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Reprinted from PHYSICAL REVIEW LETTERS Vol. 38, No. 9, 28 February 1977
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PHYSICAL REVIEW LETTERS

VOLUME 38 | 28 FEBRUARY 1977 | NUMBER 9

Model for Relativistic Two-body Interaction with Radiation Reaction*

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(Received 15 December 1976)

An exactly solvable model for relativistic two-body interaction, consisting of two harmonic oscillators coupled via a massless scalar field is studied. It is shown that prescribing the asymptotic field does not lead to a normal Cauchy problem for the particles, whereas the formulation as a Cauchy problem for particle and field is consistent. Radiation damping of the system is also discussed.

Several authors[1-4] have argued that the two-body problem in electrodynamics, when formulated in terms of action at a distance using either retarded, advanced, or some combination of the fields, does not lead to a normal Cauchy problem on the particle level. This argument is based on general considerations such as the finite propagation velocity of the interaction and the absence of planes of absolute simultaneity.[3,4] Driver[1] has studied a two-body model proposed by Synge[5] restricted to one-dimensional motion. His results seem to confirm the above statement. However, the model has the deficiency that the field is eliminated from the beginning, thereby omitting radiation reaction. This makes a comparison with the Cauchy problem of *particles* and *field* impossible.

In this Letter I would like to discuss an exactly solvable model consisting of two classical (nonrelativistic) harmonic oscillators coupled via a *relativistic* massless scalar field in Minkowski space. The system is closed in the sense that the dynamics for field and sources is derivable from a Lagrangian for which the total energy is conserved. Physically the model resembles two sources at fixed position in space (e.g., atoms in a solid) influencing each other by radiation. The complete solution to the system not only describes emission and absorption by the sources, but also takes into account *radiation reaction.* It is therefore possible to study radiation damping.

Mathematically this model gives insight into the dynamics of the relativistic two-body Cauchy problem: Elimination of the field from the dynamics of the sources by imposing boundary conditions leads to differential-difference equations for the particles. These equations can be solved to show that, in general, a solution is not uniquely determined by the initial data at some instant, but that the specification of the data on a finite time segment is necessary. In contrast to this, prescribing the field on some initial surface does lead to a normal Cauchy problem for *particles* and *field*: Given the field on an initial surface, one finds differential-difference equations of a special type for which normal initial data of the sources determine the solution uniquely.

The model is defined by the following Lagrange function

$$L=\tfrac{1}{2}[\dot{Q}_1^{\,2}(t)-\omega_0^{\,2}Q_1^{\,2}(t)+\dot{Q}_2^{\,2}(t)-\omega_0^{\,2}Q_2^{\,2}(t)]+\lambda\int d^3x[\rho_1(\vec{x})Q_1(t)+\rho_2(\vec{x})Q_2(t)]\Phi(\vec{x},t)+\tfrac{1}{2}\int d^3x\{\dot{\Phi}^2(\vec{x},t)-[\nabla\Phi(\vec{x},t)]^2\},\qquad(1)$$

where Q_1, Q_2 are the oscillator variables, $\Phi(\vec{x}, t)$ is a massless scalar field, and ω_0 is the "bare" spring constant; the dot means differentiation with respect to t, λ is a coupling constant, and ρ_1, ρ_2 are the "charge" distributions which in the point limit are taken to be

$$\rho_1(\vec{x}) = \delta^3(\vec{x} - \vec{a}), \quad \rho_2(\vec{x}) = \delta^3(\vec{x} + \vec{a}). \qquad (2)$$

Thus the oscillators couple to the field only at $\vec{x} = \pm\vec{a}$ (see Fig. 1).

From Eq. (1) one derives the dynamical equations for the system:

$$\ddot{Q}_1(t) + \omega_0^2 Q_1(t) = \lambda\Phi(\vec{a}, t), \qquad (3a)$$

$$\ddot{Q}_2(t) + \omega_0^2 Q_2(t) = \lambda\Phi(-\vec{a}, t), \qquad (3b)$$

$$\Box\Phi(\vec{x}, t) = \lambda\delta^3(\vec{x} - \vec{a})Q_1(t) + \lambda\delta^3(\vec{x} + \vec{a})Q_2(t). \qquad (3c)$$

[In the following, pairs of equations, such as (3a) and (3b), will be written as $\ddot{Q}_{1,2}(t) + \omega_0^2 Q_{1,2}(t) = \lambda\Phi(\pm\vec{a}, t)$, where the upper sign is to be associated with the first subscript.] To find solutions to the system (3), we decompose the field into retarded and incoming parts, $\Phi = \Phi^{\text{ret}} + \Phi^{\text{in}}$, where

$$\Phi^{\text{ret}} = \frac{\lambda}{4\pi}\left[\frac{Q_1(t - |\vec{x} - \vec{a}|)}{|\vec{x} - \vec{a}|} + \frac{Q_2(t - |\vec{x} + \vec{a}|)}{|\vec{x} + \vec{a}|}\right]. \qquad (4)$$

Inserting for Φ from Eq. (4) into the right-hand side of Eqs. (3a) and (3b) makes an infinite renormalization of the spring constant necessary (mathematical details can be found in Aichelburg and Beig,[7,8] where a single-source model is treated)

$$\lim_{\vec{x} \to \pm\vec{a}} \frac{Q_{1,2}(t - |\vec{x} \mp \vec{a}|)}{|\vec{x} \mp \vec{a}|} = C_\infty Q_{1,2}(t) - \dot{Q}_{1,2}(t), \qquad (5)$$

where C_∞ represents the infinite contribution in the limit. I define $\bar{\omega}^2 = \omega_0^2 - 2\Gamma C_\infty$, $a = |\vec{a}|$, and $\Gamma = \lambda^2/8\pi$, thus obtaining

