Simplified calculations for radiation reaction forces

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The Lorentz–Dirac equation of motion for the electron is derived by a new method which makes tedious power series expansions unnecessary.

1. INTRODUCTION

The Lorentz–Dirac equation of motion[1] for the electron has been treated extensively in the literature.[2] This diversity is mainly due to the fact that Maxwell equations do not determine the radiation reaction forces without ambiguity, essentially because of the divergent electron self-energy. However, the general view seems to be that Lorentz–Dirac equation describes correctly the electron dynamics. The standard derivations of these equation are rather long and tedious in spite of its simplicity.

We intend to show that radiation reaction calculations can be done very simply if appropriate geometrical objects are used in Minkowski space. A new cutoff prescription in dealing with the electron infinite self-energy based on light cones is introduced. We follow the approach of Rohrlich[3] and Teiltelboim,[4] where the total four-momentum P_μ of the electromagnetic field is the main object under study. The splitting of P_μ into its bound and radiation parts will emerge naturally from our calculation. The Lorentz–Dirac equation of motion is obtained by equating the time derivative of P_μ to the driving external force.

We now describe the way the calculation is done leaving the details for the next section: The energy–momentum tensor constructed from Lienard–Wiechert retarded potentials is[3,5]

$$T_{\mu\nu}=(e^2/4\pi)[-\kappa^{-2}((k\dot{v})^2+\dot{v}^2)k_\mu k_\nu+\kappa^{-3}(2(k\dot{v})k_\mu k_\nu -(k\dot{v})(k_\mu v_\nu+k_\nu v_\mu) +k_\mu\dot{v}_\nu+k_\nu\dot{v}_\mu)+\kappa^{-4}(-k_\mu k_\nu+k_\mu v_\nu+k_\nu v_\mu-\tfrac{1}{2}\eta_{\mu\nu})]. \qquad (1.1)$$

The following notation has been used: the x^μ are the Minkowski coordinates, $x^\mu=z^\mu(\tau)$ is the electron world line (EWL) parametrized by its proper time τ, $v^\mu=dz^\mu/d\tau=\dot{z}^\mu$, $\kappa=v_\mu(x^\mu-z^\mu(\tau))$, where τ is the retarded proper time of event x^μ, and $k^\mu=\kappa^{-1}(x^\mu-z^\mu(\tau))$ a light vector. The signature of the Minkowski tensor is -2. The velocity of light is chosen to be 1.

Following the standard procedure of field theory, we define the total four-momentum of the electromagnetic field by

$$P^\mu=\int_\sigma T^{\mu\nu}d\sigma_\nu. \qquad (1.2)$$

In Teitelboim's work[4] σ is a spacelike hyperplane that cuts the EWL orthogonally. We shall assume σ to be an arbitrary spacelike hyperplane. Such a restriction is unimportant, since the value of P^μ is independent of the detailed shape of σ outside some finite neighborhood of the EWL, as Gauss' integral theorem applied to $T^{\mu\nu}{}_{,\nu}=0$ shows. As it is well known, P^μ as defined by Eq. (1.2) is divergent; therefore, some cutoff prescription is necessary (a detailed study of the dependence of P^μ on the cutoff is given in Ref. 6). We shall use the following one: Let σ cut the EWL at proper time $\bar{\tau}$, pick any $\tau<\bar{\tau}$, and draw the future light cone C emerging from $z^\mu(\tau)$, as shown in Fig. 1. The portion of σ within C is a three-sphere as seen by an observer with four-velocity u^μ orthogonal to σ. We denote this sphere by Sp. Integral (1.2) is performed on the domain σ-Sp only, and thus it is finite. The three-dimensional picture of Sp is that of a sphere of light emited continuously from τ to $\bar{\tau}$. Within Sp, when $\tau\to\bar{\tau}$, it is assumed that the contribution to the total four-momentum is m_0v^μ, where m_0 is the bare mass, which, when added to the electromagnetic mass, gives the observed electron mass m. We therefore write

$$P^\mu=m_0v^\mu+\int_{\sigma\text{-Sp}}T^{\mu\nu}d\sigma_\nu. \qquad (1.3)$$

This last integral splits naturally into two pieces, P^μ_B and P^μ_R, which require separate computation. In order

