When Can A Classical Electron Accelerate Without Radiating?

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June 9, 1977

Abstract

Classical electron models with extended charge distributions can move and/or deform without radiating. Can a model be contrived that will undergo radiationless motion while accelerating (on the average) over a distance large compared to its size? The answer is no: we prove that the world-tube traced out in Minkowski space by the electron's charge distribution must enclose a straight time-like line, if the motion is radiationless.

I. Introduction

A point charge must radiate if it accelerates, but the same is not true of an extended charge distribution. A uniformly charged spherical shell of radius b can move in <u>arbitrary</u> periodic motion of period 2b/c without radiating[(1),(2)]; and there are other examples as well[(3)]. Although the individual pieces of charge comprising such models accelerate and emit radiation fields, their superposed radiation fields cancel in all directions.

In the known examples of radiationless motion, the "center" of the electron model oscillates about constant velocity motion over a distance only of the order of the size of the electron. This leads one to ask whether this must always be the case, or whether the charge distribution of an electron model can so deform while it moves that the "center" of the electron can accelerate over larger distances without the electron as a whole radiating. For example, might the classical nuclear atom be stable against radiative decay when the orbiting electrons are extended deformable charge distributions?

Alas, no: we prove here that the motion of the known examples is typical of the general case. For an electron model composed of charge of one sign, for which $\mathbf{j}=\rho\mathbf{v}(|\mathbf{v}|/c<1)$, we show that for radiationless motion, the "center" of the electron (suitably defined) strays no further than the electron "radius" (suitably defined) from a fictitious point undergoing constant velocity motion.

In § II, beginning briefly with an argument found else-where[3],[4], we develop a condition for a charge distribution not to radiate in the directon of the unit vector $\hat{x}$. After looking at two examples in § III, we show in § IV that if there is to be no radiation in any direction, the electron's world-tube in Minkowski space is required to contain a straight time-like line.

II. Condition For No Radiation

The asymptotic vector potential $A^\mu(x)$ in the Lorentz gauge, due to the current $j^\mu(x)$, is

$$A^\mu(x) \underset{r\to\infty}{\longrightarrow} r^{-1}\int d^4x' j^\mu(x')\delta(r-t-\tilde{x}\cdot x'), \tag{1}$$

where $\tilde{x}\equiv(1,\ \mathbf{x}/r)$, the scalar product $a\cdot b=-a^0b^0+\mathbf{a}\cdot\mathbf{b}$, $c\equiv1$, and in this section, Greek indices run from 0 to 3, Latin indices from 1 to 3 It follows that asymptotically

$$\partial_\mu A^\nu(x) \to r^{-1}\tilde{x}_\mu L^\nu(x), \tag{2a}$$

$$F^{\mu\nu} \to r^{-1}(\tilde{x}^\mu L^\nu - \tilde{x}^\nu L^\mu), \tag{2b}$$

$$T^{\mu\nu} \to (4\pi r^2)^{-1}\tilde{x}^\mu\tilde{x}^\nu L\cdot L, \tag{2c}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, $T^{\mu\nu}$ is the energy-momentum-stress density tensor, and L^μ is defined as

$$L^\mu(x)\equiv\int d^4x' j^\mu(x')\delta'(r-t-\tilde{x}\cdot x'). \tag{3}$$

If there is to be no radiation in the $\hat{\mathbf{x}}$ direction it follows from Eq.(2.c) that $L\cdot L=0$. The Lorentz gauge condition $\partial_\mu A^\mu=0$ implies that $\tilde{x}\cdot L=0$, according to Eq.(2.a). Since $\tilde{x}\cdot\tilde{x}=0$ (by definition), and since two orthogonal light-like vectors are parallel, a necessary (and easily seen to be sufficient) condition for no radiation in the $\hat{x}$-direction is

$$L^\mu(x)=\tilde{x}^\mu f(r-t,\hat{x}), \tag{4}$$

where f is an unspecified function of its arguments. Up to this point we have followed a previously published argument[4].

