PHYSICAL REVIEW D VOLUME 11, NUMBER 10 15 MAY 1975

Larmor formula in curved spaces

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(Received 4 June 1974; revised manuscript received 6 February 1975)

The energy-momentum tensor of the field produced by an electron with arbitrary movement in a curved space contains a tensor $T^r_{\mu\nu}$ with the following properties: (i) vanishing covariant divergence, (ii) null flux across the cones that come out from the world line of the electron, and (iii) positive-definite diagonal elements of $T^r_{\mu\nu}$. These properties make it possible to express the differential conservation law in an integral form over a two-dimensional surface contained in the cone with apex in a fixed point of the world line of the electron. This integral gives a measure of the energy irreversibly emitted by the electron associated to the tensor $T^r_{\mu\nu}$. The corresponding rate of radiation is given by $\frac{2}{3}e^2\ddot{z}^2$, where $\ddot{z}^2$ is the square of the covariant acceleration.

I. INTRODUCTION

DeWitt and Brehme[1] and Hobbs[2] generalized to curved spaces the well-known Lorentz-Dirac equation[3] for flat space. From the DeWitt-Brehme-Hobbs equations it follows that, even in the absence of an incoming electromagnetic field acting on the electron, the electron has a nonvanishing covariant acceleration, that is, the trajectory is not a geodesic, in contrast with the flat-space case. These equations, however, shed no light on the important problem of knowing when and how much the electron radiates when moving in a curved space. According to many authors[4,5] it seems extremely difficult to get from the equation of motion any clear information as to the value of the radiated energy. In the flat-space case Rohrlich[6] has established a Lorentz-invariant criterion (independent from the Lorentz-Dirac equation) that solves the above-mentioned problem, namely, the electron radiates if and only if its acceleration is not zero.

In a previous paper[7] (to which we shall refer as A) we have shown that the energy-momentum tensor for the electron field has a part $T^r_{\mu\nu}$ that we call the radiation tensor and that it has the following properties:

(i) $T^r_{\mu\nu}$ has null flux across the light cones with apex on the electron world line.

(ii) $T^r_{\mu\nu}$ satisfies the (covariant) conservation law

$$T^r_{\mu\nu;}{}^{\nu}=0 \tag{1.1}$$

off the world line.

$$\text{(iii)}\quad T^r_{\mu\mu}\geqslant 0 \quad (\text{no sum})\,. \tag{1.2}$$

We will show that the tensor $T^r_{\mu\nu}$ describes energy that is irreversibly emitted by the electron in the form of radiation, escaping to infinity. In particular the properties (i) and (ii) allow us to express the differential conservation law (1.1) in an integral form over an arbitrary two-dimensional closed surface contained in a light cone with apex on the world line. This integral conservation law allows us to evaluate in a covariant way the radiated energy flux associated to the tensor $T^r_{\mu\nu}$, which is

$$R=\tfrac{2}{3}e^2\ddot{z}^2\geqslant 0\,, \tag{1.3}$$

where $\ddot{z}^\alpha$ is the covariant acceleration of the electron, that is,

$$\ddot{z}^\alpha=\frac{d\dot{z}^\alpha}{d\tau}+\Gamma_{\beta\gamma}{}^{\alpha}\dot{z}^\beta\dot{z}^\gamma,\quad \dot{z}^\alpha=\frac{dz^\alpha}{d\tau}\,. \tag{1.4}$$

The scalar (1.3) becomes the well-known Larmor formula in the flat-space case.

In a flat space $T^r_{\mu\nu}$ describes, of course, all the energy radiated by the electron,[8] but in a curved space the "tail" of the solution of the Maxwell equations[1] introduces complications. Nevertheless, there are curved spaces where the tail vanishes.[9] For such spaces $T^r_{\mu\nu}$ obviously describes the whole of the radiated energy.[7] For them, then, we can generalize Rohrlich's criterion, since from (1.3) we can see that the electron radiates if and only if its covariant acceleration does not vanish. Probably there is a great variety of Riemann spaces where all the energy radiated by the electron is described by $T^r_{\mu\nu}$, but we will not attempt any analysis of this problem here.

We will be using the same symbols, notation, and conventions as DeWitt and Brehme,[1] and in the following we will refer to Ref. 1 simply as DB. Throughout the present paper we will assume that the metric of our Riemann space has the properties mentioned in Sec. II of A. From Eqs. (2.28) and (2.29) of A we have that the radiation tensor for an electron is given by

$$T^r_{\mu\nu}=\frac{e^2}{4\pi}g^{1/2}\Delta\sigma_{;\mu}\sigma_{;\nu}(\ddot{z}^2-\kappa'^2\kappa^{-2})\kappa^{-4}\,. \tag{1.5}$$

All the terms used in this equation are defined in A. The point z that appears as one of the arguments of the world function $\sigma(x, z)$ is the retarded point associated with the point x. Following DB we associate the indices α through κ of the Greek alphabet to the point z, while the indices λ through ω are always to be associated with the point x.

