturbed equation

$$x'(t) = ax'(t-1) + bx'(t-2-1/n) \tag{13}$$

has unbounded solutions. This is the case regardless of which norm used.

Proof. To show that every solution of (12) tends exponentially to a constant we use arguments from [14] and [1].

Observe, first, that the roots of the algebraic equation

$$\rho^2 - a\rho - b = 0 \tag{14}$$

are a pair of complex conjugates, $\rho, \bar{\rho} = (a \pm i\sqrt{-a^2-4b})/2$ with

$$|\rho| = |\bar{\rho}| = \sqrt{-b} < 1.$$

In particular, this shows that $1 - a - b \neq 0$.

Now note that Eq. (12) for $t \geq 0$, with absolutely continuous initial data

$$x(t) = \phi(t) \quad \text{on } [-2, 0],$$

can be rewritten as

$$x(t) = ax(t-1) + bx(t-2) + [\phi(0) - a\phi(-1) - b\phi(-2)]$$

with the same initial condition. Clearly, by the method of steps, there is a unique continuous solution of the latter problem on $[-2, \infty)$ even if ϕ is merely continuous. Since $1 - a - b \neq 0$, we can define

$$y(t) = x(t) - c \quad \text{where} \quad c = \frac{\phi(0)-a\phi(-1)-b\phi(-2)}{1 - a - b}.$$

Then y satisfies

$$y(t) = ay(t-1) + by(t-2) \quad \text{for } t \geq 0$$

with

$$y(t) = \phi(t) - c \quad \text{on } [-2, 0]$$

Next define

$$z(t) = \begin{bmatrix} y(t) \\ y(t-1) \end{bmatrix} \quad \text{for} \quad t \geq -1,$$

and one finds that

$$z(t) = Az(t-1) \quad \text{for } t \geq 0 \quad \text{where } A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}.$$

For this equation the initial condition is

$$z(t) = \begin{bmatrix} \phi(t) - c \\ \phi(t-1) - c \end{bmatrix} \quad \text{on } [-1, 0].$$

Hence, for each $k = 0, 1, \ldots$

$$z(t) = A^{k+1} \begin{bmatrix} \phi(t) - c \\ \phi(t-1) - c \end{bmatrix} \quad \text{for } k \leq t < k+1.$$

Now the eigenvalues of A are the roots of Eq. (14), and so each has modulus $= \sqrt{-b}$. Thus

$$|A^k|^{1/k} \to \sqrt{-b} < 1 \quad \text{as} \quad k \to \infty,$$

where $|\cdot|$ is the norm on A induced by any norm for vectors in R^2. It follows that $z(t) \to 0$ exponentially, and hence

$$x(t) \to c \text{ exponentially as } t \to \infty.$$

Turning now to Eq. (13), we shall prove the existence of solutions $e^{\lambda t}$ where $\lambda = \mu \pm in\pi$ with $\mu > 0$. Substituting $e^{\lambda t}$ with $\lambda = \mu \pm in\pi$ into (13) we find the requirement

$$\Delta(\mu) \equiv e^{(2+1/n)\mu} - ae^{(1+1/n)\mu} + b = 0.$$

Now $\Delta(\mu)$ depends continuously on the real variable μ and

$$\Delta(0) = 1 - a + b = 1 - |a| - |b| < 0$$

while

$$\Delta(\mu) \to +\infty \quad \text{as} \quad \mu \to +\infty.$$

Hence there does exist $\mu > 0$ such that $\Delta(\mu) = 0$. Using this value of μ we need only to set

$$x(t) = e^{\mu t} \sin(n\pi t) \quad \text{on} \quad [-2-1/n, 0]$$

to obtain the same form for the solution of Eq. (13) on $[-2-1/n, \infty)$. This solution is unbounded. ∎

The difficulty illustrated by Example 3 will not occur if $|a|+|b| < 1$. For further information and generalizations see [14] and [9].

3. The Two-Body Problem of Electrodynamics

The two-body problem of classical electrodynamics, as represented by the functional differential equations in Section 1, was studied by Driver in [3]. However the analysis given there was based on the premise that solutions of an equation of neutral type should be piecewise continuously differentiable.

In view of the discussion in Section 2, it now appears that one should only require that solutions be absolutely continuous and satisfy the neutral differential equations almost everywhere.

But this type of definition is not easily applied to Eq. (5). We can demand that $x'(t-h(t,x(t)))$ enter F only linearly. But how can we expect that, whenever x is a solution,

$$\frac{d}{dt} h_k(t,x(t)) < 1 \ ?$$

This derivative will probably not even exist everywhere if x is not differentiable everywhere. But suppose we could somehow overcome that difficulty; and let us also assume for simplicity that each g_j is positive valued, as well as each h_k. Then the method of steps applied to Eq. (5) would yield an ordinary differential system, $x'(t) = f(t,x(t))$, where $f(t,\xi)$ is discontinuous with respect to both t and ξ. This type of system has been studied by Filippov [8]. But difficulties are encountered when one tries to apply Filippov's definition and methods to the ordinary differential equations arising from a system of neutral equations. For example, it would appear that one should require x' to be essentially bounded. And even if that restriction is imposed, the appropriate conditions for uniqueness remain elusive.