Upon setting $\mu=0$ in (4), we identify f with L^0, and rewrite the nontrivial 3-vector part of (4) as

$$\int d^4x' \{j^i(x')-\hat{x}^i j^0(x')\}\delta'(r-t-\tilde{x}\cdot x)=0. \tag{5}$$

The left side of Eq.(5) is, according to (2b), the asymptotic value of the i^{th} component of the electric field multiplied by r. Even if it did not vanish, we would expect it to be transverse, i.e. we would expect its scalar product with $\hat{x}$ to vanish:

$$\frac{\partial}{\partial t}\int d^4x'\{j^0(x')-\hat{x}\cdot j(x')\}\delta(r-t+t'-\hat{x}\cdot x')=0 \tag{6}$$

Eq.(6) can be verified directly, as a consequence of charge conservation $\partial_\mu j^\mu=0$:

$$\begin{aligned}\frac{\partial}{\partial t}\int d^4x'\hat{x}\cdot j'\delta &= \int d^4x' j'\cdot\nabla'\delta = -\int d^4x'(\nabla'\cdot j')\delta \\ &= \int d^4x'\partial_{t'} j^{0\prime}\delta = \frac{\partial}{\partial t}\int d^4x' j^{0\prime}\delta\end{aligned} \tag{7}$$

Using (6), we can rewrite Eq.(5) with j^0 eliminated:

$$\frac{\partial}{\partial t}(\delta^{ik}-\hat{x}^i\hat{x}^k)\int d^4x' j^k(x')\delta(r-t-\tilde{x}\cdot x')=0 \tag{8}$$

The integral in Eq.(8) can be further rewritten, using the identity

$$\int d^4x' j^k(x')\delta=\frac{\partial}{\partial t}\int d^4x' x'^k\{j^0(x')-\hat{x}\cdot j(x')\}\delta, \tag{9}$$

which follows (as did Eq.(6) from Eq.(7)) from charge conservation and integration by parts:

$$\begin{aligned}\frac{\partial}{\partial t}\int d^4x' x'^k \hat{x}\cdot j'\delta &= -\int d^4x'\{\nabla'\cdot(x'^k j')\}\delta \\ &= -\int d^4x'[j'^k - x'^k\partial_{t'} j'^0]\delta\end{aligned} \tag{10}$$

Putting Eq.(9) into Eq.(8) gives us our final expression for the condition of no radiation in the direction $\hat{x}$:

$$(\delta^{ik}-\hat{x}^i\hat{x}^k)\frac{\partial^2}{\partial t^2}z^k(t,\hat{x})=0 \tag{11a}$$

$$z^k(t,\hat{x})\equiv\int d^4x'\,x'^k[j^0(x')-\hat{x}\cdot j(x')]\,\delta(r-t-\hat{x}\cdot x') \tag{11b}$$

To complement Eqs.(11), we define

$$z^0(t,\hat{x})\equiv\int d^4x'\,t'\{j^0(x')-\hat{x}\cdot j(x')\}\delta(r-t-\tilde{x}\cdot x'), \tag{12a}$$

and verify that

$$\frac{\partial^2}{\partial t^2}\,\tilde{x}\cdot z(t,\hat{x})=0 \tag{12b}$$

which follows from taking the second derivative with respect to t of the identity

$$0=\int d^4x'(r-t-\tilde{x}\cdot x')\{-\tilde{x}\cdot j(x')\}\delta(r-t-\tilde{x}\cdot x'), \tag{13}$$

and use of Eq.(6). We now interpret Eqs.(11), (12).

If we regard $-\tilde{x}\cdot j(x')$ as an $\hat{x}$-dependent effective charge density ρ_{eff}, Eq.(6) says that the total effective charge, located where the plane $r-t+t'-\hat{x}\cdot x'$ intersects the electron's world-tube, is conserved as t changes. This is, to be sure, an unusual charge conservation law, since the charge is not evaluated at a constant time, nor is t the time coordinate of any element of the charge: t is simply a parameter which determines where the plane (which is the asymptotic approximation to the light cone surface whose apex is at r,t) slices the world-tube.

The four-vector $z^\mu(t,\hat{x})$ is, according to its definition (11b), (12a), the "center of charge" of ρ_{eff} along the plane $r-t-\tilde{x}\cdot x'$. In what follows, we shall restrict consideration to models which are composed of charge of one sign - for definiteness, let us say $j^0>0$ (if $j^0<0$, we can define z^μ as the negative of its present definition). We also shall restrict consideration to models for which $|\mathbf{j}|<j^0$, which includes the important case $\mathbf{j}=j^0\mathbf{v}$, so that

$$\rho_{eff}\equiv j^0(x')-\hat{\mathbf{x}}\cdot \mathbf{j}(x')\geq 0. \tag{14}$$

In order to discuss the "center" of the electron and its "radius," it is convenient to suppose that in each constant time hyperplane there is a sphere which completely encloses the electron's charge. The world-tube of the electron is hereafter defined as the volume in Minkowski space occupied by these spheres. For simplicity, we can take all spheres to have the same radius b, and require the center of the spheres to trace out a continuous time-like world line.