As we will see in Sec. IV, one can compute the radiated energy associated with $T^r_{\mu\nu}$ without having to study its asymptotic behavior at large distances from the electron. With this aim, in Sec. II we give a formulation of Larmor's formula in flat spaces making use of a basic property that follows from the radiation tensor. In order to formulate this fundamental property in curved spaces, we generalize in Sec. III the world tube used by Bhabha[10] about 30 years ago in the flat-space case. With the help of this tube we establish in Sec. IV the generalized Larmor invariant associated with the tensor $T^r_{\mu\nu}$. The result does not depend on the tube used, but the generalization of the Bhabha tube is highly convenient for computational reasons.

II. DISCUSSION OF LARMOR'S FORMULA IN FLAT SPACES

We will briefly discuss in this section the derivation of Larmor's formula in flat spaces, looking for a fundamental formulation of it so as to be able to transcribe it to curved spaces. Let $z_\mu(\tau)$ be the world line of the source; τ is its proper time. We already know that the radiation tensor of a point source of massless scalar, vector, or tensor fields and of an arbitrary multipolar structure can be written in the form[11]

$$T^r_{\mu\nu} = \frac{1}{4\pi} s_\mu s_\nu \rho^{-4} Q, \tag{2.1}$$

where

$$s_\mu = x_\mu - z_\mu, \quad \rho = (x_\mu - z_\mu)v^\mu, \quad v_\mu = dz_\mu/d\tau .$$

z_μ is the retarded point associated to the point x_μ and Q is a Lorentz scalar that depends on the particular field that we are dealing with (scalar, vector, or tensor) and the multipolar structure of the source confined to the world line $z_\mu(\tau)$. The tensor (2.1) satisfies the properties (i), (ii), and (iii) mentioned in Sec. I (specialized, of course, to flat space).

Let $z_\mu(\tau^*)$ and $z_\mu(\tau)$, with $\tau > \tau^*$, be two points on the world line of the source, and let us consider now the light cones drawn from these points into the future. In addition we consider two timelike world tubes Σ_1 and Σ_2 of arbitrary radii $\epsilon_1 > 0$ and $\epsilon_2 > 0$ (defined in a Lorentz-invariant way) that surround the world line of the source and do not intercept each other. These tubes and cones define a volume Λ, and then because of property (ii) of Sec. I we have that in Λ the following relation is valid:

$$\int_\Lambda \partial^\nu T^r_{\mu\nu} d^4x = 0 . \tag{2.2}$$

If $\Delta\Sigma_1$ and $\Delta\Sigma_2$ denote the part of the tubes limited by the two cones mentioned above, then with the help of Gauss's theorem and property (i) of Sec. I we obtain from Eq. (2.2) that

$$\int_{\Delta\Sigma_1} T^r_{\mu\nu} d\Sigma^\nu_1 = \int_{\Delta\Sigma_2} T^r_{\mu\nu} d\Sigma^\nu_2 . \tag{2.3}$$

This equation is valid for arbitrary tubes Σ_1 and Σ_2. Its physical interpretation is rather obvious: It expresses the fact that the radiation emitted between the points $z(\tau^*)$ and $z(\tau)$ becomes independent from the source as soon as it is emitted and it propagates along the cones towards infinity.

Equation (2.3), however, is not valid in curved spaces, and for this reason we are going to obtain more basic information contained in this equation. To this end we choose the tubes Σ_1 and Σ_2 to be Bhabha tubes.[10] A Bhabha tube is defined by means of

$$\begin{aligned} s_\mu v^\mu &= \rho = \epsilon \quad (\epsilon > 0 \text{ fixed}), \\ s_\mu s^\mu &= 0 . \end{aligned} \tag{2.4}$$

The corresponding surface element is given by

$$d\Sigma_\nu = [\epsilon v_\nu - (1 - \kappa')s_\nu]\epsilon \, d\Omega \, d\tau . \tag{2.5}$$

All the quantities in the right-hand side are evaluated at the retarded point associated with x, $\kappa' = s_\mu \dot{v}^\mu$, and $d\Omega$ represents an element of solid angle in the rest frame of the source. Introducing Eqs. (2.1) and (2.5) in Eq. (2.3) we obtain

$$\int_{\tau^*}^{\tau} d\tau' \left(\frac{1}{4\pi\epsilon_1} \int s_\mu Q \, d\Omega\right) = \int_{\tau^*}^{\tau} d\tau' \left(\frac{1}{4\pi\epsilon_2} \int s_\mu Q \, d\Omega\right) . \tag{2.6}$$

Thus, the integral over Σ_1, for example, has been reduced to an integral over the two-dimensional surface defined by the intersection of the cone with apex in $z(\tau')$ and the tube Σ_1 followed by an integral over the retarded time.

