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Electromagnetic spin and orbital angular momenta of a classical point electron

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The bound momentum and the bound angular momentum of a point particle with arbitrary multipolar electromagnetic moments are considered. It is found that the relation $M^{(b)}_{\lambda\mu}(\tau) = 2z_{[\lambda}P^{(b)}_{\mu]}(\tau)$, which was established for a nonspinning accelerated point charge, is no longer true in the present case. An additional term emerges, which corresponds to the contribution to the bound angular momentum of the multipolar structure of the particle.

Recently, we calculated the bound and emitted angular momenta of a classical (nonspinning) accelerated point charge.[1] The main result was that the bound angular momentum is related to the bound momentum by the simple formula

$$M^{(b)}_{\lambda\mu}(\tau) = 2z_{[\lambda}P^{(b)}_{\mu]}(\tau). \tag{1}$$

which is the same relation valid for a material particle in classical mechanics. At the end of Ref. 1 we remarked that our approach could be generalized in a straightforward way to the general case of a point particle with an arbitrary electromagnetic multipolar structure. In the present paper we comment on some interesting points related to this problem; in particular, we show that Eq. (1) is no longer true for a spinning point particle.

In the general case of a point particle possessing arbitrary multipolar moments, the total electromagnetic angular momentum tensor $M_{\lambda\mu\nu}$ may still be decomposed into a radiation part and a bound part given by

$$M^{(r)}_{\lambda\mu\nu} = 2z_{[\lambda}T^{(r)}_{\mu]\nu} + 2s_{[\lambda}T^{(-3)}_{\mu]\nu}\ , \tag{2a}$$

$$M^{(b)}_{\lambda\mu\nu} = 2z_{[\lambda}T^{(b)}_{\mu]\nu} + 2s_{[\lambda}T^{(-4}_{\mu]\nu}\ , \tag{2b}$$

where

$$T^{(r)}_{\mu\nu} = T^{(-2)}_{\mu\nu}\ , \tag{3a}$$

$$T^{(b)}_{\mu\nu} = T^{(-3}_{\mu\nu} = T_{\mu\nu} - T^{(-2)}_{\mu\nu}. \tag{3b}$$

The only difference with Ref. 1 consists in replacing $T^{(-1)}_{\mu\nu}$ by $T^{(-4}_{\mu\nu}$, where the superscript notation (-4 indicates those terms in $T_{\mu\nu}$ of the fourth and higher order in the retarded distance.[2] The explicit expression for the tensor $T_{\mu\nu}$ may be obtained from the value of $F_{\mu\nu}$ given by Bhabha and Corben.[3] Integrating the tensors (3b) and (2b) over all three-space $\sigma^0(\tau)$ in the instantaneous rest frame of the particle at time τ, we obtain the bound momentum and bound angular momentum, namely,

$$P^{(b)}_{\mu} = \int_{\sigma^0(\tau)} T^{(b)}_{\mu\nu}\, d\Sigma^{\nu}\ , \tag{4a}$$

$$M^{(b)}_{\lambda\mu} = \int_{\sigma^0(\tau)} M^{(b)}_{\lambda\mu\nu}\, d\Sigma^{\nu}\ . \tag{4b}$$

To compute these integrals, we make use of the following method,[1] which is very easy to handle: We apply Gauss's integral theorem to the tensors $T^{(b)}_{\mu\nu}$ and $M^{(b)}_{\lambda\mu\nu}$ in the four-volume of Minkowski space bounded by the spacelike hyperplane $\sigma^0(\tau)$, a cylinder Σ_R with constant retarded radius R (Bhabha's tube), and the future light cone $C(\bar\tau)$ with vertex on the particle world line at the point $z(\bar\tau)$, with $\bar\tau < \tau$ (see Fig. 1). Considering first the tensor $T^{(b)}_{\mu\nu}$, it is easy to show that its divergence vanishes off the particle world line, therefore Gauss's theorem yields

$$P^{(b)}_{\mu} = \lim_{\bar\tau\to\tau} \int_{C(\bar\tau)} T^{(b)}_{\mu\nu}\, d\Sigma^{\nu} + \lim_{R\to\infty} \int_{\Sigma_R} T^{(b)}_{\mu\nu}\, d\Sigma^{\nu}\ . \tag{5}$$

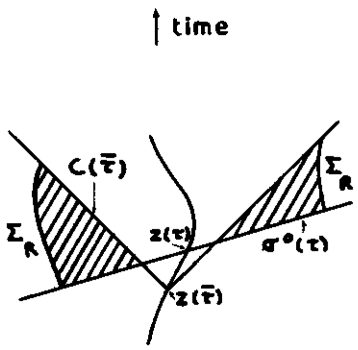

FIG. 1. Integration region considered in the evaluation of $P^{(b)}_{\mu}(\tau)$ and $M^{(b)}_{\lambda\mu}(\tau)$.