Now, because of the positive definite nature of the effective charge density (14), and the convex nature of the electron's world-tube, *$z^\mu(t,\hat{\mathbf{x}})$ traces out a world-line that lies wholly within the electron's world-tube, for any value of $\hat{\mathbf{x}}$*. This fact, together with the solution of the dynamical equations (11a), (12b)

$$z^\mu(t,\hat{\mathbf{x}})=\tilde{x}^\mu a(t,\hat{\mathbf{x}})+b^\mu(\hat{\mathbf{x}})t+c^\mu(\hat{\mathbf{x}}), \tag{15}$$

are all that we shall need in what follows. In Eq.(15), $a(t,\hat{\mathbf{x}})$ is an arbitrary function of its arguments, and $b^0=c^0=0$ can be taken with no loss of generality, while $\mathbf{b},\mathbf{c}$ are so far unrestricted vectors.

Eq.(15) states that the world-line $z^\mu(t,\hat{\mathbf{x}})$ lies in a 2-plane containing the four-vectors $\tilde{x}^\mu$, b^μ. If $a\sim t$, our task would be done, for $z^\mu(t,\hat{\mathbf{x}})$ would be a straight line lying within the world-tube. But $a\sim t$ may or may not be true, as examples testify (see the next section), so a good deal of further argument is necessary, based upon application of Eq.(15) to different directions $\hat{x}$ (§ IV).

III. Examples

Our first example of radiationless motion is a uniformly charged spherical shell whose center is at rest but whose radius is a function of time $\beta(t)$. The current density is

$$j^{\mu}=(4\pi\beta^2)^{-1}\delta(r-\beta(t))\{1,\hat{x}\dot{\beta}(t)\}. \tag{16}$$

Insertion of (16) into Eqs.(11b),(12a), and a straightforward integration yields

$$\{z^0,z\}=\frac{1}{4\pi}\int d\Omega'\{t_1,\hat{x}'\beta(t_1)\}, \tag{17}$$

where $t_1(t-r,\ \hat{x}\cdot\hat{x}')$ is the solution of

$$t_1=t-r+\hat{x}\cdot\hat{x}'\beta(t_1). \tag{18}$$

If the primed polar axis is chosen to point in the $\hat{x}$ direction, $\hat{x}\cdot\hat{x}'=\cos\theta'$ and the integration over the angle ϕ' can be performed, yielding

$$z^0=\frac{1}{2}\int_{-1}^{1}d\cos\theta' t_1 \tag{19a}$$

$$z=\hat{x}\frac{1}{2}\int_{-1}^{1}d\cos\theta'\cos\theta'\beta(t_1)=\hat{x}\left[-t+r+\frac{1}{2}\int_{-1}^{1}d\cos\theta' t_1\right] \tag{19b}$$

Eq. (18) has been used on the right side of Eq. (19b). It is clear that (19) is of the form of Eq. (15) with the identifications

$$a=\frac{1}{2}\int_{-1}^{1}d\cos\theta' t_1,\quad b=-\hat{x},\quad c=r\hat{x} \tag{20}$$

In this example, as in general, and although the world-line of z^{μ} is not straight, it is easy to see from the model's definition that the time axis is a straight line contained within the world tube.

Our last example is Schott's[1],[2] uniformly charged spherical shell of radius b, whose center at $\underset{\sim}{\zeta}(t)$ moves in arbitrary periodic motion with period 2b. In this case, the current density is

$$j^{\mu}=(4\pi b^2)^{-1}\delta(|\underset{\sim}{r}-\underset{\sim}{\zeta}(t)|-b)\{1,\dot{\underset{\sim}{\zeta}}(t)\}. \tag{21}$$

Insertion of (21) into Eqs. (11b), (12a), and integration as before yields

$$\{z^0,\underset{\sim}{z}\}=\tfrac{1}{2}\int_{-1}^{1}d\cos\theta'\{t_1,\underset{\sim}{\zeta}(t_1)\}, \tag{22}$$

where $t_1(t-r,\ \cos\theta',\ \underset{\sim}{\hat{x}})$ is the solution of

$$t_1=t-r+\cos\theta' b+\underset{\sim}{\hat{x}}\cdot\underset{\sim}{\zeta}(t_1). \tag{23}$$