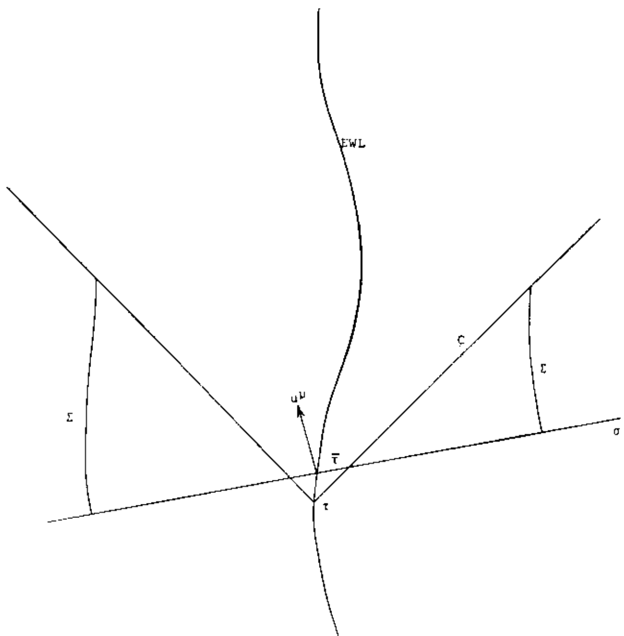

FIG. 1. The world diagram describing various hypersurfaces used in evaluating the electron four-momentum: σ is a spacelike hyperplane cutting the electron world line (EWL) at proper time $\bar{\tau}$; Σ is a time like tube surrounding the EWL; C is the future light cone with vertex at $z^\mu(\tau)$.

to exhibit this splitting we surround the EWL by a timelike tube Σ (see Fig. 1) which eventually tends to spatial infinity. In applying Gauss' integral theorem on $T^{\mu\nu}{}_{,\nu}$ in the region bounded by Σ, σ and C we get

$$P^\mu = m_0 v^\mu(\bar\tau) + P^\mu_B + P^\mu_R, \tag{1.4}$$

where

$$P^\mu_B = \int_C T^{\mu\nu} dC_\nu \tag{1.5}$$

and

$$P^\mu_R = -\int_\Sigma T^{\mu\nu} d\Sigma_\nu. \tag{1.6}$$

The integral P^μ_B over the light cone C is performed up to its intersection with σ. The second integral P^μ_R is performed over any timelike tube Σ that tends to spatial infinity, from the infinite past up to its intersection with C.

P^μ_B is identified with the four-momentum bound to the electron since it depends on the kinematical variables of the EWL at $\tau = \bar\tau$ only,[4] in the limit $\tau \to \bar\tau$. On the other hand, P^μ_R depends on the entire electron history up to τ, and is calculated out of values of $T^{\mu\nu}$ at spatial infinity. It is identified with the total radiated four-momentum up to proper time τ.

We apply this method to the electron angular momentum obtaining expressions for the bound and radiated parts.[7] Similar calculations are done for the scalar field.

2. VECTOR FIELD

In calculating the bound four-momentum we need the light-cone volume element given by[8]

$$dC^\mu = \kappa^2 d\kappa\, d\Omega k^\mu, \tag{2.1}$$

where $d\Omega$ is the solid angle seen by an observer at rest with the electron. Equation (1.1) yields the following relation:

$$T^{\mu\nu}k_\nu = (e^2/8\pi)\kappa^{-4}k^\mu, \tag{2.2}$$

and therefore

$$P^\mu_B = \frac{e^2}{8\pi}\int_{\kappa=\kappa_0}^{\infty} d\kappa\, d\Omega \kappa^{-2} k^\mu = \frac{e^2}{8\pi}\int d\Omega \kappa_0^{-1} k^\mu, \tag{2.3}$$

where κ_0 denotes the values of κ at the intersection of C and σ, which is obtained from Fig. 1 by noting that $\{\kappa_0 k^\mu - [z^\mu(\bar\tau) - z^\mu(\tau)]\}u_\mu = 0$, that is $\kappa_0^{-1} = uk/u(\bar z - z)$. P^μ_B is now given by

$$P^\mu_B = \frac{(e^2/8\pi)}{u(\bar z - z)}\int d\Omega (uk)k^\mu. \tag{2.4}$$

This expression is easily integrated to give

$$P^\mu_B = \tfrac{2}{3} e^2 \frac{(uv)v^\mu - \frac{1}{4}u^\mu}{u[z(\bar\tau) - z(\tau)]}. \tag{2.5}$$