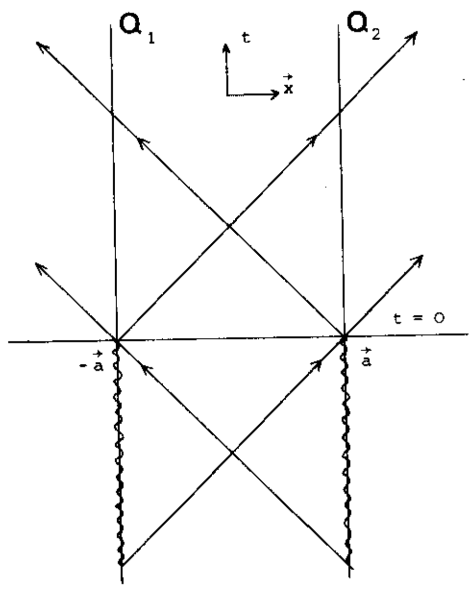

FIG. 1. The two oscillators Q_1 and Q_2 interact along the light cone; energy can be carried away by the field. If Φ^{in} is given, the data of $Q_1(t)$ and $Q_2(t)$ for $-2a \leq t \leq 0$ are necessary to determine the solution uniquely.

$$\ddot{Q}_{1,2}(t) + 2\Gamma\dot{Q}_{1,2}(t) + \bar{\omega}^2 Q_{1,2}(t) = (\Gamma/a)Q_{2,1}(t - 2a) + \Phi^{\text{in}}(\pm\vec{a}, t). \qquad (6)$$

Equations (6) are coupled differential-difference equations of retarded type for Q_1 and Q_2 with a prescribed external force Φ^{in}. First-order differential-difference equations have been extensively studied in Ref. 6. The general solution to Eqs. (6) can be obtained by the method of Laplace transform. For simplicity take $\Phi^{\text{in}} = 0$ and introduce $q_\pm = Q_1 \pm Q_2$ for which the equations decouple so that

$$\ddot{q}_\pm(t) + 2\Gamma\dot{q}_\pm(t) + \bar{\omega}^2 q_\pm(t) = \pm(\Gamma/a)q_\pm(t - 2a). \qquad (7)$$

For $t > 0$, the solution is given by

$$q_\pm(t) = (2\pi i)^{-1}\int_C ds\, e^{st} h_\pm^{-1}(s)[q_\pm(0)(s + 2\Gamma) + \dot{q}_\pm(0) \pm (\Gamma/a)e^{-2as}F_\pm(s)], \qquad (8)$$

where

$$h_\pm(s) = s^2 + 2\Gamma s + \bar{\omega}^2 \mp (\Gamma/a)e^{-2as} \qquad (9)$$

are the characteristic functions to Eqs. (7),

$$F_\pm(s) = \int_{-2a}^{0} q_\pm(t')e^{-st'}dt', \qquad (10)$$

and C is an integration path parallel to the imaginary axis in the complex s plane to the right of all zeros of $h_\pm(s)$. $F(s)$ depends on the values of $q(t)$ over the interval $-2a \leq t \leq 0$. Therefore, to obtain a unique solution to Eqs. (7) one is forced to prescribe not only the initial data $q(0)$ and $\dot{q}(0)$ but also $q_\pm(t)$

over an interval equal to the retardation time.

Instead of decomposing the field into retarded and incoming part I now write $\Phi = \Phi_I + \Phi_H$, where Φ_I is the inhomogeneous solution to Eq. (3c) because there are no constraints on the initial surface for a scalar field) and Φ_H is a homogeneous field. Formally Φ_I can be written as

$$\Phi_I(\vec{x}, t) = \frac{\lambda}{4\pi}\left\{\frac{Q_1(t - |\vec{x} - \vec{a}|)}{|\vec{x} - \vec{a}|}\Theta(t - |\vec{x} - \vec{a}|) + \frac{Q_2(t - |\vec{x} + \vec{a}|)}{|\vec{x} + \vec{a}|}\Theta(t - |\vec{x} + \vec{a}|) + \frac{Q_1(t + |\vec{x} - \vec{a}|)}{|\vec{x} - \vec{a}|}\Theta(-t - |\vec{x} - \vec{a}|) + \frac{Q_2(t + |\vec{x} + \vec{a}|)}{|\vec{x} + \vec{a}|}\Theta(-t - |\vec{x} + \vec{a}|)\right\}, \tag{11}$$

where Θ is the step function. I shall be mainly interested in the solutions for $t > 0$. Note that in this case Φ_I does not depend on the dynamics of the oscillators *prior* to $t = 0$.

The oscillator equations following from (11) now are

$$\ddot{Q}_{1,2}(t) + 2\Gamma\epsilon(t)\dot{Q}_{1,2}(t) + \bar{\omega}^2 Q_{1,2}(t) = (\Gamma/a)\{Q_{2,1}(t-2a)\Theta(t-2a) + Q_{2,1}(t+2a)\Theta(-t-2a)\} + \lambda\Phi_H(\pm\vec{a}, t), \tag{12}$$

where $\epsilon(t) = \pm 1$ for $t \gtrless 0$. Imposing, for simplicity, $\Phi_H = 0$ (no field at the initial surface) and decoupling, Eqs. (12) read for $t > 0$

$$\ddot{q}_\pm(t) + 2\Gamma\dot{q}_\pm(t) + \bar{\omega}^2 q_\pm(t) = \pm(\Gamma/a)q_\pm(t-2a)\Theta(t-2a). \tag{13}$$

For $t > 0$ these equations differ from Eqs. (7) only by the Θ function on the right-hand side. The appearance of the Θ function implies that in this case the oscillators do not influence each other for $0 \leq t < 2a$ and therefore perform free damped oscillations during this period. As a consequence the solutions to Eq. (13) are determined *uniquely* by the initial values $q_\pm(0)$ and $\dot{q}_\pm(0)$ as given by (8) with $F(s) = 0$. Note from Eqs. (12) that for $\Phi_H = 0$ the solution is time symmetric. If $\Phi_H \neq 0$ then a solution to Eqs. (12) is determined by the set of initial data of *field* and *particle* [i.e., $Q_1(0)$, $\dot{Q}_1(0)$, $Q_2(0)$, $\dot{Q}_2(0)$, $\Phi_H(\vec{x}, 0)$, $\dot{\Phi}_H(\vec{x}, 0)$].