It turns out that $\dot{\underset{\sim}{z}}$ is in fact independent of t. Using Eqs. (22), (23) we find

$$\frac{\partial}{\partial t}[z^0,\underset{\sim}{z}]=\frac{1}{2}\int_{-1}^{1}d\cos\theta'[1,\dot{\underset{\sim}{\zeta}}(t_1)](1-\underset{\sim}{\hat{x}}\cdot\dot{\underset{\sim}{\zeta}}(t_1))^{-1} \tag{24}$$

A change of integration variable from $\cos\theta'$ to t_1 using Eq.(23) results in

$$\frac{\partial}{\partial t}[z^0,\underset{\sim}{z}]=(2b)^{-1}\int_{t_-}^{t_+}dt_1[1,\dot{\underset{\sim}{\zeta}}(t_1)]=(2b)^{-1}[t_+-t_-,\underset{\sim}{\zeta}(t_+)-\underset{\sim}{\zeta}(t_-)] \tag{25}$$

where t_+ and t_- are given by

$$t_\pm=t-r\pm b+\underset{\sim}{\hat{x}}\cdot\underset{\sim}{\zeta}(t_\pm). \tag{26}$$

According to Eq.(26),

$$t_+-t_-=2b+\hat{x}\cdot\{\underset{\sim}{\zeta}(t_+)-\underset{\sim}{\zeta}(t-)\}, \tag{27}$$

so if the motion is periodic with period 2b, we see that $t_+-t_-=2b$, $\underset{\sim}{\zeta}(t_+)=\underset{\sim}{\zeta}(t_-)$, and

$$\frac{\partial}{\partial t}\{z^0,\underset{\sim}{z}\}=\{1,0\}. \tag{28}$$

Thus, from Eqs. (22), (23),(28) se see that z^{μ} is of the form of Eq.(15), with the identifications

$$a=t-r+\underset{\sim}{\hat{x}}\cdot\underset{\sim}{z},\ \ \underset{\sim}{b}=-\underset{\sim}{\hat{x}},\ \ \underset{\sim}{c}=\underset{\sim}{z}-\underset{\sim}{\hat{x}}\underset{\sim}{\hat{x}}\cdot\underset{\sim}{z}+r\underset{\sim}{\hat{x}}. \tag{29}$$

In this example, $z^{\mu}(t,\underset{\sim}{\hat{x}})$ itself is a straight line parallel to the time axis, and lying within the electron's world-tube, for any fixed direction $\hat{x}$.

Although in both examples, $b=-\hat{x}$, this is not generally true. A Lorentz-transform of these models is of course also radiationless, and leads to a straight line not parallel to the time axis threading the electron's world-tube, with a concomitant change in b.

IV. Completion of Analysis

Consider application of Eq.(15) to two different directions $\hat{x}_1$, $\hat{x}_2$ in each of which there is no radiation. We know that each world-line $z^\mu(t,\hat{x}_i)$ lies somewhere in a 2-plane, a typical point of which is given by

$$z^\mu{}_i=\tilde{x}^\mu_i\alpha_i+b^\mu_i\beta_i+c^\mu_i \qquad (i=1,2), \tag{30}$$

where $b^\mu_i\equiv b^\mu(\hat{x}_i)$, $c^\mu_i\equiv c^\mu(\hat{x}_i)$, and α_i, β_i are free parameters ranging from $-\infty$ to $+\infty$. Although we do not know the world-lines, we know that they lie within the world-tube, and therefore cannot be any farther distance than 2b apart, on a constant time hypersurface. On the constant time hypersurface $z^0_1=z^0_2=T$, we see from Eq.(30) that $\alpha_1=\alpha_2=T$, and the location of the i^{th} world-line at that instant lies somewhere on the <u>straight line</u> parallel to b_i:

$$z_i=\hat{x}_iT+b_i\beta_i+c_i \qquad (-\infty<\beta_i<\infty,\ i=1,2) \tag{31}$$

We shall show that the point on one such straight line that comes closest to another such straight line traces out a linear world-line with increasing T, and this world-line lies within the electron's world-tube.