From (2.6) we obtain, deriving with respect to τ and evaluating this derivative at τ^*, that

$$\frac{1}{4\pi\epsilon_1} \int s_\mu Q \, d\Omega = \frac{1}{4\pi\epsilon_2} \int s_\mu Q \, d\Omega . \tag{2.7}$$

This equation is valid for arbitrary ϵ_1 and ϵ_2; therefore, the integral

$$\frac{1}{4\pi\epsilon} \int s_\mu Q \, d\Omega \tag{2.8}$$

gives the same value for any two-dimensional closed surface lying inside the cone and determined by the intersection of the two three-dimensional surfaces

$$v_\mu s^\mu = \epsilon, \quad s_\mu s^\mu = 0 \tag{2.9}$$

with arbitrary ϵ. The point z is, of course, fixed in these equations.

We have thus shown that the integral (2.8) does not depend on ϵ using the particular tube defined by Eqs. (2.4). But it is easy to see that the form of the tube does not matter. In fact, let us assume that Σ_1 is the Bhabha tube (2.4) and that Σ_2 is an arbitrary tube. It is obvious that the integral over the surface $\Delta\Sigma_2$ in the right-hand side of Eq. (2.3) can be performed as an integration over the two-dimensional surface determined by the intersection of Σ_2 with the cone drawn from $z(\tau')$ into the future, followed by an integration over τ'. We can then recover in this way a fundamental equation of type (2.7) which will now be written as

$$\frac{1}{4\pi\epsilon}\int s_\mu Q\, d\Omega = \frac{1}{4\pi}\int s_\mu Q \mathfrak{a}\, d\Omega \ . \tag{2.10}$$

The scalar $\mathfrak{a}$ depends on the type of surface Σ_2; if we change it to Σ_2' the scalar $\mathfrak{a}$ changes, but the integral does not since it is again equal to the left-hand side of Eq. (2.10). We conclude, therefore, that the four-vector defined by Eq. (2.8) represents a fundamental property associated with the radiation emitted in $z(\tau)$ and that it propagates along the corresponding light cone.[12] To see more clearly its physical meaning let us evaluate from Eq. (1.5) (in the case of the flat space) the Lorentz scalar

$$R = v^\mu \left(\frac{1}{4\pi\epsilon}\int s_\mu Q\, d\Omega\right) \ . \tag{2.11}$$

A trivial computation shows that R is nothing else but Larmor's invariant,

$$R = \tfrac{2}{3} e^2 \dot{v}^2, \quad \dot{v}^\mu = \frac{d^2 z^\mu}{d\tau^2} \ , \tag{2.12}$$

which measures the rate of radiation emitted.

We will prove in Sec. IV that an equation analogous to (2.10) is valid in curved spaces. To this effect it is convenient to generalize the tube (2.4) to a curved space. Such a generalization is made in Sec. III.

III. GENERALIZATION OF BHABHA'S WORLD TUBE

In this section we are going to generalize the tube (2.4) to curved spaces. As we shall see, it is possible to carry out the integrations in an exact form for an arbitrary radius. In connection with this, notice that the tube defined by DeWitt and Brehme in Sec. 4 of their paper is not adequate for our purposes. They built their tube in the following way: They consider the points extending out a fixed distance ϵ along the geodesics orthogonal to the world line of the universe in a point $z(\tau)$, determining in this way a two-dimensional surface Π. Varying τ, Π generates the world tube of DB. An integral over this tube, then, is expressed as an integral over Π followed by an integral in τ (which obviously is not the retarded time of any point of Π). Now, if we consider two different points in Π, there are associated to them, in general, two *different* retarded points on the world line of the electron. This creates a complicated, since, as we shall see from Eq. (1.5), the tensor $T^r_{\mu\nu}$ evaluated at the point x is completely determined by a unique retarded point on the world line of the electron. An integral of $T^r_{\mu\nu}$ over Π, therefore, is practically impossible to evaluate in an exact form.

In what follows z always designates a point on the world line of the electron. Consider the future light cone emerging from the point $z(\tau)$, that is, the surface

$$\sigma(x, z) = 0 \ . \tag{3.1}$$

This surface intercepts the three-dimensional surface

$$\sigma_{;\alpha}\dot{z}^\alpha = \epsilon \quad (\epsilon > 0 \text{ fixed}) \tag{3.2}$$

in a two-dimensional surface that in the following will be denoted by Φ. When z varies on the world line of the electron, Φ generates a three-dimensional surface. This surface is the natural generalization of the Bhabha tube (2.4).