The leading divergent term of P^μ_B when $\tau \to \bar\tau$ is

$$P^\mu_B = \tfrac{2}{3} e^2 \epsilon^{-1}\{v^\mu(\bar\tau) - u^\mu/4[uv(\bar\tau)]\}, \tag{2.6}$$

where $\epsilon = \bar\tau - \tau$. This self-energy divergence cannot be incorporated to m_0 in Eq. (1.4) unless $u^\mu = v^\mu(\bar\tau)$, that is, unless σ cuts the EWL orthogonally.[9] In order to renormalize the divergence, we assume $u^\mu = v^\mu(\bar\tau)$. In this case we get[4]

$$P^\mu_B = \frac{e^2}{2\epsilon} v^\mu(\bar\tau) - \tfrac{2}{3} e^2 \dot v^\mu(\bar\tau) + O(\epsilon). \tag{2.7}$$

We now consider the radiated four-momentum P^μ_R defined by Eq. (1.6). The exact shape of the tube Σ is not important as can be seen from $T^{\mu\nu}{}_{,\nu} = 0$. We choose a tube used by Bhabha,[10] defined by the equation κ = const, its volume element is given by[10]

$$d\Sigma_\mu = \{[1 - \kappa(k\dot v)]k_\mu - v_\mu\}\kappa^2 d\tau\, d\Omega. \tag{2.8}$$

From here and Eq. (1.1) we get

$$T_{\mu\nu} d\Sigma^\nu = (e^2/4\pi)[(k\dot v)^2 + \dot v^2]k_\mu d\tau\, d\Omega, \tag{2.9}$$

in the limit $\kappa \to \infty$. A simple computation of angular integrals leads to

$$P^\mu_R = -\tfrac{2}{3} e^2 \int_{-\infty}^{\tau} \dot v^2 v^\mu d\tau, \tag{2.10}$$

which is the well-known Larmor radiation formula. The convergence of this integral requires that the acceleration vanishes at $\tau = -\infty$.

The Lorentz–Dirac equation is obtained by combining Eqs. (1.4), (2.7), and (2.10) to get

$$m\dot v_\mu - \tfrac{2}{3} e^2(\ddot v_\mu + \dot v^2 v_\mu) = F^{ex}_\mu, \tag{2.11}$$

where F^{ex}_μ is the external driving force.

The calculation of the bound and radiated angular momentum is equally simple. The total angular-momentum density is as usual given by

$$M^{\lambda\mu\nu} = x^\lambda T^{\mu\nu} - x^\mu T^{\lambda\nu}. \tag{2.12}$$

The angular momentum $M^{\lambda\mu}$ of the electron field is given by

$$M^{\lambda\mu} = \int_\sigma M^{\lambda\mu\nu} d\sigma_\nu, \tag{2.13}$$

where, as for P^μ, σ is the hyperplane that cuts orthogonally the EWL at $z(\bar\tau)$. The angular momentum (2.13) is split in its bound ($M^{\lambda\mu}_B$) and radiated ($M^{\lambda\mu}_R$) parts, in exactly the same way as the four-momentum. The following two formulas are needed:

$$M^{\lambda\mu\nu} dC_\nu = z^{[\lambda}k^{\mu]}\kappa^{-2} d\kappa\, d\Omega \tag{2.14}$$

and

$$M^{\lambda\mu\nu} d\Sigma_\nu = 2\{-(k\dot v)k^{[\lambda}v^{\mu]} + k^{[\lambda}\dot v^{\mu]} + [(k\dot v)^2 + \dot v^2]k^{[\lambda}z^{\mu]}\}, \tag{2.15}$$

where the limit $\kappa \to \infty$ has been taken in the second equation. The notation $a^{[\mu}b^{\nu]} = \frac{1}{2}(a^\mu b^\nu - a^\nu b^\mu)$ has been used.