We begin by finding the points of closest approach on the two straight lines (31), by minimizing the square of the distance vector $\underset{\sim}{D} \equiv \underset{\sim}{z}_1 - \underset{\sim}{z}_2$ with respect to the parameters β_1, β_2. The resulting equations for β_1, β_2 are

$$\begin{bmatrix} b_1^2, & -\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 \\ -\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2, & b_2^2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = -\begin{bmatrix} \underset{\sim}{b}_1 \cdot \underset{\sim}{a} \\ \underset{\sim}{b}_2 \cdot \underset{\sim}{a} \end{bmatrix}, \quad \left(\underset{\sim}{a} \equiv T(\hat{\underset{\sim}{x}}_1 - \hat{\underset{\sim}{x}}_2) + \underset{\sim}{c}_1 - \underset{\sim}{c}_2 \right) \qquad (32a,b)$$

We note that $\underset{\sim}{b}_i$ cannot vanish, for then Eq.(15) tells us that the four-velocity of the world-line $dz^\mu(t,\hat{\underset{\sim}{x}})/dz^0(t,\hat{\underset{\sim}{x}}) = \tilde{x}^\mu$ is light-like. This is not possible, as z^μ is the average position of charge elements with time-like four-velocities.

If $\underset{\sim}{b}_2$ is parallel to $\underset{\sim}{b}_1$, say $\underset{\sim}{b}_2 = \lambda \underset{\sim}{b}_1$, we of course find each point on one straight line to be equidistant from the closest point on the other straight line. Eq.(32) therefore gives us just one relation, $\beta_1 - \lambda\beta_2 = -b_1^{-2}\, \underset{\sim}{b}_1 \cdot \underset{\sim}{a}$. The distance vector between two closest points is then

$$\underset{\sim}{D}_{min.} = \underset{\sim}{a} - \hat{\underset{\sim}{b}}_1 \hat{\underset{\sim}{b}}_1 \cdot \underset{\sim}{a} \qquad (33)$$

Now $|\underset{\sim}{D}_{min.}|$ must be less than 2b for any T, else the two world-lines $z^\mu(t,\hat{\underset{\sim}{x}}_i)$ cannot lie within the electron's world tube. But since $\underset{\sim}{a}$ grows with T (see Eq.(32b)), $|\underset{\sim}{D}_{min.}|$ will increase without bound unless

$$\hat{\underset{\sim}{x}}_1 - \hat{\underset{\sim}{x}}_2 \sim \underset{\sim}{b}_1, \qquad (34)$$

i.e. $\hat{\underset{\sim}{x}}_2$ must lie in the plane containing $\hat{\underset{\sim}{x}}_1$ and $\underset{\sim}{b}_1$. So if we choose $\hat{\underset{\sim}{x}}_2$ outside of this plane, as we are free to do, and shall do hereafter, $\underset{\sim}{b}_2$ cannot be parallel to $\underset{\sim}{b}_1$. For such $\hat{\underset{\sim}{x}}_2$, Eqs (32) can be solved for β_1, β_2, since the closest points on two skew lines are uniquely defined.

We find, by putting the solution of Eq.(32a) into Eq.(31) that the point of closest approach on line 1 to line 2, which we shall call $\underset{\sim}{z}_{1(2)}$, is

$$\underset{\sim}{z}_{1(2)}=\hat{\underset{\sim}{x}}_1 T-\underset{\sim}{b}_1\underset{\sim}{P}_{1(2)}\cdot\underset{\sim}{a}+\underset{\sim}{c}_1, \tag{35}$$

where $\underset{\sim}{b}_1\underset{\sim}{P}_{1(2)}$ is the projection operator in the direction of $\underset{\sim}{b}_1$ in the skew coordinate system with axes parallel to $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_1 x \underset{\sim}{b}_2$:

$$\underset{\sim}{P}_{1(2)}\cdot\underset{\sim}{a}=|\underset{\sim}{b}_1 x\ \underset{\sim}{b}_2|^{-2}\{b_2^2\underset{\sim}{b}_1\cdot\underset{\sim}{a}-\underset{\sim}{b}_1\cdot\underset{\sim}{b}_2\underset{\sim}{b}_2\cdot\underset{\sim}{a}\}, \tag{36}$$

($\underset{\sim}{P}_{1(2)}\cdot\underset{\sim}{b}_1=1$, $\underset{\sim}{P}_{1(2)}\cdot\underset{\sim}{b}_2=0$). We note that $z^{\mu}_{1(2)}(T)\equiv(T,\underset{\sim}{z}_{1(2)}(T))$ <u>describes a straight world-line</u>, as it has a linear dependence upon T, according to Eq.(35).

