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A Two-Body Problem of Classical Electrodynamics: the One-Dimensional Case*

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The equations of motion for the two-body problem of classical electrodynamics can be formulated by substituting the expressions for the field of a moving charge, calculated from the Liénard-Wiechert potentials, into the Lorentz-Abraham force law. In this paper, radiation reaction is omitted and the charges are assumed to move along the x-axis.

Due to the finite speed of propagation, c, of electrical effects, the differential equations involve time delays, which depend upon the unknown trajectories. Thus the equations of motion are not ordinary differential equations, and one does not determine a unique solution for $t \geqslant t_0$ by merely specifying the positions and velocities of the charges at t_0.

One specifies rather arbitrary initial trajectories of the two charges over some appropriate interval $[\alpha, t_0]$, where $\alpha < t_0$. Under suitable conditions, one can then show the existence of a unique extension of these trajectories which satisfy the equations of motion for all $t > t_0$ unless and until the charges collide. Moreover the solution depends continuously on the given initial trajectories. It is also shown that two point charges of like sign cannot collide, while two point charges of opposite signs may or may not collide, depending upon the initial data. Moreover, in the event of a collision, the velocities of the two charges become $+c$ and $-c$ at the instant of collision.

The effect of an external electric field in the x-direction is also considered.

I. INTRODUCTION

The equations of motion for the two-body problem (or the N-body problem) of classical electrodynamics can be formulated by substituting the expressions for the field of a moving charge, calculated from the Liénard-Wiechert potentials, into the Lorentz-Abraham force law. Thus the equations for this problem can be considered to be at least 60 years old.

However, the resulting equations of motion are not ordinary differential equa-

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tions. Due to the finite speed of propagation of electrical effects, the differential equations involve time delays which depend upon the unknown trajectories. Until recently (*1*, *2*), these systems of delay-differential equations had not received any systematic treatment. Thus even the simplest two-body problem of classical electrodynamics remained unsolved. In particular, the question of what type of initial data (if any) would determine a unique solution was unanswered.

The present paper gives a detailed analysis of a special case of the two-body problem in which radiation reaction is omitted and both charges are assumed to move along the x-axis. An outline of the existence theorem for the case of three-dimensional motion has appeared elsewhere (*3*).

The results presented here are based on my Ph.D. thesis written at the University of Minnesota in 1960 (*1*). I am indebted to Professor Lawrence Markus for directing that work and contributing many valuable ideas. I also want to thank Professor P. C. Rosenbloom for first suggesting the two-body problem of electrodynamics to me.

II. EQUATIONS OF MOTION

Let $x_i(t)$ $(i = 1, 2)$ be the positions of the two point charges on the x-axis in a given inertial system at time t, the time of an observer in that system. Let $v_i(t) = x_i'(t)$ $(i = 1, 2)$ be the velocities of the charges. We omit radiation reaction but allow an external electric field, $E_{\text{ext}}(t, x)$, in the x-direction, possibly due to other charges moving on the x-axis. Then the equation of motion of charge i is

$$\frac{m_i v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = q_i E_j(t, x_i(t)) + q_i E_{\text{ext}}(t, x_i(t)) \qquad (i = 1, 2), \quad (1)$$

where m_i is the rest mass and q_i is the magnitude of charge i, c is the speed of light, and $E_j(t, x)$ is the electric field at (t, x) due to the other charge, j $(j = 1, 2;$ $j \neq i)$. The magnetic field of charge j is not involved in this one-dimensional case. To avoid writing each equation twice, we will continue to consider the motion of charge i due to the influence of charge j with the understanding that $(i, j) = (1, 2)$ or $(2, 1)$.

The field at time t and at the point $x_i(t)$ produced by charge j is assumed to be that computed from the Liénard-Wiechert potentials. The expression for this field involves a retarded time, $t - \tau_{ji}(t)$, representing the instant at which a light signal would have had to leave charge j in order to arrive at $x_i(t)$ at the instant t. Mathematically, the delay, $\tau_{ji}(t)$, must be a solution of the functional equation

$$\tau_{ji}(t) = |x_i(t) - x_j(t - \tau_{ji}(t))|/c. \quad (2)$$

This equation is not written explicitly in this form in standard treatments involving retarded potentials and fields.

Whenever Eq. (2) has a unique solution, $E_j(t, x_i(t))$ can be expressed in terms of $x_i(t)$, $\tau_{ji}(t)$, $x_j(t - \tau_{ji}(t))$, and $v_j(t - \tau_{ji}(t))$. Before writing out this expression, let us agree to consider only those trajectories, $x_1(t)$, $x_2(t)$, which are continuous and differentiable with $|v_1(t)| < c$ and $|v_2(t)| < c$ on appropriate intervals. Then one can obtain the following information about the solutions of (2).

LEMMA 1. *Let two numbers, α, β, be given with $-\infty \leqq \alpha < \beta < \infty$. Let $x_j(t)$ be continuous on the interval $[\alpha, \beta]$ (to be considered as $(-\infty, \beta]$ in case $\alpha = -\infty$) and differentiable with $|v_j(t)| < c$ on (α, β). Then for any finite $t \in [\alpha, \beta]$, a solution, $\tau_{ji}(t)$, of (2) exists if and only if, for some finite $\bar{\alpha} \in [\alpha, t]$*

$$|x_i(t) - x_j(\bar{\alpha})| \leqq c(t - \bar{\alpha}).$$

The solution of (2), if it exists, is unique and satisfies the inequalities

$$|x_i(t) - x_j(t)|/2c \leqq \tau_{ji}(t) \leqq t - \bar{\alpha}.$$

The following inequalities characterize $\tau_{ji}(t)$ and provide a further method of estimation: For $s \in [\alpha, t]$,

$$|x_i(t) - x_j(s)| < c(t - s) \quad \textit{whenever} \quad \alpha \leqq s < t - \tau_{ji}(t)$$

$$|x_i(t) - x_j(s)| > c(t - s) \quad \textit{whenever} \quad t - \tau_{ji}(t) < s \leqq t.$$

If, at the instant t, $x_2(t) \geqq x_1(t)$ and a solution of (2) exists, then

$$(-1)^i[x_i(t) - x_j(t - \tau_{ji}(t))] = c\tau_{ji}(t) \geqq [x_2(t) - x_1(t)]/2 \geqq 0.$$

PROOF. *Existence.* Whenever Eq. (2) has a solution, $\tau_{ji}(t)$, one can immediately satisfy the condition $|x_i(t) - x_j(\bar{\alpha})| \leqq c(t - \bar{\alpha})$ by taking $\bar{\alpha} = t - \tau_{ji}(t)$.

Now suppose $|x_i(t) - x_j(\bar{\alpha})| \leqq c(t - \bar{\alpha})$ for some finite $\bar{\alpha} \in [\alpha, t]$. Consider the function

$$D(\tau) \equiv c\tau - |x_i(t) - x_j(t - \tau)|,$$

which is well-defined and continuous for $0 \leqq \tau \leqq t - \bar{\alpha}$. By use of the mean value theorem, we observe that

$$|x_i(t) - x_j(t)| \leqq |x_i(t) - x_j(\bar{\alpha})| + |x_j(\bar{\alpha}) - x_j(t)| \leqq 2c(t - \bar{\alpha}).$$

We now compute

$$D(|x_i(t) - x_j(t)|/2c)$$
$$= |x_i(t) - x_j(t)|/2 - |x_i(t) - x_j(t - |x_i(t) - x_j(t)|/2c)|$$
$$\leqq |x_i(t) - x_j(t)|/2 - |x_i(t) - x_j(t)| + \sup_{\alpha<s<\beta} |v_j(s)|\,|x_i(t) - x_j(t)|/2c \leqq 0,$$

and

$$D(t - \bar{\alpha}) = c(t - \bar{\alpha}) - |x_i(t) - x_j(\bar{\alpha})| \geqq 0.$$

Thus there is at least one number τ such that $D(\tau) = 0$ and

$$|x_i(t) - x_j(t)|/2c \leqq \tau \leqq t - \bar{\alpha}.$$

This τ is a solution of Eq. (2).