I have written the dynamical equations for the sources in two different but equivalent forms, namely Eqs. (6) and (12). Equations (6) are suitable if the asymptotic field (in this case the incoming field) is prescribed. In this case however, the dynamics of the sources are not in general determined by the normal Cauchy data. In contrast to this, Eqs. (12) show that given initial data for the fields lead to a well-defined Cauchy problem on the particle level.

Let us look briefly into the long-time behavior of the system. The asymptotic behavior of the homogeneous differential-difference equations, Eqs. (7) for $\Phi_{in} = 0$ and Eqs. (13) for $\Phi_H = 0$, is determined by the zeros of $h_\pm(s)$ in the solution (8). A sufficient condition for

$$\lim_{t\to\infty} q_\pm(t) = 0 \tag{14}$$

is that $h_\pm(s)$ has only zeros with negative real part. Analyzing the characteristic functions shows that $h_\pm(s)$ has, in general, infinitely many zeros which all have negative real part iff $\Gamma/a < \bar{\omega}^2$. Because Eqs. (7) and (13) have the same characteristic functions, Eqs. (14) are valid for both solutions. Note that $\Phi_H = 0$ is a time symmetric condition; nevertheless the asymptotic behavior (14) follows (the arrow-of-time problem in radiating systems is discussed in Refs. 7 and 8). If $\Gamma/a < \bar{\omega}^2$, the total conserved energy of the system (1) can be expressed as a sum of quadratic terms. Thus, the energy initially stored by the oscillators is eventually completely transferred to the field.

If $\Gamma/a > \bar{\omega}^2$ the system (1) becomes *unstable* (even though radiation damping is included). $h_+(s)$ develops a real positive zero leading to an exponentially growing amplitude of the oscillators. However, if we think of two atoms with electromagnetic interaction one has for optical transitions $\bar{\omega} \simeq 10^{15}$ sec^{-1}, $\Gamma \simeq 10^{8}$ sec^{-1}, and a delay time $a \simeq 10^{-18}$ sec, so that $\Gamma/a > \bar{\omega}^2$ is not realistic.

Finally I remark that differential-difference equations of retarded type can also be solved by the method of "integration of steps," i.e., in this case integrating over intervals of the time $2a$. This, together with an analysis of the solutions and the proofs for the results given above, will be presented in a more detailed publication.

I thank Dr. H. Grosse and Dr. H. Urbantke for helpful discussions.

*Supported by Fonds zur Förderung der wissenschaftlichen Forschung in Österreich, Projekt Nr. 2911, and Fundacion Federico S. A. AR 76/2.

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Radiation damping and the expansion of the universe*†

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(Received 22 July 1976; revised manuscript received 27 September 1976)

An exactly soluble model consisting of a massless scalar field coupled to a harmonic oscillator considered previously in Minkowski space is generalized to spatially open Robertson-Walker cosmologies. It is shown that the dissipative behavior displayed by the system in Minkowski space when the total energy is finite is improved (weakened) by the expansion (contraction) of the universe. Further, we investigate the existence of solutions with purely retarded or purely advanced fields. We show that the system embedded in a universe which originates ("big bang") is similar to the open (external forces acting) system in Minkowski space, thus allowing for retarded but not advanced solutions.

I. INTRODUCTION

In recent times there has been renewed interest in explaining "the arrow of time" inherent in different kinds of dissipative processes.[1,2] Concerning the "electromagnetic arrow of time," there has been a special school of thought[2] which advocated the idea that there should be a strong connection between the observed dissipative features of electromagnetic radiation, e.g., radiation damping and the expansion of the universe.[3-6] A question very often left unanswered by those speculations is: What is really meant by a "connection" between those two phenomena? Does it mean that there is an obvious disequilibrium between matter and radiation in our present universe (a fact, though in a rather indirect way, "caused" by the cosmic expansion)? Or does it mean that the expansion of the universe by itself somehow gives an *a priori* reason for local electromagnetic processes "choosing" the retarded rather than the advanced interaction? The second viewpoint would lead to the conclusion that if our universe were to start contracting the arrow of time of local processes would be reversed.[5]

In our view, the large variety of opinions in the present context is mainly due to the fact that the notion of dissipative processes being "induced by" or "connected with " the cosmic expansion is meaningless unless the theoretical framework within which one looks for such a relation is specified. In the case of electrodynamics a controversial point is whether one should work within conventional Maxwellian electrodynamics[4] or an action-at-a-distance electrodynamics.[7,8]