From Eqs. (3.1) and (3.2) it follows that the vector δx^μ is on the surface of the generalized Bhabha tube if and only if

$$\sigma_{;\mu}\delta x^\mu = -\epsilon\, d\tau \ , \tag{3.3}$$

$$\hat{\sigma}_\mu \delta x^\mu = -(\chi + \kappa')d\tau \ . \tag{3.4}$$

Here we have introduced the simplifying notation

$$\hat{\sigma}_\mu = \sigma_{;\mu\alpha}\dot{z}^\alpha \ . \tag{3.5}$$

In Eq. (3.4) χ and κ' have the same meaning as in A.

Let us determine now the directed (vector density) surface element of the tube defined by Eq. (3.2), that is,

$$d\Sigma_\mu = \epsilon_{\mu\nu\sigma\tau}\,\delta_1 x^\nu \delta_2 x^\sigma \delta_3 x^\tau \ , \tag{3.6}$$

where $\delta_1 x^\nu$, $\delta_2 x^\sigma$, and $\delta_3 x^\tau$ are three linearly independent displacements contained in the tube (3.2) and $\epsilon_{\mu\nu\sigma\tau}$ is the four-dimensional permutation symbol. In order to do this it is necessary to give a convenient parametrization of the points of the tube (3.2).

Let us consider first the surface Φ defined by Eqs. (3.1) and (3.2) with z fixed. It is obvious that if we take a vector, lying in the cone that is drawn from the retarded point z towards the future, then this direction and the point z uniquely define a null geodesic passing through z. This geodesic intercepts the surface Φ in a well-defined point and therefore we can parametrize the points of the surface Φ by means of a null vector at the retarded point z. We choose this vector to be $\sigma_{;\alpha}$.

Similarly to what DeWitt and Brehme have done, we introduce at z a tetrad of orthonormal vectors; one of them is $\dot{z}^\alpha$, while the other three we designate as n_i^α $(i=1,2,3)$. Then we have

$$n_i^\alpha n_{j\alpha}=\delta_{ij}, \quad n_{i\alpha}\dot{z}^\alpha=0 . \tag{3.7}$$

In what follows and up to Eq. (3.20) we assume that z is a point that remains fixed on the world line of the electron.

If we denote by $\sigma_\alpha^\perp$ the component of $\sigma_{;\alpha}$ orthogonal to $\dot{z}^\alpha$, then

$$\sigma_\alpha^\perp=\sigma_{;\alpha}+(\sigma_{;\beta}\dot{z}^\beta)\dot{z}_\alpha=\sigma_{;\alpha}+\epsilon\dot{z}_\alpha . \tag{3.8}$$

From this equation it follows that if $\sigma_\alpha^\perp$ is known we obtain $\sigma_{;\alpha}$ and vice versa; therefore, we can parametrize the points on the two-dimensional surface Φ by means of $\sigma_\alpha^\perp$. Now, since this vector is in the space defined by the three vectors n_i^α, we can introduce a set of three direction cosines Ω_i that completely determine the direction of $\sigma_\alpha^\perp$ relative to the n_i^α. These satisfy the obvious relation

$$\Omega_i\Omega_i=1 . \tag{3.9}$$

From Eq. (3 8) it follows, on the other hand, that

$$\sigma_\alpha^\perp\sigma^{\perp\alpha}=\epsilon^2 . \tag{3.10}$$

Then Eqs. (3.7), (3.9), and (3.10) allow us to write

$$\sigma_\alpha^\perp=-\epsilon n_{i\alpha}\Omega_i , \tag{3.11}$$

clearly showing that $\sigma_\alpha^\perp$ is completely determined with just two direction cosines. We use them as the two parameters that allow us to identify any point in the two-dimensional surface Φ.

Consider now the problem of defining two displacement vectors $\delta_1 x^\mu$ and $\delta_2 x^\mu$ contained in Φ. If the direction cosines Ω_i determine a point x^μ in Φ, then a variation $\delta\Omega_i$ changes the point x^μ to $x^\mu+\delta x^\mu$. The vector δx^μ is determined by means of Eq. (3.11). In fact,

$$\sigma_{\alpha;\mu}^\perp\delta x^\mu=-\epsilon n_{i\alpha}\delta\Omega_i . \tag{3.12}$$

We see from Eq. (3.8) that

$$\sigma_{\alpha;\mu}^\perp=\sigma_{;\alpha\mu} , \tag{3.13}$$

since in the change $\Omega_i\to\Omega_i+\delta\Omega_i$ the point z remains fixed.