Uniqueness and characteristic inequalities. If the solution of (2) were not unique there would have to be a positive solution. Let $\tau_{ji}(t)$ be any positive solution of (2). We will prove uniqueness by showing that there can be no smaller solution. For any s such that $t - \tau_{ji}(t) < s \leqq t$, we find that

$$|x_i(t) - x_j(s)| \geqq |x_i(t) - x_j(t - \tau_{ji}(t))| - |x_j(t - \tau_{ji}(t)) - x_j(s)|$$
$$> c\tau_{ji}(t) - c|s - t + \tau_{ji}(t)| = c(t - s).$$

It follows that the solution is unique.

Very similar computations prove the characteristic inequalities.

Algebraic sign of $x_i(t) - x_j(t - \tau_{ji}(t))$. Now add the condition $x_2(t) \geqq x_1(t)$. Consider $(j, i) = (2, 1)$. Suppose (for contradiction) that

$$x_1(t) - x_2(t - \tau_{21}(t)) > 0.$$

Then $\tau_{21}(t) > 0$ and hence

$$c\tau_{21}(t) = x_1(t) - x_2(t - \tau_{21}(t)) \leqq x_2(t) - x_2(t - \tau_{21}(t)) < c\tau_{21}(t),$$

a contradiction. This, together with a similar computation for $(j, i) = (1, 2)$, completes the proof of the lemma. Q.E.D.

It might appear that, if $x_j(t)$ were continuous on $(-\infty, \beta]$ and differentiable with $|v_j(t)| < c$ on $(-\infty, \beta)$, the condition $|x_i(t) - x_j(\bar{\alpha})| \leqq c(t - \bar{\alpha})$ would necessarily be satisfied for some finite $\bar{\alpha}$. The following example shows that this is not the case.

EXAMPLE (adapted from Havas (*4*)). Consider the "hyberbolic" trajectories $x_1(t) = a_1 - [c^2(t - t_1)^2 + b_1^2]^{1/2}$ and $x_2(t) = a_2 + [c^2(t - t_2)^2 + b_2^2]^{1/2}$ for all t, with $a_2 - a_1 \geqq c|t_2 - t_1|$, $b_1 \neq 0$, and $b_2 \neq 0$. Then, at any instant t, the assumption that (2) is satisfied for $(j, i) = (2, 1)$ gives

$$c\tau_{21}(t) = -a_1 + [c^2(t - t_1)^2 + b_1^2]^{1/2} + a_2 + [c^2(t - \tau_{21}(t) - t_2)^2 + b_1^2]^{1/2}$$
$$> c|t_2 - t_1| + c|t - t_1| + c|t - \tau_{21}(t) - t_2| \geqq c\tau_{21}(t),$$

a contradiction. A similar contradiction is found when $(j, i) = (1, 2)$. Thus (2) has no solution for any t, even though the trajectories are continuously differentiable with $|v_j(t)| < c$ for all t.

When Eq. (2) has no solution we will say that the field $E_j(t, x_i(t)) = 0$.

This simply means that the past trajectories of the charges have been such that the field produced by charge j has not yet reached the point $x_i(t)$.

With this convention, it can easily be verified by substitution that the trajectories of the above example give a four-parameter family of solutions of Eq. (1) with $E_{\text{ext}}(t, x) = E$, a constant, provided $|b_1| = -m_1c^2/q_1E$, and $|b_2| = m_2c^2/q_2E$ (cf. Born (*5*)). In fact, as shown by Schott (*6*), these trajectories continue to satisfy the equation of motion when the radiation reaction terms are included in Eq. (1). In the present example the two-body problem has broken down into two one-body problems, and the difficulties of retarded arguments are eliminated. As Havas points out, Milner (*7*) gave a special case of this example but thought it was only an approximate solution of the two-body problem.

So long as the charges do not collide, there is no loss of generality in assuming $x_2(t) > x_1(t)$. We make this assumption and also assume $|v_i(t)| < c$, and $x_j(s)$ continuous on $[\alpha, t]$ and differentiable with $|v_j(s)| < c$ on (α, t). Making use of the algebraic sign of $x_i - x_j(t - \tau_{ji}(t))$ determined in Lemma 1, we can now write the expression for $E_j(t, x_i(t))$ in a fairly simple form. Eq. (1) becomes (in mks units)

$$\frac{v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = \frac{(-1)^i q_1 q_2}{4\pi\epsilon_0 m_i c^2 \tau_{ji}^2(t)} \frac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))} + q_i E_{\text{ext}}(t, x_i(t))/m_i, \qquad (i, j) = (1, 2), (2, 1), \tag{3a}$$

when (2) has a solution. If (2) has no solution then the first term on the right hand side is omitted. These equations are to be considered along with the pair

$$x_i'(t) = v_i(t) \qquad i = 1, 2. \tag{3b}$$

If we were considering the case of three-dimensional motion Eq. (3a) would be much more complicated. In particular, the acceleration of charge j at the retarded time $t - \tau_{ji}(t)$ would occur on the right hand side, making the existence and uniqueness problems much harder (*3*).

III. THE EXISTENCE AND UNIQUENESS THEOREM AND CONTINUOUS DEPENDENCE OF THE SOLUTIONS ON THE INITIAL DATA

Before tackling the existence and uniqueness theorem for the entire system of functional and functional-differential equations (2), (3) = (2), (3a), (3b), we will obtain further properties of the solutions of Eq. (2).

THEOREM 1 (Existence and properties of solutions of Eq. (2)). *Let $x_j(t)$ be continuous on $[\alpha, \beta]$, where $\alpha < \beta < \infty$, and differentiable with $|v_j(t)| \leqq c$ on (α, β). For some $t_0 \in (\alpha, \beta)$, let $x_i(t)$ be continuous on $[t_0, \beta]$ and differentiable with $|v_i(t)| < c$ on (t_0, β), and let $x_2(t) \geqq x_1(t)$ for all $t \in [t_0, \beta)$. Suppose* Eq. (2) *has a solution at t_0.*

Then, for all $t \in [t_0, \beta]$, (i) *a unique solution, $\tau_{ji}(t)$, of* (2) *exists,* (ii) *$t -$*

$\tau_{ji}(t)$ is strictly increasing, (iii) *$\tau_{ji}(t)$ is continuous, and* (iv) *$\tau_{ji}(\beta) = 0$ if and only if $x_2(\beta) = x_1(\beta)$—a collision. Also, for all $t \in [t_0, \beta]$,*

$$\frac{x_2(t) - x_1(t)}{c + b_j(t)} \leqq \tau_{ji}(t) \leqq \frac{x_2(t) - x_1(t)}{c + a_j(t)},$$

where $a_j(t)$ and $b_j(t)$ are any numbers such that $-c \leqq a_j(t) \leqq (-1)^j v_j(s) \leqq b_j(t) \leqq c$ for all $s \in (\alpha, t)$. (This estimate can give new information only when $-c < a_j(t)$ or $b_j(t) < c$.)

Moreover, if $x_2(t) > x_1(t)$ on (t_0, β), $v_j(t)$ is continuous on (α, β), and $v_i(t)$ is continuous on (t_0, β), then, for all $t \in (t_0, \beta)$, $\tau_{ji}(t)$ is continuously differentiable and satisfies the delay-differential equation

$$\tau'_{ji}(t) = \frac{(-1)^i v_i(t) - (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}. \tag{4}$$

This also shows that $\tau'_{ji}(t) < 1$.

PROOF: From the existence of a solution at t_0, we find, for any $t \in (t_0, \beta]$,

$$|x_i(t) - x_j(t_0 - \tau_{ji}(t_0))| \leqq |x_i(t) - x_i(t_0)| + |x_i(t_0) - x_j(t_0 - \tau_{ji}(t_0))|$$
$$< c(t - t_0) + c\tau_{ji}(t_0) = c(t - t_0 + \tau_{ji}(t_0)).$$

The existence and uniqueness of the solution of (2) for all $t \in [t_0, \beta]$ now follows from Lemma 1 with $\bar{\alpha} = t_0 - \tau_{ji}(t_0)$.

For $t \in (t_0, \beta]$, one could in fact take $\bar{\alpha} = t_0 - \tau_{ji}(t_0) + \delta$, where δ is an appropriate small positive number. This shows that $t - \tau_{ji}(t) > t_0 - \tau_{ji}(t_0)$, or, using an arbitrary $t_1 \in [t_0, \beta)$ in place of t_0, the same calculation shows that $t - \tau_{ji}(t)$ is strictly increasing.