It is the purpose of this paper to obtain a better understanding of these problems by studying the time development of a simple *field-theoretical* radiating system, where the field is included in the Cauchy problem. In a previous paper[9] we studied a model, first considered by Schwabl and Thirring,[10] describing a harmonic nonrelativistic oscillator interacting with a massless scalar field in Minkowski space-time. The Cauchy problem for the coupled system was set up and solved. The main conclusion to be drawn from this paper is that, without any time asymmetrical assumptions such as boundary conditions leading to retarded or advanced fields, the system shows a dissipative behavior: For initial configurations, where the field on the initial spacelike slice decreases with $|\vec{x}|=r\to\infty$ in such a way that the total energy E of the system is finite, the oscillator loses its energy completely with time $t\to\infty$. By time symmetry this has to hold also for $t\to-\infty$. We thus arrive at a picture where an incoming wave of finite energy excites an (because of $E<\infty$ initially quiescent) oscillator which subsequently reradiates the energy thus acquired. However, if the field is demanded to be purely retarded the oscillator has to perform damped oscillations for all times: Its energy blows up exponentially with $t\to-\infty$ giving rise to exponential growth of the field with $r\to\infty$, hence violating $E<\infty$. It is now essential to observe that it has been tacitly assumed that the system is *closed* in the sense that the field and particle equations are supposed to hold for all times. If one allows the oscillator equation to be violated in the past of the initial surface ("open system") then the oscillator can be excited by a mechanism not accounted for in the original equations ("by hand"). Now purely retarded solutions exist for $E<\infty$, whereas purely advanced solutions do not. This is no contradiction to the time symmetry of the original equations, since the very notion of an open system violates the latter. A short review of the model in Minkowski space is given in Sec. II.

In Sec. III the model is generalized to Robertson-Walker space-times. The gravitational field is treated as external. The scalar field equation is chosen to be conformally invariant in analogy to electrodynamics. Since in this paper we are mainly interested in studying the influence on the system of the time dependence of the metric rather than that of a topology different from R^3, we restrict ourselves to open universes ($k=0$ or -1). The case $k=+1$ will be treated by one of the authors (R.B.) in a future publication. In Sec. III the case $k=0$ (flat-space sections) is treated in detail. We give the general solution of the Cauchy problem for the coupled system of equations.

In Sec. IV the damping behavior of the oscillator for different kinds of dynamics of the universe is investigated. Einstein's equations are not imposed. Our conclusion is that in general damping of the oscillator, present already in Minkowski space-time, is *improved* in an *expanding* universe and *weakened* in a *contracting* universe. The expansion can even ensure damping when this is not the case in the corresponding situation in Minkowski space-time, e.g., while in the latter fields which are periodic over the whole Cauchy surface do not lead to damping, they do for expanding universes. In an exponentially contracting universe the oscillator loses its energy if the field displays a certain decrease with $r\to\infty$.

In Sec. V a rather obvious generalization of the notion of a retarded field for the considered universe is adopted. Again solutions to the system are looked for with purely retarded (advanced) radiation. It turns out that universes which originate play a similar role for the system as "opening" the system in Minkowski space. For models which have a past horizon this is almost trivial. For those which have no past horizon a little more care is needed. We give several examples. The standard Friedmann models especially allow purely retarded but no advanced solutions to the system. The case $k=-1$ is roughly sketched in Sec. VI. All results remain valid.

II. THE MODEL IN MINKOWSKI SPACE

We first consider a massless scalar field in Minkowski space, coupled to a nonrelativistic one-dimensional harmonic oscillator. The dynamics of this system can be derived from the action integral

$$W=\tfrac{1}{2}\int dt(\dot{Q}^2-\omega_0^2Q^2)+\lambda\int d^4x\,\rho(\vec{x})\Phi(\vec{x},t)Q(t)+\tfrac{1}{2}\int d^4x\,\Phi_{,\mu}\Phi_{,\nu}\eta^{\mu\nu}, \tag{2.1}$$

where the oscillator has unit mass and ω_0 is the "bare" spring constant. [Our notation is $x^\mu=(t,\vec{x})$, $\eta^{\mu\nu}=\text{diag}(1,-1,-1,-1)$, $\Phi_{,\nu}\equiv\partial\Phi/\partial x^\nu$, $d^4x=dt\,d^3x$, and the overdot means derivative with respect to t.] The interaction between the oscillator and the field is determined by a "charge" density $\rho(\vec{x})$, satisfying $\int d^3x\,\rho(\vec{x})=1$, and the coupling constant λ.

In this paper we shall be mainly interested in the "point limit," i.e., the case where $\rho(\vec{x})=\delta^3(\vec{x})$ (for the detailed treatment of the extended model, see Ref. 9). In this limit the field couples to the oscillator only at $\vec{x}=0$.

From the action integral (2.1) we obtain a coupled system of equations:

$$\ddot{Q}(t)+\omega_0^2Q(t)=\lambda\Phi(\vec{0},t), \tag{2.2a}$$

$$\Box\Phi(\vec{x},t)=\lambda\delta^3(\vec{x})Q(t). \tag{2.2b}$$

Given the initial values $Q(t_0)$, $\dot{Q}(t_0)$, $\Phi(\vec{x},t_0)$, and $\dot{\Phi}(\vec{x},t_0)$ on the hypersurface $t=t_0$, we want to know $Q(t)$ and $\Phi(\vec{x},t)$ for $t\gtrless t_0$.