Following DB we introduce the notation

$$D_{\mu\alpha}=-\sigma_{;\mu\alpha} . \tag{3.14}$$

From the hypothesis that we have made concerning the metric it is easy to see that the bivector $D^{-1}{}_{\mu\alpha}$ exists (DB, p. 231), that is,

$$D_{\mu\alpha}D^{-1\,\nu\alpha}=\delta_\mu^\nu . \tag{3.15}$$

With the help of $D^{-1\,\nu\alpha}$ and Eq. (3.13), we obtain from Eq. (3.12) that an arbitrary displacement δx^μ in the surface Φ is determined by

$$\delta x^\mu=\epsilon D^{-1\,\mu}{}_\alpha n_i^\alpha\delta\Omega_i . \tag{3.16}$$

This equation is formally identical to Eq. (4.7) of DB, but, of course, there is a fundamental difference, since our geodesics are the null geodesics that generate the future light cone emerging from the point z. Instead, DB make use of spacelike geodesics to build their tube. It is obvious to verify that the displacements (3.16) satisfy Eqs. (3.3) and (3.4) with $d\tau=0$.

A pair of independent variations $\delta_1\Omega_i$ and $\delta_2\Omega_i$ in the direction cosines define an element $d\Omega$ of solid angle by the relation

$$\Omega_i\, d\Omega=\epsilon_{ijk}\delta_1\Omega_j\delta_2\Omega_k , \tag{3.17}$$

where ϵ_{ijk} is the three-dimensional antisymmetric permutation symbol. If we denote by $\delta_1 x^\mu$ and $\delta_2 x^\mu$ the corresponding displacements in the surface Φ produced by $\delta_1\Omega_i$ and $\delta_2\Omega_i$, then using (3.16) and (3.17) we get

$$\delta_1 x_\nu\delta_2 x_\sigma-\delta_1 x_\sigma\delta_2 x_\nu=\epsilon^2 D^{-1}{}_{\nu\alpha}D^{-1}{}_{\sigma\beta}n_i^\alpha n_j^\beta\epsilon_{ijk}\Omega_k\, d\Omega . \tag{3.18}$$

This equation will be useful for computational purposes later on.

To obtain a complete parametrization of the points of the tube defined by Eq. (3.2) it is not sufficient, of course, to have the two direction cosines that define $\sigma_\alpha^\perp$ and the proper time τ. It is also necessary to give a transport law for the tetrad $\dot{z}^\alpha$, n_i^α when τ varies. For our purpose, however, such specification is superfluous. All the information about the tetrad $\dot{z}^\alpha$, n_i^α that we need is in Eqs. (3.7). The following equations can be inferred from them [DB, Eqs. (4.16) and (4.19)]:

$$n_i^\alpha n_i^\beta=g^{\alpha\beta}+\dot{z}^\alpha\dot{z}^\beta , \tag{3.19}$$

$$\epsilon_{\alpha\beta\gamma\delta}n_i^\gamma n_j^\delta=g^{-1/2}(z)\epsilon_{ijk}(n_{k\beta}g_{\alpha\theta}-n_{k\alpha}g_{\beta\theta})\dot{z}^\theta . \tag{3.20}$$

Let us now build a displacement $\delta_3 x^\mu$ on the tube (3.2) such that it is orthogonal to the linearly independent displacements $\delta_1 x^\mu$ and $\delta_2 x^\mu$ on the surface Φ, that is, $\delta_3 x^\mu$ should satisfy the following equations:

$$\begin{aligned} &\delta_1 x^\mu \delta_3 x_\mu = 0 \ , \\ &\delta_2 x^\mu \delta_3 x_\mu = 0 \ , \\ &\sigma^{;\mu}\delta_3 x_\mu = -\epsilon\, d\tau \ , \\ &\hat{\sigma}^\mu \delta_3 x_\mu = -(\chi+\kappa')d\tau \ . \end{aligned} \tag{3.21}$$

Of course, this linear system has a unique solution and it is easily seen to be

$$\delta_3 x^\mu = \epsilon^{\mu\nu\omega\lambda}\delta_1 x_\nu \delta_2 x_\omega [\epsilon\hat{\sigma}_\lambda - (\chi+\kappa')\sigma_{;\lambda}]M^{-1}d\tau \ , \tag{3.22}$$

where

$$M = -\epsilon^{\mu\nu\omega\lambda}\delta_1 x_\mu \delta_2 x_\nu \sigma_{;\omega}\hat{\sigma}_\lambda \ . \tag{3.23}$$

The evaluations of M, $\delta_3 x^\mu$, and $d\Sigma_\mu$ are quite similar and we illustrate them by showing the evaluation of M. In contrast with DeWitt and Brehme, we will not make use of the operation of "homogenization" of indices since it will not be necessary to make covariant expansions at small distances of the world line of the electron. We calculate $d\Sigma_\mu$ for arbitrary ϵ.