Now, since $t - \tau_{ji}(t)$ is monotone and $t_0 - \tau_{ji}(t_0) \leqq t - \tau_{ji}(t) \leqq \beta$ for $t_0 \leqq t \leqq \beta$, it follows that $\lim_{t\to t_1} [t - \tau_{ji}(t)]$ exists for all $t_1 \in [t_0, \beta]$ (one-sided limits being understood at t_0 and β). Thus

$$a_{ji}(t_1) = \lim_{t\to t_1} \tau_{ji}(t)$$

also exists. Due to the continuity of $x_j(t)$ on $[\alpha, \beta]$, we can take the limit in (2) as $t \to t_1$ (cf. Landau (*8*), Theorem 77) and find

$$a_{ji}(t_1) = |x_i(t_1) - x_j(t_1 - a_{ji}(t_1))|/c.$$

But this equation has the unique solution $a_{ji}(t_1) = \tau_{ji}(t_1)$. Therefore

$$\lim_{t\to t_1} \tau_{ji}(t) = \tau_{ji}(t_1),$$

which shows that $\tau_{ji}(t)$ is continuous.

The conclusion that $\tau_{ji}(\beta) = 0$ if and only if $x_j(\beta) = x_i(\beta)$ now follows from inspection of (2) together with the knowledge that the solution is unique.

Now let $a_j(t)$ and $b_j(t)$ be numbers such that $-c \leqq a_j(t) \leqq (-1)^j v_j(s) \leqq b_j(t) \leqq c$ for all $s \in (\alpha, t)$, where t is some number in $[t_0, \beta]$. By the mean value theorem, there exists a number ξ between $t - \tau_{ji}(t)$ and t such that

$$
\begin{aligned}
c\tau_{ji}(t) &= (-1)^i[x_i(t) - x_j(t - \tau_{ji}(t))] \\
&= (-1)^i[x_i(t) - x_j(t) + v_j(\xi)\tau_{ji}(t)] \\
&= x_2(t) - x_1(t) - (-1)^j v_j(\xi)\tau_{ji}(t).
\end{aligned}
$$

Therefore

$$\tau_{ji}(t) = \frac{x_2(t) - x_1(t)}{c + (-1)^j v_j(\xi)}.$$

The estimates of the theorem follow immediately from this.

In order to show that $\tau_{ji}(t)$ is continuously differentiable, we consider Eq. (2) in the form

$$f(t, \tau_{ji}(t)) = 0,$$

where $f(t, \tau) \equiv c\tau - |x_i(t) - x_j(t - \tau)|$. For each $t_1 \in (t_0, \beta)$, since $x_j(t_1) \neq x_i(t_1)$, the solution, $\tau_{ji}(t_1)$, is positive. Hence $\alpha < t_1 - \tau_{ji}(t_1) < t_1$ and $|x_i(t_1) - x_j(t_1 - \tau_{ji}(t_1))| > 0$. Due to the continuity of $x_j(t)$ and $x_i(t)$, there is an open rectangle, R, about the point $(t_1, \tau_{ji}(t_1))$ such that, for all $(t, \tau) \in R$,

$$t_0 < t < \beta, \qquad \alpha < t - \tau < t, \quad \text{and} \quad |x_i(t) - x_j(t - \tau)| > 0.$$

In R, $f(t, \tau)$ is a differentiable function of t and τ, and

$$\frac{\partial f(t, \tau)}{\partial \tau} = c - \frac{[x_i(t) - x_j(t - \tau)]v_j(t - \tau)}{|x_i(t) - x_j(t - \tau)|} > 0.$$

If $v_j(t)$ and $v_i(t)$ are now assumed continuous, $f(t, \tau)$ is continuously differentiable. The differentiability of $\tau_{ji}(t)$ and Eq. (4) then follow from the implicit function theorem (cf. Landau (*8*), Theorem 315) and Lemma 1. Equation (4) itself then shows that $\tau_{ji}'(t)$ is continuous.

Finally the fact that $\tau_{ji}'(t) < 1$ follows from (4) since the right hand side is increased when $(-1)^i v_i(t)$ is replaced by c. Q.E.D.

In the proof of the existence theorem for the two-body problem, we will arrange to replace Eq. (2) by Eq. (4). We then consider a system of six first order delay-differential equations, (3), (4) = (3a), (3b), (4). If the dependent variables could be considered to be the six quantities $x_i(t)$, $v_i(t)$, $\tau_{ji}(t)$, $(j, i) = (2, 1), (1, 2)$, then we would have a delay-differential system in which the retarded arguments are explicit (rather than implicit) functions of the independent variable, t, and the dependent variables.

The general form of such a system (2) is

$$y'(t) = f(t, y(t), y(g_2(t, y(t))), \cdots, y(g_m(t, y(t)))), \tag{5}$$

where $y(t) = (y_1(t), \cdots, y_n(t))$, an n-dimensional, vector-valued function, $f = (f_1, \cdots, f_n)$ is a given vector-valued function, and the $g_j(t, y)$ $(j = 2, \cdots, m)$ are given functions with the property that $\alpha \leqq g_j(t, y) \leqq t$ for all values of t and y under consideration. By analogy to difference-differential systems, the problem posed for such a delay-differential system is as follows. One specifies an initial function $y(t) = \phi(t)$ on a segment $[\alpha, t_0]$, and then tries to extend $y(t)$ continuously to values of $t > t_0$ so as to satisfy the delay-differential system there.

With this motivation, we will now define the two-body problem to be considered, and prove an existence and uniqueness theorem for that problem.

PROBLEM (One-dimensional, two-body problem). *Let $E_{ext}(t, x)$ be a given continuous function in a domain D^2 (a connected open set) of the t, x plane. Let initial trajectories, $x_1(t)$ and $x_2(t)$, be given which are continuously differentiable with $|x_1'(t)|, |x_2'(t)| < c$ on an interval $[\alpha, t_0]$, where $-\infty \leqq \alpha < t_0 < \infty$. ($x_i'(\alpha)$ and $x_i'(t_0)$ are one-sided derivatives.) We seek extensions, $x_1(t)$ and $x_2(t)$, of the original trajectories to $[\alpha, \beta)$, where $\beta > t_0$, such that each $x_i(t)$ and $v_i(t) = x_i'(t)$ is continuous for $\alpha \leqq t < \beta$, and, for $t_0 < t < \beta$ and $i = 1, 2$,*

(a) *$(t, x_i(t)) \in D^2$ and $x_2(t) > x_1(t)$,*

(b) *$|v_i(t)| < c$, and*

(c) *$x_i(t)$ and $v_i(t)$ satisfy* Eq. (3a), *where $\tau_{ji}(t)$ is defined by* Eq. (2). *In case* Eq. (2) *has no solution, the first term on the right hand side of* Eq. (3a) *is omitted.*

Remarks. If, for the given initial trajectories, Eq. (2) has a solution at t_0 for $(j, i) = (2, 1)$ and $(1, 2)$ (as will be assumed in the next theorem), then, without loss of generality, one can assume $\alpha > -\infty$. This is justified as follows. In case a solution of the above-stated problem exists, it follows from Theorem 1 that each $t - \tau_{ji}(t) \geqq t_0 - \tau_{ji}(t_0) > -\infty$ for $t_0 \leqq t < \beta$. Thus, if a solution of the problem exists when $\alpha = -\infty$, α can be increased without affecting that solution. On the other hand, if no solution exists when $\alpha = -\infty$ then certainly no solution will exist if α is increased.

The statement of the problem and of the existence theorem to follow could be generalized by permitting different ranges of definition for the two initial trajectories. However, this would make the notation more cumbersome.

THEOREM 2 (Extended existence and uniqueness theorem for the one-dimensional, two-body problem). *Let $E_{ext}(t, x)$ be a given continuous function which is Lipschitz continuous with respect to x in every compact subset of D^2. Let $x_i(t)$ $(i = 1, 2)$ be given continuous functions on $[\alpha, t_0]$, where $-\infty \leqq \alpha < t_0 < \infty$, such that each $v_i(t) = x_i'(t)$ is Lipschitz continuous on $[\alpha, t_0]$, and such that*

(1) *each $(t_0, x_i(t_0)) \in D^2$ and $x_2(t_0) > x_1(t_0)$,*

(2) *each* $|v_i(t)| < c$ *on* $[\alpha, t_0]$, *and*

(3) Eq. (2) *has a solution*, $\tau_{ji}(t_0)$, *at* t_0 *for* $(j, i) = (2, 1)$ *and* $(1, 2)$.