Before solving Eqs. (2.2) it is convenient to introduce the following notation: For a solution of Eq. (2.2b) with given initial values at t_0 we write

$$\Phi(\vec{x},t)=\Phi_I(\vec{x},t;t_0)+\Phi_H(\vec{x},t;t_0), \tag{2.3}$$

where Φ_I is the unique solution of Eq. (2.2b) with

$$\Phi_I(\vec{x},t_0;t_0)=0,\quad \dot{\Phi}_I(\vec{x},t_0;t_0)=0, \tag{2.4}$$

and Φ_H satisfies the homogeneous wave equation

$$\Box\Phi_H(\vec{x},t;t_0)=0 \tag{2.5}$$

with initial values

$$\Phi_H(\vec{x},t_0;t_0)=\Phi(\vec{x},t_0),\quad \dot{\Phi}_H(\vec{x},t_0;t_0)=\dot{\Phi}(\vec{x},t_0). \tag{2.6}$$

Obviously, in the limit $t_0\to-\infty$ and $t_0\to+\infty$ we obtain the usual retarded and advanced fields

$$\lim_{t_0\to-\infty\,(+\infty)}\Phi_I(\vec{x},t;t_0)=\Phi_{\text{ret (adv)}}(\vec{x},t). \tag{2.7}$$

We define the formal incoming (outgoing) field $\Phi_{\text{in (out)}}$ by

$$\Phi_{\text{in (out)}}(\vec{x},t)=\Phi(\vec{x},t)-\Phi_{\text{ret (adv)}}(\vec{x},t). \tag{2.8}$$

In the more general context studied later we shall use this "t_0 procedure" for the *definition* of the retarded (advanced) and incoming (outgoing) fields. If, for instance, the space-time manifold were bounded in the past by some spacelike hypersurface $t=T$, we would call $\Phi_I(\vec{x},t;T)$ the retarded field.

We further note that Eq. (2.7) holds independent of any assumption about the source for $t\to-\infty$ $(+\infty)$, (e.g., boundedness). Hence, the relations corresponding to Eq. (2.4) will not hold for the retarded (advanced) field, i.e., in general

$$\lim_{t\to-\infty\,(+\infty)}\Phi_{\text{ret (adv)}}(\vec{x},t)\neq 0.$$

Therefore, we shall have no right to think of the formal incoming (outgoing) field as the "field at past (future) infinity" without restricting the behavior of the source with $t\to-\infty$ $(+\infty)$. In our case such restrictions on the asymptotic motion of the source will automatically follow from restrictions on the asymptotic behavior of $\Phi(\vec{x},t_0)$, $\dot{\Phi}(\vec{x},t_0)$ with $|\vec{x}|=r\to\infty$ by the initial-value solution of the total system.

Solving Eq. (2.2b) gives formally

$$\Phi_I(\vec{x},t;t_0)=\frac{\lambda}{4\pi r}[\theta(t-t_0-r)Q(t-r)+\theta(-t+t_0-r)Q(t+r)],\quad \theta(x)=\begin{cases}1 & \text{for } x>0\\ 0 & \text{for } x<0\end{cases} \tag{2.9}$$

while the homogeneous part of the field can be expressed by the initial values at $t=t_0$ as usual:

$$\Phi_H(\vec{x},t;t_0)=\int\frac{d^3x'}{4\pi|\vec{x}-\vec{x}'|}\left\{\theta(t-t_0)\left[\frac{d}{dt}\,\delta(t-t_0-|\vec{x}-\vec{x}'|)\Phi(\vec{x}',t_0)+\delta(t-t_0-|\vec{x}-\vec{x}'|)\dot{\Phi}(\vec{x}',t_0)\right]\right.$$
$$\left.+\,\theta(t_0-t)\left[\frac{d}{dt}\,\delta(t-t_0+|\vec{x}-\vec{x}'|)\Phi(\vec{x}',t_0)-\delta(t-t_0+|\vec{x}-\vec{x}'|)\dot{\Phi}(\vec{x}',t_0)\right]\right\}. \tag{2.10}$$

Because of the rotational symmetry of the source, only the spherical part of the field couples to the oscillator. Defining

$$\psi_{t_0}(r)=\frac{r}{4\pi}\int d\Omega\,\Phi(\vec{x},t_0),$$
$$\chi_{t_0}(r)=\frac{r}{4\pi}\int d\Omega\,\dot{\Phi}(\vec{x},t_0), \tag{2.11}$$

and inserting into Φ_H we obtain $[\epsilon(t)=\pm1,\text{ for } t\gtrless 0]$

$$\Phi_H(\vec{x},t;t_0)=\frac{1}{2r}\left[\epsilon(t-t_0)\int_{||t-t_0|-r|}^{|t-t_0|+r}dr'\chi_{t_0}(r')+\psi_{t_0}(|t-t_0|+r)-\epsilon(|t-t_0|-r)\psi_{t_0}(||t-t_0|-r|)\right]$$
$$+\text{nonspherical parts}. \tag{2.12}$$

Inspection of the total field, as given by Eqs. (2.9) and (2.12), shows that it is discontinuous along $r=|t-t_0|$. This "jump" in the field and its first derivative comes from the point structure of the source. We can achieve a regular solution (for $r\neq 0$) to Eq. (2.2b) by constraining the Cauchy data of Φ to display a certain singular behavior at $r=0$:

$$\psi_{t_0}(0)=\frac{\lambda}{4\pi}Q(t_0),$$
$$\chi_{t_0}(0)=\frac{\lambda}{4\pi}\dot{Q}(t_0). \tag{2.13}$$

Henceforth, we shall assume the continuity conditions (2.13) to be valid, which implies that we cannot completely dispose of ψ and χ anymore. Nevertheless if in what follows we write "$\psi=0$," this must always be understood as being outside an arbitrary small neighborhood of $r=0$.

Inserting for the total field from Eqs. (2.9) and (2.12) into Eq. (2.2a) leads to the radiation-reaction equation of the oscillator. Again, because of the point coupling, the self-field of the oscillator diverges and makes an infinite renormalization of the spring constant necessary:

$$\overline{\omega}^2=\lim_{r\to 0}\left(\omega_0^{\,2}-\frac{\lambda^2}{4\pi r}\right)>0. \tag{2.14}$$

Taking into account this renormalization and the continuity conditions (2.13) we obtain for the oscillator

$$\ddot{Q}(t)+\epsilon(t-t_0)2\Gamma\dot{Q}(t)+\overline{\omega}^2Q(t)=\lambda[\epsilon(t-t_0)\chi_{t_0}(|t-t_0|)+\psi'_{t_0}(|t-t_0|)] \tag{2.15}$$

(ψ' means derivative of ψ with respect to the argument).