But a satisfactory analysis will become possible in our case, thanks to certain special features of the equations of motion for two charged particles which are not present in general in Eq. (5).

Let initial data for the two-body problem, Eqs. (2), (3), (4), be specified,

$$\left.\begin{aligned} x_a(t) = \phi_a(t), \quad v_a(t) = \psi_a(t) \\ x_b(t) = \phi_b(t), \quad v_b(t) = \psi_a(t) \end{aligned}\right\} \quad \text{on } [\alpha,\ 0], \tag{15}$$

where ϕ_a, ϕ_b, ψ_a, and ψ_b are absolutely continuous, $\phi_a(0) \neq \phi_b(0)$, and $|\psi_a(t)| < c$, $|\psi_b(t)| < c$ on $[\alpha, 0]$, and Eqs. (1) have solutions $\tau_{ba}(0)$ and $\tau_{ab}(0)$ at $t = 0$. Generally one can expect that $\psi_a = \phi_a'$ and $\psi_b = \phi_b'$; but this will not matter. Note that we do not assume any Lipschitz-type conditions whatever on ϕ_a, ϕ_b, ψ_a, ψ_b.

Now observe that we can proceed by the method of steps. For Eqs. (2_a), (3_a), and (4_a) become a system of seven ordinary differential equations for x_a, ψ_a, and τ_{ba} in some sufficiently small neighborhood of the point $(0, \phi_a(0), \psi_a(0), \tau_{ba}(0)) \in R^8$. Moreover, these equations are "decoupled" from the seven analogous equations for x_b, v_b, and τ_{ab}.

If we introduce two vector unknowns

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \equiv \begin{bmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \\ \tau_{ba} \end{bmatrix} \quad \text{and} \quad v = v_a = \begin{bmatrix} v_{a1} \\ v_{a2} \\ v_{a3} \end{bmatrix},$$

then Eqs. (2_a), (3_a), and (4_a), with the initial conditions (15) substituted, can be considered as a special case of

$$\begin{aligned} x'(t) &= f(t-g(x(t)), x(t), v(t)) \\ v'(t) &= \tilde{f}^{(1)}(t-g(x(t)), x(t), v(t)) \\ &\quad + \tilde{f}^{(2)}(t-g(x(t)), x(t), v(t))\psi'(t-g(x(t))). \end{aligned} \tag{16}$$

Here f, $\tilde{f}^{(1)}$, and $\tilde{f}^{(2)}$ are known continuous functions on a neighborhood N of $(-g(\phi_a(0),\tau_{ba}(0)),\phi_a(0),\tau_{ba}(0),\psi_a(0)) \in R^8$, each is locally Lipschitzian on N with respect to all arguments but the first, $\psi' = \psi_b'$ is integrable, g is a scalar-valued continuously-differentiable function, and

$$\langle (\text{grad } g)(\xi),\ f(s,\xi,\eta) \rangle \equiv \sum_{i=1}^{4} \frac{\partial g(\xi)}{\partial \xi_i} f_i(s,\xi,\eta) < 1. \tag{17}$$

(Note that f is a 4-vector-valued function, $\tilde{f}^{(1)}$ is 3-vector-valued, and $\tilde{f}^{(2)}$ is 3×3-matrix valued.)

We seek an absolutely continuous solution of (16) for $t \geq 0$ (a.e.) subject to the initial conditions

$$x(0) = x_0 = \begin{bmatrix} \phi_{a1}(0) \\ \phi_{a2}(0) \\ \phi_{a3}(0) \\ \tau_{ba}(0) \end{bmatrix}, \quad v(0) = v_0 = \begin{bmatrix} \psi_{a1}(0) \\ \psi_{a2}(0) \\ \psi_{a3}(0) \end{bmatrix}. \tag{18}$$

To treat this system we shall make a change of variables which was used previously by Travis [15] and Winston [16].