Then the one-dimensional, two-body problem has a unique solution for $\alpha \leqq t < \beta$, *where either* $\beta = +\infty$ *or else one of the following occurs as* $t \to \beta - 0$:

$$(t, x_i(t)) \to \text{the boundary of } D^2 \text{ for } i = 1 \text{ or } 2, \text{ or}$$

$$\lim x_1(t) = \lim x_2(t) \text{—a collision.}$$

Preliminary Remarks. This theorem gives about as much information about the trajectories as one could expect without considering the algebraic sign of q_1q_2. In particular we observe that, so long as each $(t, x_i(t))$ stays away from the boundary of D^2 and the charges do not collide, the range of validity of the solution will not be restricted because of speeds approaching c or because of delays approaching zero.

The continuity conditions on $E_{\text{ext}}(t, x)$ are automatically satisfied if one thinks of this field as a solution of Maxwell's equations. The domain D^2 is then determined by the singularities of that field (e.g. the trajectories of other charges), the location of which must be estimated somehow. In the true two-body problem $E_{\text{ext}}(t, x) \equiv 0$ and all references to D^2 are omitted.

The Lipschitz condition on $v_i(t)$ for $\alpha \leqq t \leqq t_0$ could undoubtedly be weakened, however it could probably not be simply omitted if the solution is to be unique. This is suggested by the following simple example of a scalar delay-differential equation: Consider

$$y'(t) = -2y(t - y(t)) + 5 \quad \text{for} \quad t > 0,$$

where the functions f and g (cf. Eq. (5)) are considered in the domain

$$\{-\infty < t < \infty, \quad y > 0, \quad y(\text{delayed}) > 0\}.$$

Clearly f and g are analytic. However if one chooses the continuous (but not Lipschitz continuous) initial function

$$y(t) = \phi(t) = |4 + t|^{1/2} + 2 \quad \text{on} \quad -\infty < t \leqq 0,$$

both

$$y(t) = 4 + t \quad \text{for} \quad t \geqq 0$$

and

$$y(t) = 4 + t - t^2 \quad \text{for} \quad 0 \leqq t \leqq 2$$

provide solutions.

It might be possible to give the existence and uniqueness theorem without the condition 3 that Eq. (2) have a solution at t_0. We have already considered an example of such trajectories (following Lemma 1).

PROOF OF THEOREM 2: Without loss of generality, we can assume

$$-\infty < \alpha < \min_{(j,i)=(2,1),(1,2)} [t_0 - \tau_{ji}(t_0)].$$

If this is not the case, α can be increased or decreased and the initial data extended, if necessary, so that the given conditions remain satisfied. We will see that the manner of extension has no effect on the solution.

For simplicity, we will also define $\tau_{ji}(t) \equiv \tau_{ji}(t_0)$ for $\alpha \leqq t \leqq t_0$. This is not really necessary, but it will make the problem look more like that for difference-differential equations.

Now observe that we have a set of six initial functions,

$$x_i(t), v_i(t), \tau_{ji}(t) \qquad (j, i) = (2, 1), (1, 2),$$

on $\alpha \leqq t \leqq t_0$ for the dependent variables of the six delay-differential equations (3) and (4). This initial data is of the type required for systems like (5). We will make use of the fact that the initial functions, particularly $v_i(t)$, are Lipschitz continuous.

The argument of the function f appearing in this delay-differential system can be considered to lie in E^9, a 9-dimensional Euclidean space, a typical point of which is

$$(t, \tau_{21}, \tau_{12}, x_1, x_2, v_1, v_2, v_{2(1)}, v_{1(2)}).$$

The last two components represent the terms $v_2(t - \tau_{21}(t))$ and $v_1(t - \tau_{12}(t))$ respectively.

We will consider the delay-differential system in that domain D of E^9 determined by the conditions

$$0 < \tau_{21} < t - \alpha, \quad 0 < \tau_{12} < t - \alpha,$$

$$(t, x_1), \quad (t, x_2) \in D^2,$$

$$x_2 > x_1,$$

$$|v_1| < c, \quad |v_2| < c, \quad \text{and}$$

$$|v_{1(2)}| < c, \quad |v_{2(1)}| < c,$$

together with the requirement that

$$(t_0, \tau_{21}(t_0), \tau_{12}(t_0), x_1(t_0), x_2(t_0), v_1(t_0), v_2(t_0),$$
$$v_2(t_0 - \tau_{21}(t_0)), v_1(t_0 - \tau_{12}(t_0))) \in D.$$

The first group of five conditions defines an open set in E^9. If this set is not connected (as could be the case if D^2 is not convex) then the last condition picks out the one connected component of the set to be called D.

Observe that the function f of (5), representing the right hand sides of (3) and (4), is analytic so long as the argument lies in D. Also the functions $g_j(t, y)$ of (5), representing $t - \tau_{21}$ and $t - \tau_{12}$, are analytic and satisfy the condition $\alpha < g_j(t, y) < t$.

Now, since we have strict inequalities, $t - \tau_{ji} < t$, it is easy to obtain a local existence and uniqueness theorem for the system (3), (4). One observes that in some neighborhood of the starting point

$$\alpha < t - \tau_{ji} < t_0 .$$

Thus, for small $t - t_0 > 0$, one has the ordinary differential system obtained by replacing $v_j(t - \tau_{ji}(t))$ in (3) and (4) by the given initial function. The assumed continuity and Lipschitz conditions now assure the existence of a unique solution on some interval $[t_0, t_0 + h]$, where $h > 0$.

The extended existence and uniqueness theorem for the system (3), (4) is now found by an analysis similar to that used for ordinary differential equations. The theorem is proved in detail in (*2*), even for the case of weak inequalities $g_j(t, y) \leqq t$. The conclusion is that a unique solution exists for $\alpha \leqq t < \beta$, where either $\beta = +\infty$ or else the argument of f of Eq. (5) comes arbitrarily close to the boundary of D as t approaches β.

In our case this means that if $\beta < \infty$ and cannot be increased, then at least one of the following occurs as $t \to \beta - 0$:

(i) $\liminf [x_2(t) - x_1(t)] = 0$,
(ii) $(t, x_i(t))$ comes arbitrarily close to the boundary of D^2 for $i = 1$ or 2,
(iii) $\limsup |v_i(t)| = c$ for $i = 1$ or 2,
(iv) $\limsup |v_j(t - \tau_{ji}(t))| = c$ for $(j, i) = (2, 1)$ or $(1, 2)$,
(v) $\liminf \tau_{ji}(t) = 0$ for $(j, i) = (2, 1)$ or $(1, 2)$, or
(vi) $\liminf (t - \tau_{ji}(t)) = \alpha$ for $(j, i) = (2, 1)$ or $(1, 2)$.

We can observe at once that (vi) will not occur, since for $|v_i(t)| < c$ on $\alpha < t < \beta$ $(i = 1, 2)$ we have seen that any solution of Eq. (4) has the property $\tau'_{ji}(t) < 1$. This also shows that each retarded argument $t - \tau_{ji}(t) \geqq t_0 - \tau_{ji}(t_0)$ on $[t_0, \beta)$. Thus the manner of extension of the initial data to times before $\min_{(j,i)=(2,1),(1,2)} [t_0 - \tau_{ji}(t_0)]$, if this was done, had no effect on the solution. Similarly, the extension of $\tau_{ji}(t)$ back to α was immaterial.

We now prove that we have indeed obtained a unique solution of the one-dimensional, two-body problem for $\alpha \leqq t < \beta$. A rearrangement of Eq. (4) gives

$$c\tau'_{ji}(t) = (-1)^i[v_i(t) - v_j(t - \tau_{ji}(t))(1 - \tau'_{ji}(t))].$$

Integrating this from t_0 to t and using the fact that (2) is satisfied at t_0 gives

$$c\tau_{ji}(t) = (-1)^i[x_i(t) - x_j(t - \tau_{ji}(t))]$$

for all $t \in [t_0, \beta)$. This shows that Eq. (2) is satisfied and we thus have a solution of the two-body problem.