The general solution of Eq. (2.15) can readily be written down as

$$Q(t)=e^{-\Gamma|t-t_0|}\left\{\left[\cos\omega(t-t_0)+\frac{\Gamma}{\omega}\sin\omega|t-t_0|\right]Q(t_0)+\frac{\sin\omega(t-t_0)}{\omega}\dot{Q}(t_0)\right\}$$
$$+\lambda\theta(t-t_0)\int_{t_0}^{t}dt'G_+(t-t')[\chi_{t_0}(t')+\psi'_{t_0}(t')]-\lambda\theta(t_0-t)\int_{t_0}^{t}dt'G_-(t-t')[-\chi_{t_0}(t')+\psi'_{t_0}(t')], \tag{2.16}$$

where

$$G_{\pm}(t)=e^{\mp\Gamma t}\frac{\sin\omega t}{\omega}, \quad \omega=(\bar{\omega}^2-\Gamma^2)^{1/2}, \quad 2\Gamma=\frac{\lambda^2}{4\pi}.$$

(We assume that $\bar{\omega}^2>\Gamma^2$ throughout this paper.) This is the complete solution for $Q(t)$ in terms of the initial values. Inserting Eq. (2.16) into Eq. (2.9) gives together with Eq. (2.12) the initial-value solution for the field. We have thus obtained the time development of the coupled system (2.2a) and (2.2b) in terms of the Cauchy data at $t=t_0$.

We now may ask the following question: Under which conditions does the energy of the oscillator go to zero for $t\to\infty$? The answer is simple: If we demand the field on the initial surface to fall off at spatial infinity such that

$$\lim_{r\to\infty}\psi'_{t_0}(r)=\lim_{r\to\infty}\chi_{t_0}(r)=0 \qquad (2.17)$$

[which implies that also $\lim_{r\to\infty}\psi'_t(r)=\lim_{r\to\infty}\chi_t(r)=0$ at any t], then we show in the Appendix that the integral in Eq. (2.16) vanishes with $t\to\pm\infty$ and therefore that

$$\lim_{t\to\pm\infty}Q(t)=\lim_{t\to\pm\infty}\dot{Q}(t)=0. \qquad (2.18)$$

More precise information about the decrease of the oscillator energy may be obtained from the Appendix and will be used in Sec. IV.

Now we look for solutions of our system with $\Phi_{in}=0$. Taking the limit $t_0\to-\infty$ in Eq. (2.9) one gets the usual retarded field

$$\Phi(\vec{x},t)=\Phi_{ret}(\vec{x},t)=\frac{\lambda}{4\pi}\frac{Q(t-r)}{r}. \qquad (2.19)$$

From the oscillator Eq. (2.15) we have for $\Phi_{in}=0$

$$\ddot{Q}(t)+2\Gamma\dot{Q}(t)+\bar{\omega}^2Q(t)=0. \qquad (2.20)$$

The solution of this equation behaves like $e^{-\Gamma t}$; hence $\Phi(\vec{x},t)\sim e^{-\Gamma(t-r)}/r$. This field diverges for $r\to\infty$ and clearly violates our asymptotic conditions (2.17). A similar argument holds if one requires $\Phi_{out}=0$. Therefore no solutions to the system, Eqs. (2.2a) and (2.2b), exist for which either $\Phi_{in}=0$ or $\Phi_{out}=0$ and conditions (2.17) are satisfied.

For obtaining the above results we assumed that the system is *closed*, i.e., Eqs. (2.2a) and (2.2b) are valid for all t. In an *open* system external forces not included in the dynamics may act on the particle for $t<t_0$, thus violating Eq. (2.2a). Let us restrict $Q(t)$ for $t<t_0$ by

$$\lim_{t\to-\infty}Q(t)=\lim_{t\to-\infty}\dot{Q}(t)=0. \qquad (2.21)$$

Demanding now that $\Phi=\Phi_{ret}$ for $t<t_0$, it is clear that (2.13) is valid. The time evolution for $t>t_0$ is now determined by Eqs. (2.2a) and (2.2b). Inspection of Eqs. (2.9) and (2.12) shows that $\Phi=\Phi_{ret}$ is propagated by these equations. From (2.15) it follows that

$$\lim_{t\to\infty}Q(t)=\lim_{t\to\infty}\dot{Q}(t)=0. \qquad (2.22)$$

Hence solutions with $\Phi_{in}=0$ exist for the open system. Solutions with $\Phi_{out}=0$ give rise to exponential growth of $Q(t)$ with $t\to\infty$ and are ruled out for the open system on the same grounds as for the closed one. Note that this notion of an open system introduces a time asymmetry.

III. THE MODEL IN ROBERTSON-WALKER (RW) SPACES WITH $k=0$

As a next step we consider our system to be embedded in a homogeneous isotropic universe with flat-space sections (RW space with $k=0$). The gravitational field is treated as external, i.e., the back reaction of our system on the metric of space-time is neglected. The symmetry of this space-time makes it possible to choose preferred coordinates $(t,\vec{x})$ such that the hypersurfaces $t=$ const are the surfaces of homogeneity and t measures the proper time along geodesics of particles being at rest with respect to these surfaces:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=dt^2-\mathcal{R}^2(t)d\vec{x}^2. \qquad (3.1)$$

$\mathcal{R}(t)\geq 0$ is a prescribed function *not* constrained to make $g_{\mu\nu}$ fulfill Einstein's equations with some standard source.