Assume, for the moment, that system (16) has a solution on an interval $[0, t_1)$ satisfying (18) at $t = 0$. Of course, this implies that t_1 is small enough so that

$$(t-g(x(t)),\ x(t),\ v(t)) \in N \quad \text{for } 0 \leq t < t_1.$$

It follows from (17) that

$$\frac{d}{dt}[t - g(x(t))] > 0 \quad \text{on } [0, t_1),$$

and this derivative is continuous. Thus, if we define

$$\sigma(t) = \sigma_x(t) \equiv t - g(x(t)) \quad \text{on } [0, t_1), \tag{19}$$

then σ^{-1} is uniquely defined on $\sigma[0, t_1) = [-g(x_0), s_1)$ and is continuously differentiable with

$$\frac{d}{ds}\sigma^{-1}(s) = \frac{1}{\sigma'(\sigma^{-1}(s))} > 0.$$

Let us now define

$$y(s) \equiv x(\sigma^{-1}(s)) \quad \text{and} \quad w(s) \equiv v(\sigma^{-1}(s)) \tag{20}$$

on $[-g(x_0), s_1)$. The absolute continuity of y and w follows from the absolute continuity of x and v, plus the continuity and positivity of $d\sigma^{-1}(s)/ds$. One then finds

$$\begin{aligned} y'(s) &= \frac{f(s,y(s),w(s))}{1 - \langle (\text{grad } g)(y(s)),\ f(s,y(s),w(s))\rangle} \\ w'(s) &= \frac{\tilde{f}^{(1)}(s,y(s),w(s)) + \tilde{f}^{(2)}(s,y(s),w(s))\psi'(s)}{1 - \langle (\text{grad } g)(y(s)),\ f(s,y(s),w(s))\rangle} \end{aligned} \tag{21}$$

with

$$y(-g(x_0)) = x_0 \quad \text{and } w(-g(x_0)) = v_0. \tag{22}$$

Note that system (21) contains no reference to the particular solution x, v of (16) or to the function σ_x. Moreover, by virtue of the hypotheses on the given functions f, $\tilde{f}^{(1)}$, $\tilde{f}^{(2)}$, g, and ψ', system (21) satisfies all the requirements of the Carathéodory theory for existence and uniqueness. Thus there is one and only one absolutely continuous solution y, w of (21) and (22) on $[-g(x_0), \tilde{s}_1)$ for some $\tilde{s}_1 > -g(x_0)$.

It remains to reverse the process and <u>uniquely</u> recover x and v as the solution of (16) and (18). So let y and w be the unique solution of (21) and (22) on $[-g(x_1), \tilde{s}_1)$. And consider the equation

Now,

$$\frac{\partial}{\partial\sigma}\,[\sigma - t + g(y(\sigma))]$$

$$= \frac{1}{1 - \langle (\mathrm{grad}\ g)(y(\sigma)),\ f(\sigma, y(\sigma), w(\sigma)) \rangle} > 0,$$

and Eq. (23) has a solution, $\sigma = -g(x_0)$, at $t = 0$. Thus it follows, from the implicit function theorem, that Eq. (23) defines a function $\sigma(t)$ uniquely on an interval $[0, \tilde{t}_1)$. Moreover, $\sigma'(t)$ is continuous and positive there. On $[0, \tilde{t}_1)$ let us define

$$x(t) \equiv y(\sigma(t)) \quad \text{and} \quad v(t) \equiv w(\sigma(t)). \tag{24}$$

Then x and v are absolutely continuous and, from $x'(t) = y'(\sigma(t))\sigma'(t)$ and $v'(t) = w'(\sigma(t))\sigma'(t)$, one readily discovers that x and v satisfy Eqs. (16) a.e. Hence Eqs. (24) define the unique absolutely continuous solution of (16) and (18) on $[0, \tilde{t}_1)$.

The above, with an analogous argument for particle b, establishes <u>local</u> existence and uniqueness for the two-body problem represented by Eqs. (2), (3), (4), and (15). The solution can be extended by a method of steps, as in [6] and [3] to yield the following.

<u>Theorem</u>. Equations (1), (2), and (3) together with the initial conditions (15) have a unique solution x_a, x_b on an interval $[\alpha, \beta)$ such that $v_a = x_a'$ and $v_b = x_b'$ are absolutely continuous, with Eqs. (2) perhaps only satisfied a.e. on $[0, \beta)$, and with

$$x_a(t) \neq x_b(t), \quad |v_a(t)| < c, \quad |v_b(t)| < c\,.$$

And either $\beta = \infty$ or, as $t \to \beta$, one of the following occurs:

(i) $\lim x_a(t) = \lim x_b(t)$ - a collision, or

(ii) $\limsup |v_a(t)| = c$ or $\limsup |v_b(t)| = c$.

Actually a stronger theorem is true. Namely, in the above the final conclusion is that either $\beta = \infty$ or $\lim_{t\to\beta} x_a(t) = \lim_{t\to\beta} x_b(t)$ -- a collision. See [4] or [3]. However, the proof of this assertion requires more detailed knowledge of the functions $f^{(1)}$ and $f^{(2)}$ in Eqs. (3) than was given here.

<u>Remark</u>. Unfortunately, the existence and uniqueness proof given here fails to work for the three-body problem. For then each decoupled set of ordinary differential equations contains two different "delays", and the change of variables. (19) and (20), does not help.

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