Conversely, if we had a solution of the two-body problem (2), (3), Eq. (4) would be satisfied according to Theorem 1. Thus we would have a solution of the system (3), (4) with the given initial data. Since that solution is unique, the solution of the two-body problem is also unique on $[\alpha, \beta)$.

We will next show that conditions (iii), (iv), and (v) cannot occur unless (i) or (ii) occurs.

Suppose (for contradiction) that $\beta < \infty$ and cannot be increased, and yet neither (i) nor (ii) occurs as $t \to \beta - 0$. This implies that $x_2(t) - x_1(t) \geqq \delta > 0$ and that $(t, x_1(t))$ and $(t, x_2(t))$ remain inside a compact set $F \subset D^2$ for $t_0 \leqq t < \beta$.

It now follows from Lemma 1 or Theorem 1 that each $\tau_{ji}(t) \geqq \delta/2c > 0$, which eliminates (v).

Since $|v_j(s)| < c$ on the compact set $[\alpha, \beta - \delta/2c]$ it follows that $|v_j(s)| \leqq \bar{c} < c$ there. This eliminates (iv).

Using these bounds together with the fact that $E_{ext}(t, x)$, being continuous, is bounded on F, Eq. (3a) yields

$$-H_1 \leqq \frac{v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} \leqq H_2 \quad \text{for} \quad t_0 < t < \beta,$$

where H_1 and H_2 are positive constants. Integration (or application of the mean value theorem) now gives

$$-H_1(\beta - t_0) \leqq \frac{v_i(t)}{[1 - v_i^2(t)/c^2]^{1/2}} - \frac{v_i(t_0)}{[1 - v_i^2(t_0)/c^2]^{1/2}} \leqq H_2(\beta - t_0)$$

for $t_0 \leqq t < \beta$. These upper and lower bounds can only hold if $|v_i(t)| \leqq \bar{c}$ for some $\bar{c} < c$. This eliminates (iii). (Note that this argument depends upon the fact that the exponent on the bracketed quantity in the denominator of the left hand side of (3a) is $3/2$, or at least greater than 1.)

To get the exact statement of the theorem from conditions (i) and (ii), we use the fact that $\lim_{t \to \beta - 0} x_i(t)$ exists for $i = 1, 2$. This follows from the Lipschitz continuity of $x_i(t)$ (Lipschitz constant c). Q.E.D.

Having established existence and uniqueness, it is natural to ask about the dependence of the solutions on the initial data and the external field. The following theorem asserts that this dependence is continuous and that the solution of a slightly modified problem exists almost as far as that of the original problem.

THEOREM 3 (Continuous dependence of the solutions on the initial data and the external field). *Let* $x_1(t)$, $x_2(t)$ *on* $[\alpha, \beta)$ *be the unique solution of the two-body problem assuming the conditions of Theorem* 2. *Assume that*

$$\alpha < \min_{(j,i)=(2,1),(1,2)} [t_0 - \tau_{ji}(t_0)].$$

Then for every $\bar{\beta} \in (t_0, \beta)$ *and every* $\epsilon > 0$, *there exist positive numbers*, δ_1, δ_2, δ_3, δ_4, *and* δ_5, *with the following properties. The modified two-body problem formed by using an external field* $\hat{E}_{\text{ext}}(t, x)$ *and initial functions* $\hat{x}_1(t)$, $\hat{x}_2(t)$ *defined for* $\alpha \leqq t \leqq \hat{t}_0$ *satisfying the requirements of Theorem* 2 *except conditions* (1), (2), *and* (3) *together with the conditions*

$|\hat{E}_{\text{ext}}(t, x) - E_{\text{ext}}(t, x)| \leqq \delta_1$ *for all* $(t, x) \in D^2$,
$\hat{t}_0 \in (\alpha, \bar{\beta})$ *with* $\hat{t}_0 \geqq t_0 - \delta_2$,
$|\hat{x}_i(t) - x_i(t)| \leqq \delta_3$ *for* $\alpha \leqq t \leqq \hat{t}_0$, $i = 1, 2$, *and*
$|\hat{x}_i'(t) - x_i'(t)| \leqq \delta_4$ *for* $\alpha \leqq t \leqq \hat{t}_0$, $i = 1, 2$

has a unique solution, $\hat{x}_1(t)$, $\hat{x}_2(t)$, *valid for* $\alpha \leqq t < \bar{\beta}$, *and* $|\hat{x}_i(\hat{t}) - x_i(t)| < \epsilon$ *and* $|\hat{x}_i'(\hat{t}) - x_i'(t)| < \epsilon$ *whenever* $\hat{t}, t \in [\alpha, \bar{\beta})$ *and* $|\hat{t} - t| \leqq \delta_5$.

Preliminary Remark. Here we have assumed that slightly more initial data is given than would be necessary for the original problem alone. This is necessary to permit comparison with the new problem in which the values of $\hat{t}_0 - \hat{\tau}_{ji}(\hat{t}_0)$ may be slightly smaller than in the original problem. Of course, as observed in the proof of Theorem 2, we could always extend the given initial data back rather arbitrarily. However, the values of δ_2, δ_3, and δ_4 would depend on the extension.

PROOF OF THEOREM 3: In the proof of Theorem 2, we showed the existence and uniqueness of the solution of the two-body problem by considering an equivalent problem based on the system (3), (4) with appropriate initial data. A similar technique works here.

Without loss of generality we again assume $\alpha > -\infty$. Then there exists a number $\bar{c} \in (0, c)$ such that $|v_i(t)| \leqq \bar{c}$ for $\alpha \leqq t \leqq \bar{\beta}$, $i = 1, 2$. The number $\bar{c}$ will serve as a convenient Lipschitz constant for both $x_1(t)$ and $x_2(t)$ over $\alpha \leqq t \leqq \bar{\beta}$.

We will now list some preliminary restrictions, to be imposed on δ_2, δ_3, and δ_4, and their consequences:

$$\delta_2 < t_0 - \alpha \quad \text{and} \quad 2\bar{c}\delta_2 < x_2(t_0) - x_1(t_0)$$

imply that $x_2(t) - x_1(t) > 0$ for $t_0 - \delta_2 \leqq t \leqq t_0$. Now we can require

$$2\delta_3 < \inf_{t_0 - \delta_2 \leqq t \leqq \bar{\beta}} [x_2(t) - x_1(t)],$$

which implies that $\hat{x}_2(\hat{t}_0) - \hat{x}_1(\hat{t}_0) > 0$. $\delta_4 < c - \bar{c}$ implies that $|\hat{v}_i(t)| < c$ for $\alpha \leqq t \leqq \hat{t}_0$, $i = 1, 2$. Similarly, a further restriction of δ_2 and δ_3 will assure $(\hat{t}_0, \hat{x}_i(\hat{t}_0)) \in D^2$ for $i = 1, 2$.

Now, since $\alpha < t - \tau_{ji}(t)$ for $t_0 \leqq t \leqq \bar{\beta}$, it follows from Lemma 1 that

$$|x_i(t) - x_j(\alpha)| < c(t - \alpha) \text{ for } t_0 \leqq t \leqq \bar{\beta}, \qquad (j, i) = (2, 1), (1, 2).$$

Add the restriction

$$2c\delta_2 + 2\delta_3 < \min_{t_0 \leqq t \leqq \bar{\beta}} \{c(t - \alpha) - \max_{(j,i)=(2,1),(1,2)} |x_i(t) - x_j(\alpha)|\}.$$

Then it follows that, if $t_0 - \delta_2 \leqq \hat{t}_0 \leqq t_0$,

$$\begin{aligned} c(\hat{t}_0 - \alpha) - |\hat{x}_i(\hat{t}_0) - \hat{x}_j(\alpha)| &\geqq c(\hat{t}_0 - \alpha) - |\hat{x}_i(\hat{t}_0) - x_i(\hat{t}_0)| \\ &\quad - |x_i(\hat{t}_0) - x_i(t_0)| - |x_i(t_0) - x_j(\alpha)| - |x_j(\alpha) - \hat{x}_j(\alpha)| \\ &\geqq -c\delta_2 + c(t_0 - \alpha) - \delta_3 - c\delta_2 - |x_i(t_0) - x_j(\alpha)| - \delta_3 > 0, \end{aligned}$$

and, if $t_0 \leqq \hat{t}_0 < \bar{\beta}$,

$$\begin{aligned} c(\hat{t}_0 - \alpha) - |\hat{x}_i(\hat{t}_0) - \hat{x}_j(\alpha)| &\geqq c(\hat{t}_0 - \alpha) - |\hat{x}_i(\hat{t}_0) - x_i(\hat{t}_0)| \\ &\quad - |x_i(\hat{t}_0) - x_j(\alpha)| - |x_j(\alpha) - \hat{x}_j(\alpha)| > 2c\delta_2 > 0. \end{aligned}$$

Thus, again using Lemma 1, Eq. (2) has a unique solution, $\hat{\tau}_{ji}(\hat{t}_0)$, at $\hat{t}_0$ for the modified problem. A similar, but slightly simpler, calculation shows that, under the same condition on δ_2 and δ_3, Eq. (2) also has a unique solution, $\tau_{ji}(\hat{t}_0)$, for the original problem at the instant $\hat{t}_0$.