By means of Eq. (3.18), M can be put in the form

$$\begin{aligned} M &= -\tfrac{1}{2}\epsilon^2\epsilon^{\mu\nu\omega\lambda}D^{-1}{}_{\mu\alpha}D^{-1}{}_{\nu\beta}n_i^\alpha n_j^\beta \epsilon_{ijk}\sigma_{;\omega}\hat{\sigma}_\lambda \Omega_k d\Omega \\ &= -\tfrac{1}{2}\epsilon^2\epsilon_{\alpha\beta\delta\gamma}|D^{-1}{}_{\mu\alpha}|D^{\lambda\gamma}D^{\omega\delta}n_i^\alpha n_j^\beta \epsilon_{ijk}\Omega_k \sigma_{;\omega}\hat{\sigma}_\lambda d\Omega \ . \end{aligned} \tag{3.24}$$

Making use of Eq. (3.20) and the abbreviation

$$\Omega^\alpha = n_i^\alpha \Omega_i \ , \tag{3.25}$$

we write M in the form

$$M = -\epsilon^2 |D^{-1}{}_{\mu\alpha}| g^{-1/2}(z) \times(\sigma_{;\alpha}\dot{z}^\alpha \sigma^{;\omega\beta}\Omega_\beta \hat{\sigma}_\omega - \Omega_\alpha \sigma^{;\alpha}\hat{\sigma}_\omega \hat{\sigma}^\omega) \ . \tag{3.26}$$

Here we have come back to the bivector $\sigma_{;\mu\alpha}$ instead of $D_{\mu\alpha}$ defined in Eq. (3.14). Notice that M in Eq. (3.23) (as well as $\delta_1 x^\mu$ and $\delta_2 x^\mu$) is formally identical to that in DB [see DB, Eq. (4.24)], and since we have only used the fundamental property of the world function σ

$$\sigma_{;\alpha\omega}\sigma^{;\omega} = \sigma_{;\alpha} \tag{3.27}$$

in obtaining Eq. (3.26) it is also valid in the case of DB.

We now state the properties of the tube (3.2) and of our parametrization in Eq. (3.26). In particular we have

$$\Omega_\alpha = -\epsilon^{-1}\sigma_{;\alpha} - \dot{z}_\alpha \ , \tag{3.28}$$

which is nothing else but Eq. (3.11) combined with Eq. (3.8) and definition (3.25). It is also easy to derive from Eq. (3.15) that

$$|D^{-1}{}_{\mu\alpha}| = -g(x)g(z)D^{-1} \ . \tag{3.29}$$

When we relate this equation to the equations [see DB, Eqs. (1.60) and (1.62)]

$$\Delta = \bar{g}^{-1}D \ , \tag{3.30}$$

$$g^{1/2}(x)g^{1/2}(z) = \bar{g} \ , \tag{3.31}$$

we obtain

$$|D^{-1}{}_{\mu\alpha}| = -g^{1/2}(x)g^{1/2}(z)\Delta^{-1} \ . \tag{3.32}$$

Introducing Eqs. (3.28) and (3.32) in Eq. (3.26) we finally obtain

$$M = -\epsilon^3 g^{1/2}(x)\Delta^{-1}d\Omega \ . \tag{3.33}$$

Analogously, from (3.22) we obtain

$$\delta_3 x^\mu = [\epsilon^{-1}(\chi+\kappa'-\hat{\sigma}_\omega\hat{\sigma}^\omega)\sigma^{;\mu} + \hat{\sigma}^\mu]d\tau \ . \tag{3.34}$$

Introducing $\delta_3 x^\mu$ and $\delta_1 x^\mu$ and $\delta_2 x^\mu$ defined through Eq. (3.16) in Eq. (3.6), one finds that the vector-density surface element of the tube (3.2) can be expressed in the form

$$d\Sigma_\mu = \epsilon g^{-1/2}(x)\Delta^{-1}[\epsilon\hat{\sigma}_\mu - (\chi+\kappa')\sigma_{;\mu}]\,d\Omega\,d\tau \ . \tag{3.35}$$

Thus we have expressed $d\Sigma_\mu$ in a covariant form for arbitrary ϵ without making use of special coordinate systems. The key point that facilitates such construction is the famous Synge world function $\sigma(x,z)$. We must keep in mind that z in Eq. (3.35) is the retarded point corresponding to the point x on the tube (3.2). This fact introduces considerable simplifications, as we shall see in Sec. IV.