The second integral in (5) is evaluated over the tube Σ_R whose surface element is given by[5]

$$d\Sigma^{\nu} = [v^{\nu} - (1 - k\dot{v}R)k^{\nu}]R^2 d\Omega d\tau . \tag{6}$$

From this formula and taking into account the relation [see Ref. 2, Eq. (26)]

$$T^{(-3)}_{\mu\nu} k^{\nu} = 0 , \tag{7}$$

we obtain the result

$$T^{(b)}_{\mu\nu}\, d\Sigma^{\nu} = O(R^{-1}) \quad \text{as } R \to \infty . \tag{8}$$

Note that this behavior does not guarantee the vanishing of the integral over Σ_R in the limit $R \to \infty$. Since the total area of this tube behaves like $O(R)$ as $R \to \infty$, the integral tends to a finite value in this limit. Here we need to introduce the asymptotic condition of uniform motion in the remote past.[6] In fact, when $\dot{v} = 0$ the tensor $T^{(-3)}_{\mu\nu}$ identically vanishes and, furthermore, the surface element (6) reduces to

$$d\Sigma^{\nu} = (v^{\nu} - k^{\nu})R^2 d\Omega d\tau , \tag{9}$$

i.e., it grows only like $O(R^2)$. This implies that $T_{\mu\nu} d\Sigma^{\nu} = O(R^{-2})$, ensuring the vanishing of the second integral in Eq. (5). Hence, the bound momentum is given by

$$P^{(b)}_{\mu}(\tau) = \lim_{\bar{\tau} \to \tau} \int_{C(\bar{\tau})} T^{(b)}_{\mu\nu}\, d\Sigma^{\nu} . \tag{10}$$

From this result it follows that $P^{(b)}_{\mu}(\tau)$ is a "state function" of the particle[6] (it only depends on the present time τ). Indeed, in the limit $\bar{\tau} = \tau$, the integral (10) is evaluated over the future light cone with vertex at $z(\tau)$ and, since all quantities in $T^{(b)}_{\mu\nu}$ are retarded, only the present time τ contributes.

In a similar fashion we compute the bound angular momentum $M^{(b)}_{\mu\nu}(\tau)$. It is easy to verify that the tensor $M^{(b)}_{\lambda\mu\nu}$ is divergenceless off the particle world line. Besides, from the known relation [see Ref. 2, Eq. (46)]

$$T^{(-4)}_{\mu\nu} k^{\nu} \propto k_{\mu} , \tag{11}$$

it follows that

$$M^{(b)}_{\lambda\mu\nu}\, d\Sigma^{\nu} = O(R^{-1}) \text{ as } R \to \infty . \tag{12}$$

At this step, we introduce the asymptotic condition $\dot{v} = 0$ at $\tau \to -\infty$ to guarantee the vanishing of the integral over Bhabha's tube Σ_R in the limit $R \to \infty$. In fact, when $\tau \to -\infty$, $d\Sigma^{\nu} = O(R^2)$, so that there is no contribution to the bound angular momentum flux through Bhabha's tube of this portion of the world line. On the other hand, for all points lying at a finite distance, Eq. (12) implies the vanishing of the corresponding flux. Therefore, we can write

$$M^{(b)}_{\lambda\mu}(\tau) = \lim_{\bar{\tau} \to \tau} \int_{C(\bar{\tau})} M^{(b)}_{\lambda\mu\nu}\, d\Sigma^{\nu} . \tag{13}$$

On replacing in this equation the value of $M^{(b)}_{\lambda\mu\nu}$ given by Eq. (2b) we arrive at

$$M^{(b)}_{\lambda\mu}(\tau) = 2 z_{[\lambda} P^{(b)}_{\mu]}(\tau) + 2 \lim_{\bar{\tau} \to \tau} \int_{C(\bar{\tau})} s_{[\lambda} T^{(-4)}_{\mu]\nu}\, d\Sigma^{\nu} . \tag{14}$$

Note that the integral over the light cone $C(\bar{\tau})$ cannot vanish for a spinning particle because $T^{(-5)}_{\mu\nu} k^{\nu}$ is not proportional to k_{μ} (see Ref. 3).

The first term in Eq. (14), which is also present in the case of a nonspinning charged particle [see Eq. (1)], may be identified as the orbital bound angular momentum. On the other hand, the second term is a consequence of the internal electromagnetic structure of the point particle and, therefore, may be consistently interpreted as the bound spin angular momentum of a classical electron.

In practice, the explicit computation of the quantities given by Eqs. (10) and (14) is very easy to perform. The reason is that all points on the light cone $C(\bar{\tau})$ have the same retarded proper time $\bar{\tau}$.

As a last point we want to mention that Van Weert[2] has worked out expressions for $P^{(b)}_{\mu}(\tau)$ and $M^{(b)}_{\lambda\mu}(\tau)$ by applying Stokes's theorem to certain tensors whose divergences coincide with (2b) and (3b). However, as he referred the bound angular momentum to the present position of the particle, he only obtained the second term of Eq. (14).

[1] C. A. López and D. Villarroel, Phys. Rev. D 11, 2724 (1975). We use the notation of this reference.
[2] Ch. G. Van Weert, Physica 76, 345 (1974). This author exhibits a splitting which differs from ours in that the first term in Eq. (2a) is missing.
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