Now, using the knowledge of algebraic signs from Lemma 1, we can compute

$$\begin{aligned} c\,|\hat{\tau}_{ji}(\hat{t}_0) - \tau_{ji}(\hat{t}_0)| &= |\hat{x}_i(\hat{t}_0) - \hat{x}_j(\hat{t}_0 - \hat{\tau}_{ji}(\hat{t}_0)) - x_i(\hat{t}_0) + x_j(\hat{t}_0 - \tau_{ji}(\hat{t}_0))| \\ &\leqq |\hat{x}_i(\hat{t}_0) - x_i(\hat{t}_0)| + |\hat{x}_j(\hat{t}_0 - \hat{\tau}_{ji}(\hat{t}_0)) - x_j(\hat{t}_0 - \hat{\tau}_{ji}(\hat{t}_0))| \\ &\quad + |x_j(\hat{t}_0 - \hat{\tau}_{ji}(\hat{t}_0)) - x_j(\hat{t}_0 - \tau_{ji}(\hat{t}_0))| \\ &\leqq 2\delta_3 + \bar{c}\,|\hat{\tau}_{ji}(\hat{t}_0) - \tau_{ji}(\hat{t}_0)|. \end{aligned}$$

Therefore,

$$|\hat{\tau}_{ji}(\hat{t}_0) - \tau_{ji}(\hat{t}_0)| \leqq \frac{2\delta_3}{c - \bar{c}}.$$

By additional restrictions on δ_1, δ_2, δ_3, and δ_4, we can now make the changes in $E_{\text{ext}}(t, x)$ and the initial data for the system (3), (4) as small as desired. The fact that this makes only a small change in the solution, as claimed, now follows from the corresponding (nontrivial) theorem for system (5). See ref. *2*. Q.E.D.

IV. COLLISIONS, NONCOLLISIONS, AND OTHER PROPERTIES OF THE TRAJECTORIES

In this section we will obtain conditions under which the solutions of the one-dimensional, two-body problem will or will not terminate in a collision. We will also obtain conditions under which the speeds of the charges will or will not approach the speed of light and conditions under which the distance between the charges is unbounded.

As one would expect, the possibility of a collision depends upon the algebraic signs of the charges, or simply upon the sign of the product q_1q_2. However, we will use several inequalities which can be stated independently of this sign.

LEMMA 2. *Let $x_1(t)$, $x_2(t)$ be a solution of the one-dimensional, two-body problem for $\alpha \leqq t < \beta$ with Eq. (2) satisfied at t_0 (and hence for $t_0 \leqq t < \beta$).*

Then, for $t_0 \leqq t < \beta$,

$$\frac{(-1)^i}{q_1 q_2}\left\{\frac{v_i(t)}{[1 - v_i^2(t)/c^2]^{1/2}} - \frac{v_i(t_0)}{[1 - v_i^2(t_0)/c^2]^{1/2}}\right\}$$
$$\geqq \frac{1}{4\pi\epsilon_0 m_i c^2} \max\left[0, \frac{1}{\tau_{ji}(t)} - \frac{1}{\tau_{ji}(t_0)}\right] + \frac{(-1)^i}{q_j m_i}\int_{t_0}^{t} E_{\text{ext}}(s, x_i(s))\, ds \tag{6}$$
$$(j, i) = (2, 1), (1, 2).$$

If, in addition, we assume that $|v_i(t)| \leqq \bar{c} < c$ when $\alpha \leqq t < t_1 (i = 1, 2)$ for some $t_1 \in (t_0, \beta]$, then, defining $x(t) = x_2(t) - x_1(t)$, we have for $t_0 < t < t_1$

$$\frac{x''(t)}{q_1 q_2} \geqq \frac{K_1(m_1^{-1} + m_2^{-1})}{4\pi\epsilon_0 x^2(t)} + \frac{1}{q_1 q_2}\sum_{i=1,2} (-1)^i[1 - v_i^2(t)/c^2]^{3/2} \cdot q_i E_{\text{ext}}(t, x_i(t))/m_i, \tag{7}$$

where $K_1 = (1 - \bar{c}^2/c^2)^{1/2}(1 - \bar{c}/c)^4$, and

$$\frac{x''(t)}{q_1 q_2} \leqq \frac{K_2(m_1^{-1} + m_2^{-1})}{4\pi\epsilon_0 x^2(t)} + \frac{1}{q_1 q_2}\sum_{i=1,2} (-1)^i[1 - v_i^2(t)/c^2]^{3/2} \cdot q_i E_{\text{ext}}(t, x_i(t))/m_i, \tag{8}$$

where $K_2 = (1 + \bar{c}/c)^3/(1 - \bar{c}/c)$.

Preliminary Remark. The estimates of this lemma are, admittedly, crude and could certainly be sharpened. They will, however, serve to show what types of behavior can occur with respect to collision or noncollision in the one-dimensional, two-body problem. They will also provide information about the limiting values of the speeds of the charges.

PROOF OF LEMMA 2. Since $-(-1)^i v_i(t) < c$, Eq. (4) yields

$$-\tau_{ji}'(t) < \frac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}$$

for $t_0 < t < \beta$. Substituting this inequality into Eq. (3a) gives

$$\frac{(-1)^i v_i'(t)}{q_1 q_2[1 - v_i^2(t)/c^2]^{3/2}} > \frac{1}{4\pi\epsilon_0 m_i c^2}\max\left[0, \frac{-\tau_{ji}'(t)}{\tau_{ji}^2(t)}\right] + \frac{(-1)^i}{q_j m_i} E_{\text{ext}}(t, x_i(t))$$

for $t_0 < t < \beta$. An application of the mean value theorem gives (6).

Now consider Eq. (3a) on $t_0 < t < t_1$ under the assumption that $|v_i(t)| \leqq \bar{c} < c$ for $\alpha \leqq t < t_1 (i = 1, 2)$. Using the estimates for $\tau_{ji}(t)$ from Theorem 1 with $-a_j(t) = \bar{c} = b_j(t)$, we find

$$\frac{K_1/m_i}{4\pi\epsilon_0 x^2(t)} \leqq \frac{(-1)^i v_i'(t)}{q_1 q_2} - \frac{(-1)^i[1 - v_i^2(t)/c^2]^{3/2} q_i E_{\text{ext}}(t, x_i(t))}{q_1 q_2 m_i} \leqq \frac{K_2/m_i}{4\pi\epsilon_0 x^2(t)}.$$

Adding these inequalities for $i = 1$ to the corresponding inequalities for $i = 2$ gives (7) and (8). Q.E.D.

THEOREM 4 (One-dimensional trajectories of two point charges of like sign). *Let q_1 and q_2 have the same sign and let $x_1(t)$, $x_2(t)$ on $[\alpha, \beta)$ be a solution of the one-dimensional, two-body problem with* Eq. (2) *satisfied at t_0 (β could be $+\infty$). Let $E_{\text{ext}}(t, x_i(t))$ be bounded for $t_0 \leqq t < \beta$ and let $\int_{t_0}^{\beta} |E_{\text{ext}}(t, x_i(t))| \, dt$ exist ($i = 1, 2$).*

Then $|v_i(t)| \leqq \bar{c} < c$ for $\alpha \leqq t < \beta$, $i = 1, 2$, and $x_2(t) - x_1(t)$ is bounded away from zero for $t_0 \leqq t < \beta$.