From Eq. (2.14) we see that

$$M^{\lambda\mu}_B = z^\lambda P^\mu_B - z^\mu P^\lambda_B, \tag{2.16}$$

which is the result one expects for material particles. From Eq. (2.15), after angular integrations are performed, we get

$$M^{\lambda\mu}_R = -\tfrac{4}{3} e^2 \int_{-\infty}^{\tau} \dot v^2 z^{[\lambda}v^{\mu]} d\tau + \tfrac{4}{3} e^2 \int_{-\infty}^{\tau} v^{[\lambda}\dot v^{\mu]} d\tau. \tag{2.17}$$

This result was obtained by López and Villarroel.[7]

3. SCALAR FIELD

In this section we apply the same technique to the massless scalar field of a point source. The field equation to be solved is

$$\Box \Phi = 4\pi g \int_{-\infty}^{+\infty} \delta^4(x - z(\tau))\, d\tau. \quad (3.1)$$

The retarded solution to this equation is known to be $\Phi = g\kappa^{-1}$. The energy-momentum tensor is given by

$$4\pi T_{\mu\nu} = \Phi_{,\mu}\Phi_{,\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Phi_{,\alpha}\Phi_{,}{}^{\alpha}, \quad (3.2)$$

which gives

$$T_{\mu\nu} = \frac{g^2}{4\pi}\left[\kappa^{-2}(k\dot{v})^2 k_\mu k_\nu + \kappa^{-3}(-2k_\mu k_\nu + \dot{v}_\mu k_\nu + \dot{v}_\nu k_\mu - \eta_{\mu\nu}) + \kappa^{-4}(k_\mu k_\nu - v_\mu k_\nu - v_\nu k_\mu + v_\mu v_\nu + \tfrac{1}{2}\eta_{\mu\nu})\right], \quad (3.3)$$

when $\Phi = g\kappa^{-1}$. From this expression we obtain

$$T^{\mu\nu} dC_\nu = (v^\mu - \tfrac{1}{2}k^\mu)\kappa^{-2}\, d\kappa\, d\Omega \quad (3.4)$$

and

$$T^{\mu\nu} d\Sigma_\nu = (k\dot{v})^2 k^\mu\, d\tau\, d\Omega, \quad \text{as } \kappa \to \infty, \quad (3.5)$$

which, after angular integrations are performed, gives

$$P_B^\mu = (g^2/2\epsilon)v^\mu - (g^2/3)\dot{v}^\mu + O(\epsilon) \quad (3.6)$$

and

$$P_R^\mu = -\tfrac{1}{3}g^2 \int_{-\infty}^{\tau} \dot{v}^2 v_\mu\, d\tau. \quad (3.7)$$

The equation of motion that follows from (3.6) and (3.7), after mass renormalization, is

$$m\dot{v}_\mu - \tfrac{1}{3}g^2(\ddot{v}_\mu + \dot{v}^2 v_\mu) = F_\mu^{\text{ex}}. \quad (3.8)$$

The bound angular momentum is given by the same expression (2.16) as for the electron. The radiated angular momentum that results is obtained from (2.17) by replacing $\frac{2}{3}e^2$ by $\frac{1}{3}g^2$.

[1]P. A. M. Dirac, Proc. Roy. Soc. London A 167, 148 (1938).

[2]F. Rohrlich, in *The Physicist's Conception of Nature*, edited by T. Mehra (Reidel, Dordrecht, Holland, 1973), pp. 331–69.

[3]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[4]C. Teitelboim, Phys. Rev. D 1, 1572 (1970).

[5]A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Macmillan, New York, 1964).

[6]R. Tabensky, "Electrodynamics and the electron equation of motion," to be published.

[7]C. A. López and D. Villarroel, "Bound and emitted angular momenta of a classical accelerated point charge," to be published in Phys. Rev. D.

[8]J. L. Synge, Ann. Mat. 84, 33 (1970).

[9]R. Tabensky and D. Villarroel, "Classical electron mass renormalization," to be published.

[10]H. J. Bhabha, Proc. Roy. Soc. London A 172, 384 (1939).