The minimum distance vector $\underset{\sim}{D}_{min.}\equiv\underset{\sim}{z}_{1(2)}-\underset{\sim}{z}_{2(1)}$ is, by Eq.(35),

$$\underset{\sim}{D}_{min}=\underset{\sim}{a}-\underset{\sim}{\mathbb{P}}_{12}\cdot\underset{\sim}{a}, \tag{37}$$

where $\underset{\sim}{\mathbb{P}}_{12}\equiv\underset{\sim}{b}_1\underset{\sim}{P}_{1(2)}+\underset{\sim}{b}_2\underset{\sim}{P}_{2(1)}$ is the projection operator on the plane containing $\underset{\sim}{b}_1$, $\underset{\sim}{b}_2$. Again we argue that $|\underset{\sim}{D}_{min}|$ must be less than 2b for any T, so by Eq.(37) and the definition (32b) of $\underset{\sim}{a}$, $\hat{\underset{\sim}{x}}_1-\hat{\underset{\sim}{x}}_2$ <u>must lie in the plane containing $\underset{\sim}{b}_1$, $\underset{\sim}{b}_2$</u>. When this is so, $\underset{\sim}{D}_{min.}$ is a constant vector (independent of T), and <u>$z^{\mu}_{1(2)}(T)$ is parallel to $z^{\mu}_{2(1)}(T)$</u>.

Indeed, the $z^{\mu}_{i(j)}(T)$ constructed in this way must <u>all</u> be parallel world-lines. For, we argue that the world-line of the closest approach point on line 1 to line 2 must be parallel to the world-line of the closest approach point on line 1 to line 3. The acutal world-line $z^{\mu}(t,\hat{\underset{\sim}{x}}_1)$ must lie a finite distance from any closest approach point on line 1. (This is because it must be within a distance 2b of any other world-line which intersects line 2. Since line 2 is skew to line 1, if the world line $z^{\mu}(t,\hat{x}_1)$ gets too far along line 1 from the closest approach point $z_{1(2)}$ on line 1, it will be further than 2b from any point on line 2.) But the actual world-line $z^{\mu}(t,\hat{x}_1)$ cannot be a finite distance from two different closest approach points on line 1 that separate without limit as T increases. Thus the points cannot separate as T increases which means that $z^{\mu}_{1(2)}(T)$ is parallel to $z^{\mu}_{1(3)}(T)$.

Since $z^{\mu}_{i(j)}$ is parallel to $z^{\mu}_{i(k)}$, and since we have previously shown that $z^{\mu}_{i(j)}$ is parallel to $z^{\mu}_{j(i)}$, all $z^{\mu}_{i(j)}$'s are parallel.

Thus the tangent vectors $v \equiv dz_{i(j)}/dT$ calculated from Eq.(35) are all equal:

$$v = \hat{x}_i - \lambda_i b_i \qquad (\text{arbitrary } \hat{x}_i), \tag{36}$$

where $\lambda_i \equiv p_{i(j)} \cdot (\hat{x}_i - \hat{x}_j)$ is actually independent of $\hat{x}_j$. Furthermore, $|v|<1$, for if this was not so, the world-line $z^{\mu}(t,\hat{x}_i)$ could not remain a finite distance from a light-like or space-like line and still be a time-like world-line. But $|v| \leq 1$ is precisely the condition that it be possible to choose the directions $\hat{x}_i$, $\hat{x}_j$ so that b_i is orthogonal to b_j, since upon solving

$$0=(v-\hat{x}_i)\cdot(v-\hat{x}_j)=v^2 - v\,\hat{v}\cdot(\hat{x}_i-\hat{x}_j)+\hat{x}_i\cdot\hat{x}_j \tag{37}$$

for v, we find $v \leq \frac{1}{2}|\hat{x}_i+\hat{x}_j| \leq 1$.

Suppose we choose two directions $\hat{x}_1$, $\hat{x}_2$ so that $b_1 \cdot b_2=0$. The actual world-lines $z^{\mu}(t,\hat{x}_1)$, $z^{\mu}(t, \hat{x}_2)$ are in the electron's world-tube, so each can be no farther apart at time T than the distance b from the location of the center of the electron at time T. Furthermore, each world-line at time T lies on its respective straight line (31), where we have chosen the two straight lines to be orthogonal. A moments consideration of the geometry will convince one that the center of the electron can lie no farther than the distance b from the point of closest approach to either straight line. For example, if the two orthogonal straight lines actually intersect, an imaginary rod of length 2b whose ends (representing the world-line locations) can slide along the two straight lines always has its center a distance b from the intersection of the two lines.

Thus, the center of the electron at any time T is no farther than the distance b from any closest approach point $\underset{\sim}{z}_{i(j)}$, which means that the electron's world-tube must contain the time-like straight world-line $z^{\mu}{}_{i(j)}$. This concludes our proof that radiationless motion of a classical electron model must be "close" to constant velocity motion.

References

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