If $E_{\text{ext}}(t, x) \equiv 0$ and $\beta = \infty$, then $\lim_{t\to\infty} [x_2(t) - x_1(t)] = \infty$.

COROLLARY. *If q_1 and q_2 have the same sign, the conditions of Theorem 2 are satisfied, and $E_{\text{ext}}(t, x)$ is bounded in $D^2 = E^2$ (the entire t, x plane), then a unique solution exists for all $t \geqq \alpha$,* i.e. *the charges do not collide in a finite time.*

If $E_{\text{ext}}(t, x) \equiv 0$ then $\lim_{t\to\infty} [x_2(t) - x_1(t)] = \infty$ and $|v_i(t)| \leqq \bar{c} < c$ for all $t \geqq \alpha$.

PROOF OF THEOREM 4. We introduce positive constants M and N such that

$$|E_{\text{ext}}(t, x_i(t))| \leqq M \quad \textit{for} \quad t_0 \leqq t < \beta \quad \textit{and} \int_{t_0}^{\beta} |E_{\text{ext}}(t, x_i(t))| \, dt \leqq N$$

for $i = 1, 2$. (Of course the second condition would be an automatic consequence of the first if β were finite.) Again we let $x(t) = x_2(t) - x_1(t)$.

Now, since the right hand side of (6) is bounded from below, it follows that $(-1)^i v_i(t) \geqq -\hat{c} > -c$ for $t_0 \leqq t < \beta (i = 1, 2)$, where $\hat{c}$ is a constant. Using the fact that α can be considered finite and $|v_i(t)| < c$ on $\alpha \leqq t \leqq t_0$, we can, by increasing $\hat{c}$ if necessary, assert that $(-1)^i v_i(t) \geqq -\hat{c}$ for $\alpha \leqq t < \beta$, $i = 1, 2$. Applying this estimate, together with the estimate $\tau_{ji}(t) \geqq x(t)/2c$ for $t_0 \leqq t < \beta$, to (3a) gives

$$\frac{(-1)^i v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} \leqq \frac{q_1 q_2}{\pi\epsilon_0 m_i x^2(t)} \frac{c + \hat{c}}{c - \hat{c}} + \frac{(-1)^i q_i E_{\text{ext}}(t, x_i(t))}{m_i} \tag{9}$$

for $t_0 < t < \beta$, $i = 1, 2$. (Note that we could have reduced the hypothesis for (8) to $(-1)^i v_i(t) \geqq -\bar{c}$ if we had defined $K_2 = 4(1 + \bar{c}/c)/(1 - \bar{c}/c)$.)

We next observe from (3a) that

$$(-1)^i \left\{ v_i(t) - \frac{q_i}{m_i} \int_{t_0}^{t} \left[1 - \frac{v_i^2(s)}{c^2}\right]^{3/2} E_{\text{ext}}(s, x_i(s))\, ds \right\}$$

is a nondecreasing function for $t_0 < t < \beta$, $i = 1, 2$. It is also bounded from above by $c + |q_i| N/m_i$. Thus the expression has a limit as $t \to \beta - 0$. Since the integral is continuous with respect to its upper limit, it follows that

$$\lim_{t \to \beta - 0} v_i(t)$$

exists.

Suppose (for contradiction) that $\lim_{t\to\beta-0} (-1)^i v_i(t) = c$ for $i = 1$ or 2. Choose any number $c_1 \in (\hat{c}, c)$. Then there exists a number $\beta_1 \in (t_0, \beta)$ such that $(-1)^i v_i(t) \geqq c_1$ for $\beta_1 < t < \beta$. On this interval, $x'(t) = (-1)^i v_i(t) + (-1)^j v_j(t) \geqq c_1 - \hat{c} > 0$. Thus $x(t) \geqq x(\beta_1) + (c_1 - \hat{c})(t - \beta_1)$. Substituting this into (9) and applying the mean value theorem between β_1 and $t \in (\beta_1, \beta)$, we find

$$\frac{(-1)^i v_i(t)}{[1 - v_i^2(t)/c^2]^{1/2}} - \frac{(-1)^i v_i(\beta_1)}{[1 - v_i^2(\beta_1)/c^2]^{1/2}} \leqq \frac{q_1 q_2}{\pi\epsilon_0 m_i} \frac{c + \hat{c}}{c - \hat{c}} \frac{1}{(c_1 - \hat{c}) x(\beta_1)} + |q_i| N/m_i .$$

Since the right hand side is a finite constant, $(-1)^i v_i(t)$ cannot approach c. Thus $-\hat{c} \leqq \lim_{t\to\beta-0} (-1)^i v_i(t) < c$ for $i = 1, 2$. This implies that $|v_i(t)| \leqq \bar{c} < c$ for $\alpha \leqq t < \beta$, $i = 1, 2$.

Let $x_0 = x(t_0) = x_2(t_0) - x_1(t_0)$. Since $x'(t)$ is continuous, the only way $x(t)$ can ever become $< x_0$ is for $x'(t)$ to be negative on some interval. Assume $x(t_1) = x_0$ and $x'(t) < 0$ on an interval $t_1 < t < t_2$, where $t_0 \leqq t_1 < t_2 \leqq \beta$. Then, multiplying (7) by $q_1 q_2 x'(t)$ we obtain

$$x''(t)x'(t) \leqq \frac{q_1 q_2 K_1(m_1^{-1} + m_2^{-1})}{4\pi\epsilon_0} \frac{x'(t)}{x^2(t)} - \left(\frac{|q_1|}{m_1} + \frac{|q_2|}{m_2}\right) M x'(t).$$

Applying the mean value theorem between t_1 and a point $t \in (t_1, t_2)$ gives

$$\begin{aligned} -2c^2 &\leqq [x'(t)]^2/2 - [x'(t_1)]^2/2 \\ &\leqq \frac{q_1 q_2 K_1(m_1^{-1} + m_1^{-1})}{4\pi\epsilon_0} \left[\frac{1}{x_0} - \frac{1}{x(t)}\right] + \left(\frac{|q_1|}{m_1} + \frac{|q_2|}{m_2}\right) M x_0 . \end{aligned}$$

Because of the lower bound, there must exist a number $\bar{x}_0 \in (0, x_0)$ such that $x(t) > \bar{x}_0$ for all $t_1 \leqq t < t_2$ and hence for all $t_0 \leqq t < \beta$.

Now in the special case $E_{\text{ext}}(t, x) \equiv 0$ with $\beta = \infty$, $x'(t)$ cannot remain $\leqq 0$ for all $t > t_0$. For if it did, then $x(t)$ would be bounded and hence, by (7), $x''(t)$ would be positive and bounded away from zero. But as soon as $x'(t)$ becomes positive for some $t > t_0$ it will remain positive (and continue to increase since $x''(t) > 0$). Hence $x(t) \to \infty$ as $t \to \infty$. Q.E.D.

Next we consider the possible trajectories of two charges of opposite signs. In this we will simplify the conditions and arguments by assuming $E_{\text{ext}}(t, x) \equiv 0$.

THEOREM 5 (One-dimensional trajectories of two point charges of opposite signs). *Let q_1 and q_2 have opposite signs and let $E_{\text{ext}}(t, x) \equiv 0$ on $D^2 = E^2$. Let $x_1(t)$, $x_2(t)$ on $[\alpha, \beta)$ be a solution of the one-dimensional, two-body problem with Eq. (2) satisfied at t_0.*

If $\beta = +\infty$, then $x_2(t) - x_1(t) \to \infty$ monotonely as $t \to \infty$, i.e. *the charges "escape". Also $|v_i(t)| \leqq \bar{c} < c$ for all $t \geqq \alpha$, $i = 1, 2$.*

If $\liminf_{t\to\beta-0} [x_2(t) - x_1(t)] = 0$, *then β is finite, the charges collide at $t = \beta$, $\lim_{t\to\beta-0} \tau_{ji}(t) = 0$ for $(j, i) = (2, 1)$ and $(1, 2)$, and $\lim_{t\to\beta-0} v_i(t) = (-1)^{i+1}c$ for $i = 1, 2$,* i.e. *the two charges are traveling toward each other at the speed of light at the instant of collision.*

Moreover, both of these cases are possible, i.e. *appropriate initial functions can be chosen so as to ensure either "escape" or collision.*

Preliminary Remark. In view of the first conclusion of the theorem, the assumption $\liminf_{t\to\beta-0} [x_2(t) - x_1(t)] = 0$ is much stronger than necessary to conclude that β is finite. However, since we have not assumed that the initial data satisfies the conditions of Theorem 2 or that the solutions are extended as far as possible, we need the strong assumption stated in order to get the conclusions about the collision.