If we proceed in a similar fashion to obtain Eq. (3.35) but for the tube of DeWitt and Brehme we obtain

$$d\Sigma_\mu = \epsilon g^{-1/2}(x)\Delta^{-1}(\chi+\kappa')\sigma_{;\mu}d\Omega\,d\tau \ . \tag{3.36}$$

This formula is apparently simpler than (3.35); however, complications arise because the geodesics employed are the ones orthogonal to the world line of the electron and therefore τ is not the retarded time at which the fields are usually evaluated. In particular, for example, the Δ appearing in (3.36) is different from the Δ of Eq. (1.5).

V. GENERALIZATION OF LARMOR'S FORMULA

Let us consider the following auxiliary construction. Let Σ_1 and Σ_2 be two generalized Bhabha tubes (3.2) of radii ϵ_1 and ϵ_2, respectively, $\epsilon_1 > \epsilon_2$. From the points $z(\tau)$ and $z(\tau^*)$ we draw future light cones. We designate by Λ the domain bounded by these four-surfaces. Let $\bar{g}_{\mu\alpha}(x, z(\tau^*))$ be the bivector of parallel translation over the geodesic that goes from x to $z(\tau^*)$. Consider now the integral

$$\int_\Lambda (\bar{g}^\mu{}_\alpha T^r_{\mu\nu})^{;\nu} d^4x \,. \tag{4.1}$$

The integral obviously defines a vector at $z(\tau^*)$, and we can transform it by means of Gauss's theorem and properties (i) and (ii) of Sec. I, obtaining

$$\int_{\Delta\Sigma_1} \bar{g}^\mu{}_\alpha T^r_{\mu\nu} d\Sigma^\nu - \int_{\Delta\Sigma_2} \bar{g}^\mu{}_\alpha T^r_{\mu\nu} d\Sigma^\nu = \int_\Lambda \bar{g}^\mu{}_\alpha{}^{;\nu} T^r_{\mu\nu} d^4x \,. \tag{4.2}$$

In contrast with the flat-space case we have in the present case that the right-hand side does not vanish. Since $T^r_{\mu\nu}$ as well as $d\Sigma^\nu$ is evaluated at the retarded point associated with x, we have from Eqs. (1.5) and (3.35) and the relations

$$\sigma_{;\mu}\sigma^{;\mu} = 0 \,, \tag{4.3}$$

$$\hat{\sigma}_\mu \sigma^{;\mu} = \sigma_{;\mu\alpha}\dot{z}^\alpha \sigma^{;\mu} = \sigma_{;\alpha}\dot{z}^\alpha = \epsilon \tag{4.4}$$

that

$$T^r_{\mu\nu} d\Sigma^\nu_1 = \frac{e^2}{4\pi}\epsilon_1 \sigma_{;\mu}(\ddot{z}^2 - \kappa'^2\kappa^{-2})\kappa^{-4} d\Omega\, d\tau \,. \tag{4.5}$$

Notice that the complicated factor Δ which appears in $T^r_{\mu\nu}$ has been eliminated. Thus, the integrand of the surface integrals in Eq. (4.2) can be written [in this case $\kappa = \epsilon$; see Eq. (3.2)] as

$$\frac{e^2}{4\pi}\bar{g}_{\mu\alpha}\sigma^{;\mu}(\ddot{z}^2 - \kappa'^2\epsilon_i{}^{-2})\epsilon_i{}^{-1} d\Omega\, d\tau \quad (i = 1, 2) \,. \tag{4.6}$$

Since the geodesics of $\bar{g}_{\mu\alpha}$ and $\sigma^{;\mu}$ are not the same, we cannot make use in (4.6) of the relation [DB, Eq. (1.34)]

$$\bar{g}_{\mu\alpha}\sigma^{;\mu} = \sigma_{;\alpha} \,. \tag{4.7}$$

To abbreviate let us introduce the notation

$$\psi_i = \frac{e^2}{4\pi}\epsilon^{-1}(\ddot{z}^2 - \kappa'^2\epsilon_i{}^{-2}) \quad (i = 1, 2) \,. \tag{4.8}$$

Then Eq. (4.2) can be written, in the case of the tube (3.2), in the form

$$\int_{\tau^*}^{\tau} d\tau' \int \bar{g}_{\mu\alpha}\sigma^{;\mu}\psi_1 d\Omega = \int_{\tau^*}^{\tau} d\tau' \int \bar{g}_{\mu\alpha}\sigma^{;\mu}\psi_2 d\Omega + \int_\Lambda \bar{g}^\mu{}_\alpha{}^{;\nu} T^r_{\mu\nu} d^4x \,. \tag{4.9}$$