In order to specify our system we consider the oscillator to be nonrelativistic with respect to t. The scalar field should fulfill the conformally invariant wave equation in the space-time (3.1). This leads to the action

$$W=\tfrac{1}{2}\int dt\,(\dot{Q}^2-\omega_0{}^2Q^2)$$
$$+\lambda\int d^4x\sqrt{-g}\,\rho(\vec{x},t)Q(t)\Phi(\vec{x},t)$$
$$+\tfrac{1}{2}\int d^4x\sqrt{-g}\,(\Phi_{,\mu}\Phi_{,\nu}g^{\mu\nu}-\tfrac{1}{6}R\Phi^2), \qquad (3.2)$$

where $g\equiv\det g_{\mu\nu}$ and R is the (four-dimensional) curvature scalar.

We want the invariant size of the "charge distribution" not to be affected by the time development of the universe (as is the case for atoms, stars, etc.). This means that we require $\rho(\vec{x},t)$ to be normalized such that

$$\int d^3x\sqrt{-g}\,\rho(\vec{x},t)=\int d^3x\,\mathcal{R}^3(t)\rho(\vec{x},t)=1. \qquad (3.3)$$

Again taking the point limit in the coupling term $\mathcal{R}^3(t)\rho\to\delta^{(3)}(\vec{x})$ we obtain from Eq. (3.2)

$$\ddot{Q}(t)+\omega_0{}^2Q(t)=\lambda\Phi(\vec{0},t), \qquad (3.4a)$$

$$(\Box_g + \tfrac{1}{6}R)\Phi(\vec{x},t) = \lambda\,\frac{\delta^3(\vec{x})}{\mathcal{R}^3(t)}\,Q(t)\,, \tag{3.4b}$$

$$\Box_g \equiv \frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\,g^{\mu\nu}\,\frac{\partial}{\partial x^\nu}\right).$$

The solution of the scalar wave equation is most easily obtained by taking advantage of conformal invariance. Introducing a new time coordinate τ by

$$\tau = f_{t_0}(t) \equiv \int_{t_0}^{t} \frac{dt'}{\mathcal{R}(t')}\,, \tag{3.5}$$

the metric takes the form

$$ds^2 = \mathcal{R}^2(f_{t_0}^{-1}(\tau))(d\tau^2 - d\vec{x}^2) \tag{3.6}$$

[f^{-1} denotes the inverse function of f, $f_{t_0}^{-1}(f_{t_0}(t)) = t$].
Transforming Φ to $\overline{\Phi}$ by

$$\overline{\Phi}(\vec{x},\tau) \equiv \mathcal{R}(f_{t_0}^{-1}(\tau))\Phi(\vec{x}, f_{t_0}^{-1}(\tau)) \tag{3.7}$$

yields

$$\left(\frac{\partial^2}{\partial\tau^2} - \Delta\right)\overline{\Phi}(\vec{x},\tau) = \lambda\delta^{(3)}(\vec{x})Q(f_{t_0}^{-1}(\tau)) \tag{3.8}$$

where

$$\Delta = \frac{\partial}{\partial x^i}\,\frac{\partial}{\partial x^i}\,.$$

We therefore may quickly find Φ_I and Φ_H defined above by solving Eq. (3.8) for $\overline{\Phi}$. After defining

$$\begin{aligned}\psi_{t_0}(r) &= \mathcal{R}(t_0)r\int \frac{d\Omega}{4\pi}\,\Phi(\vec{x},t_0)\,,\\ \overline{\chi}_{t_0}(r) &= \frac{d}{dt}\left[\mathcal{R}(t)r\int\frac{d\Omega}{4\pi}\,\Phi(\vec{x},t)\right]\bigg|_{t=t_0}\,,\end{aligned} \tag{3.9}$$

the result is

$$\Phi_I(\vec{x},t;t_0) = \frac{\lambda}{4\pi\mathcal{R}(t)r}\left[\theta(f_{t_0}(t)-r)Q(f_{t_0}^{-1}(f_{t_0}(t)-r)) + \theta(f_{t_0}(t)+r)Q(f_{t_0}^{-1}(f_{t_0}(t)+r))\right] \tag{3.10}$$

and

$$\Phi_H(\vec{x},t;t_0) = \frac{1}{2r\mathcal{R}(t)}\left(\epsilon(t-t_0)\int_{||f_{t_0}(t)|-r|}^{|f_{t_0}(t)|+r} dr'\,\overline{\chi}_{t_0}(r')\mathcal{R}(t_0) + \psi_{t_0}(|f_{t_0}(t)|+r) - \epsilon(|f_{t_0}(t)|-r)\psi_{t_0}(||f_{t_0}(t)|-r|)\right)$$
$$+\ \text{nonspherical part.} \tag{3.11}$$

We shall henceforth omit the subscript t_0 in the expressions $f_{t_0}^{-1}(f_{t_0}(t)\pm r)$ since they can be shown to be independent of t_0. Continuity of $\Phi = \Phi_I + \Phi_H$ at $|f_{t_0}(t)| = r$ again implies

$$\begin{aligned}\psi_{t_0}(0) &= \frac{\lambda}{4\pi}\,Q(t_0)\,,\\ \overline{\chi}_{t_0}(0) &= \frac{\lambda}{4\pi}\,\dot{Q}(t_0)\,.\end{aligned} \tag{3.12}$$