Proof of Theorem 5. Again we let $x(t) = x_2(t) - x_1(t)$. From (3a), it follows that $v_1'(t) > 0$ and $v_2'(t) < 0$, and hence $x''(t) < 0$ for $t_0 < t < \beta$.

Suppose $x'(t) > 0$ for $t_0 < t < t_1$. Then it follows that

$$v_1(t_0) \leqq v_1(t) < v_2(t) \leqq v_2(t_0) \text{ for } t_0 \leqq t < t_1 .$$

But, by the continuity of $x_i'(t)$ and the fact that α can be considered finite, $|v_i(t)| \leqq \bar{c} < c$ for $\alpha \leqq t \leqq t_0$ $(i = 1, 2)$. These two results together imply

$$|v_i(t)| \leqq \bar{c} < c \quad \text{for} \quad \alpha \leqq t < t_1 \qquad (i = 1, 2).$$

We will use these conclusions in the four parts of the proof below.

1. Let $\beta = +\infty$. Suppose (for contradiction) that for some $t_1 \geqq t_0$, $x'(t_1) \leqq 0$. Choose any number $t_2 > t_1$. Then for all $t \geqq t_2$, $x'(t) \leqq x'(t_2) < 0$. Thus $x(t)$ is bounded for all $t > t_2$, and yet $x'(t)$ is negative and bounded away from zero. This implies that $x(t)$ becomes zero in a finite time—a contradiction. Thus we conclude that $x'(t) > 0$ for all $t \geqq t_0$, and hence $|v_i(t)| \leqq \bar{c} < c$ for all $t \geqq \alpha$, $i = 1, 2$.

By (7), we see that $x(t)$ could not be bounded, or else $x''(t)$ would be negative and bounded away from zero, implying $x'(t) \leqq 0$ after a finite time. Thus we conclude that $x(t) \to \infty$ monotonely as $t \to \infty$.

2. Assume $\liminf_{t\to\beta-0} x(t) = 0$. Then β is finite, and $\lim_{t\to\beta-0} x_i(t)$ exists for $i = 1, 2$ because $|x_i'(t)| < c$. Thus $x_i(t)$ can be extended to $[\alpha, \beta]$. Then Theorem 1 shows that $\lim_{t\to\beta-0} \tau_{ji}(t) = 0$ for $(j, i) = (2, 1), (1, 2)$, and inequality (6) shows that $\lim_{t\to\beta-0} (-1)^i v_i(t) = -c$ for $i = 1, 2$.

3. We exhibit a set of initial data which will lead to "escape" ($x(t) \to \infty$ as $t \to \infty$) by requiring that the conditions of Theorem 2 be satisfied and

$$x'(t_0) = v_2(t_0) - v_1(t_0) \geqq \left\{ \frac{-q_1 q_2 K_2(m_1^{-1} + m_2^{-1})}{2\pi\epsilon_0[x_2(t_0) - x_1(t_0)]} \right\}^{1/2}, \tag{10}$$

where $K_2 = (1 + \bar{c}/c)^3/(1 - \bar{c}/c)$ with $\bar{c} \equiv \max_{\alpha \leq t \leq t_0, i=1,2} |v_i(t)|$.

Let $t_1 = \sup\{t \in [t_0, \beta) : x'(s) > 0 \text{ for } t_0 \leqq s < t\}$. Suppose (for contradiction) that $t_1 < \beta$. Then $x'(t) > 0$ for $t_0 \leqq t < t_1$ and $x'(t_1) = 0$. For $t_0 < t < t_1$, (8) holds. Multiplying both sides of that inequality by $-q_1q_2x'(t)$ (a positive quantity) and integrating gives

$$-\frac{[x'(t)]^2}{2} + \frac{[x'(t_0)]^2}{2} \leqq \frac{-q_1 q_2 K_2(m_1^{-1} + m_2^{-1})}{4\pi\epsilon_0} \left[\frac{1}{x(t_0)} - \frac{1}{x(t)}\right]$$

for $t_0 \leqq t \leqq t_1$. This last inequality together with (10) shows that $[x'(t)]^2 > 0$ for $t_0 \leqq t \leqq t_1$—a contradiction. We thus conclude that $t_1 = \beta$.

Now a collision is impossible because $x'(t) > 0$ for $t_0 \leqq t < \beta$. Thus, since the conditions of Theorem 2 are satisfied, $\beta = +\infty$.

4. Finally, we exhibit a set of initial data which will lead to a collision by requiring that the conditions of Theorem 2 be satisfied and

$$x'(t_0) = v_2(t_0) - v_1(t_0) < \left\{ \frac{-q_1 q_2 K_1(m_1^{-1} + m_2^{-1})}{2\pi\epsilon_0[x_2(t_0) - x_1(t_0)]} \right\}^{1/2}, \tag{11}$$

where $K_1 = (1 - \bar{c}^2/c^2)^{1/2}(1 - \bar{c}/c)^4$ with $\bar{c} \equiv \max_{\alpha \leqq t \leqq t_0, i=1,2} |v_i(t)|$.

Suppose (for contradiction) that $x'(t) > 0$ for $t_0 \leqq t < \beta$, where β is as described in Theorem 2. Then $x(t)$ is an increasing function and there is no collision. Hence $\beta = \infty$, and (7) holds for all $t > t_0$. Multiplying both sides of that inequality by $q_1q_2x'(t)$ (a negative quantity) and integrating gives

$$\frac{[x'(t)]^2}{2} - \frac{[x'(t_0)]^2}{2} \leqq \frac{-q_1 q_2 K_1(m_1^{-1} + m_2^{-1})}{4\pi\epsilon_0} \left[\frac{1}{x(t)} - \frac{1}{x(t_0)}\right]$$

for $t \geqq t_0$. But $x(t)$ increases without bound, thus eventually, using (11), $[x'(t)]^2$ becomes negative—a contradiction.

We have thus shown that $x'(t_1) \leqq 0$ for some $t_1 \in [t_0, \beta)$. By the same argument used in the beginning of part 1 of this proof, $x(t)$ becomes zero in a finite time. Q.E.D.

V. DISCUSSION

The theorems of the last two sections give a rather complete qualitative analysis of the one-dimensional, two-body problem under the assumption that, for the given initial trajectories, the charges are already within each others influence at t_0, i.e., Eq. (2) has a solution at t_0. From Theorems 4 and 5 one can see that if $E_{ext}(t, x) \equiv 0$, the conditions of Theorem 2 are satisfied, and β is as described in Theorem 2, then

(1) there are no bounded solutions for all $t \geqq t_0$,

(2) a bounded solution occurs only for unlike charges, and in this case $\beta < \infty$ and a collision occurs at $t = \beta$,

(3) both charges reach the speed of light at the instant of a collision, and

(4) if a collision does not occur, then $\beta = \infty$ and $x_2(t) - x_1(t) \to \infty$ as $t \to \infty$.

Observe that the "escape" condition (10) and the collision condition (11) do not cover all possible initial trajectories. In particular, since $v_2(t_0) - v_1(t_0) < 2c$, the escape condition (10) is applicable only when $x_2(t_0) - x_1(t_0)$ is sufficiently large. This does not necessarily mean that there is no escape condition when $x_2(t_0) - x_1(t_0)$ is small, but rather that the estimates of Lemma 2 are too crude to produce it.

Apart from the possibility of improving these estimates, there are two larger unsolved problems for the system (2), (3): Does a unique solution exist if, in Theorem 2, one omits assumption 3 that Eq. (2) has a solution at t_0? The second question pertains to a different formulation of the problem. Do solutions of (2), (3) exist for all $t < \beta$, and, if so, what is required to determine them? For example, could one determine a unique solution on $-\infty < t < \beta$ by specifying appropriate data at one or two points, t_0 or t_{01} and $t_{02} \in (-\infty, \beta)$? This latter question has been discussed heuristically by Synge (*9*).

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