The integration over Λ can be performed in the following way. First, for τ' fixed between τ^* and τ we consider the cone drawn from $z(\tau')$ into the future; then we perform the integration over that part of this cone which is bounded by the tubes Σ_1 and Σ_2, followed by the integration over τ'. As integration variables over the cone we can choose, for example, a pair of direction cosines and the radius ϵ of Eq. (3.2) varying from ϵ_1 to ϵ_2. Such a procedure has been recently used by Synge[13] in Minkowski space. We can write, therefore,

$$\int_\Lambda \bar{g}^\mu{}_\alpha{}^{;\nu} T^r_{\mu\nu} d^4x = \int_{\tau^*}^{\tau} d\tau' \int_{\epsilon_1 \le \epsilon \le \epsilon_2} \bar{g}^\mu{}_\alpha{}^{;\nu} T^r_{\mu\nu} g^{-1/2}(x) \Xi\, d\epsilon\, d\Omega \,, \tag{4.10}$$

where Ξ is a biscalar that is irrelevant to our purposes. Introducing this expression in Eq. (4.9) and taking the derivative with respect to τ and evaluating it at τ^* we have

$$\int \bar{g}_{\mu\alpha}\sigma^{;\mu}\psi_1 d\Omega = \int \bar{g}_{\mu\alpha}\sigma^{;\mu}\psi_2 d\Omega + \int \bar{g}^\mu{}_\alpha{}^{;\nu} T^r_{\mu\nu} g^{-1/2} \Xi\, d\epsilon\, d\Omega \,. \tag{4.11}$$

Now we can make use of Eq. (4.7), and since $T^r_{\mu\nu}$ is proportional to $\sigma_{;\nu}$ and it is known [DB, Eq. (1.31)] that

$$\bar{g}_{\mu\alpha;\nu}\sigma^{;\nu} = 0 \tag{4.12}$$

the integral over the cone of Eq. (4.11) is null and we obtain

$$\int \sigma_{;\alpha}\psi_1 d\Omega = \int \sigma_{;\alpha}\psi_2 d\Omega \,, \tag{4.13}$$

showing that Eq. (2.7) can be generalized to curved spaces and that the integral

$$\int \sigma_{;\alpha}\psi\, d\Omega \tag{4.14}$$

is independent of the radius of the tube (3.2). This shows that the radiation emitted at the point z associated with $T^r_{\mu\nu}$ becomes independent of the source and propagates along the light cone with apex at this point.

According to the discussion of Sec. II, the integral (4.14) is related to the energy radiated at z, and in analogy to what was shown there it can be seen that the form of the tube which intercepts the cone with apex at z is irrelevant as far as the flux of radiated energy is concerned. In addition, the surface that intercepts the cone can be any timelike or spacelike surface. According to the analysis of Sec. II, the Larmor generalized invariant associated with $T^r_{\mu\nu}$ is

$$R = \dot{z}^\alpha \int \sigma_{;\alpha}\psi\, d\Omega = \epsilon \int \psi\, d\Omega \,; \tag{4.15}$$

one recalls that the point z remains fixed in the integration process. Introducing Eq. (4.8) in Eq. (4.15) we obtain

$$R = e^2\left(\ddot{z}^2 - \frac{1}{4\pi}\int \kappa'^2 \epsilon^{-2} d\Omega \right) \quad . \tag{4.16}$$

With the help of Eqs. (3.8) and (3.11) we have

$$\begin{aligned} \kappa' &= \sigma_{;\alpha}\ddot{z}^\alpha \\ &= (\sigma_\alpha^\perp - \epsilon \dot{z}_\alpha)\ddot{z}^\alpha \\ &= -\epsilon n_{i\alpha}\ddot{z}^\alpha \Omega_i \quad . \end{aligned} \tag{4.17}$$

Therefore

$$\frac{1}{4\pi}\int \epsilon^{-2}\kappa'^2 d\Omega = \tfrac{1}{3} n_i^\alpha n_i^\beta \ddot{z}_\alpha \ddot{z}_\beta \ . \tag{4.18}$$

With the help of Eq. (3.19) we see that the right-hand side of this equation is equal to $\frac{1}{3}\ddot{z}^2$. Thus, Larmor's generalized invariant (4.16) becomes

$$R = \tfrac{2}{3}e^2\ddot{z}^2 \ . \tag{4.19}$$

R obviously is positive-definite.

We can apply the same method to the massless scalar field with a point source. In this case the radiation tensor is [see A, Eq. (A7)]

$$T^r_{\mu\nu} = \frac{Gm^2}{4\pi} g^{1/2}\sigma_{;\mu}\sigma_{;\nu}\,\Delta\kappa'^2\kappa^{-6} \ , \tag{4.20}$$

yielding the generalized Larmor formula given by

$$R = \tfrac{1}{3}Gm^2\ddot{z}^2 \ . \tag{4.21}$$

ACKNOWLEGMENT

I would like to thank Professor A. O. Barut and Dr. P. Cordero for useful discussions and for reading the manuscript.

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