To insert Φ into the right-hand side of the oscillator equation (3.4a) we again expand the argument of Q for small r,

$$\begin{aligned}\mathcal{R}^{-1}(t)Q(f^{-1}(f(t)-r)) &= \mathcal{R}^{-1}(t)[Q(t) - \dot{Q}(t)f^{-1\prime}(f(t))r + O(r^2)]\\ &= \mathcal{R}^{-1}(t)Q(t) - \dot{Q}(t)r + O(r^2)\,,\end{aligned} \tag{3.13}$$

and take the limit $r\to 0$. After a spring-constant renormalization

$$\overline{\omega}^2 = \lim_{r\to 0}\left(\omega_0^{\,2} - \frac{\lambda^2}{4\pi\mathcal{R}r}\right)\,, \tag{3.14}$$

this leads to

$$\ddot{Q}(t) + 2\Gamma\epsilon(t-t_0)\dot{Q}(t) + \overline{\omega}^2 Q(t) = \frac{\lambda}{\mathcal{R}(t)}[\psi'_{t_0}(|f_{t_0}(t)|) + \epsilon(t-t_0)\mathcal{R}(t_0)\overline{\chi}_{t_0}(|f_{t_0}(t)|)] \equiv g_{t_0}(t)\,. \tag{3.15}$$

Obviously the homogeneous part of Eq. (3.15) is identical to that of the corresponding equation in Minkowski space-time, whereas on the right-hand side $g_{t_0}(t)$, space-time curvature, manifests itself in two ways:

(i) An extra factor $\mathcal{R}^{-1}(t)$ enters.

(ii) The argument of ψ' and $\overline{\chi}$ is modified by the deformed null rays. The solution of (3.15) is

$$Q(t) = e^{-\Gamma|t-t_0|}\left\{\left[\cos\omega(t-t_0) + \frac{\Gamma}{\omega}\,\sin\omega|t-t_0|\right]Q(t_0) + \frac{\sin\omega(t-t_0)}{\omega}\,\dot{Q}(t_0)\right\}$$
$$+\ \lambda\theta(t-t_0)\int_{t_0}^{t} dt'G_+(t-t')g_{t_0}(t') + \lambda\theta(t_0-t)\int_{t}^{t_0} dt'G_-(t-t')g_{t_0}(t')\,. \tag{3.16}$$

IV. ASYMPTOTIC BEHAVIOR

In this section we study the long-time behavior of our system. It is of special interest to investigate under which conditions the motion of the oscillator is damped out. Clearly the oscillator can completely radiate its energy away only for $t\to\infty$. Therefore, we concentrate in this section on space-times for which t is not bounded in the future.

As in Minkowski space, we have to choose some asymptotic behavior for ψ and $\bar\chi$ at spatial infinity. We take

$$\begin{aligned}\psi'_{t_0}(r)&=O(r^{-\eta}),\\ \bar\chi_{t_0}(r)&=O(r^{-\eta}),\end{aligned}\tag{4.1}$$

where $\eta\geq 0$. This choice is again consistent with the time development, as can be checked from Eq. (3.11). Restricting ourselves to $\eta>0$ would confine the field to configurations where the "charge" occupies a privileged position in space. However, $\eta=0$ contains cases such as $\Phi\sim\cos(\vec{k}\cdot\vec{x})$, which are more in the spirit of the cosmological principle.

Now let us return to Eq. (3.16). The asymptotic behavior of $Q(t)$ is determined by the asymptotic behavior of

$$g_{t_0}(t)\equiv\frac{\psi'_{t_0}(f_{t_0}(t))+\bar\chi_{t_0}(f_{t_0}(t))\Re(t_0)}{\Re(t)}.\tag{4.2}$$

It follows from Eq. (4.1) that

$$\begin{aligned}g_{t_0}(t)&=O(\Re^{-1}(t)[f_{t_0}(t)]^{-\eta})\\ &=O\left(\frac{d}{dt}[f_{t_0}(t)]^{1-\eta}\right).\end{aligned}\tag{4.3}$$

Applying the results of the Appendix we infer that if $\lim_{t\to\infty}g_{t_0}(t)=0$, then

$$\lim_{t\to\infty}Q(t)=\lim_{t\to\infty}\dot Q(t)=0.\tag{4.4}$$

From this we see that the asymptotic behavior of the oscillator depends in general on the dynamics of the universe. In what follows we discuss this dependence in more detail.

First let us consider the case where the universe expands only to a finite "radius."

(1) Let

$$0<\lim_{t\to\infty}\Re(t)<\infty$$

and $\eta=0$. Then

$$\lim_{t\to\infty}g_{t_0}(t)\neq 0,$$

and therefore, in general, $Q(t)$ and $\dot Q(t)$ will *not* tend to zero. Since the limit does not always exist, one can look at the average limit to show that

$$\begin{aligned}\operatorname*{LIM}_{t\to\infty}Q(t)&\equiv\lim_{\gamma\to 0}\gamma\int_0^\infty dt\,e^{-\gamma t}Q(t)\\ &=\frac{1}{\Gamma^2+\omega^2}\operatorname*{LIM}_{t\to\infty}g_{t_0}(t),\\ \operatorname*{LIM}_{t\to\infty}\dot Q(t)&=\frac{1}{\Gamma^2+\omega^2}\operatorname*{LIM}_{t\to\infty}\dot g_{t_0}(t).\end{aligned}\tag{4.5}$$

(2) If

$$0<\lim_{t\to\infty}\Re(t)<\infty,$$

but $\eta>0$, we have

$$\lim_{t\to\infty}g_{t_0}(t)=0$$

and Eqs. (4.4) follow. The rate at which Q and $\dot Q$ tend to zero depends on that of $g_{t_0}(t)$. From our assumptions we deduce

$$[f_{t_0}(t)]^{-1}=O(t^{-1}),\tag{4.6}$$

and therefore, according to the Appendix that

$$Q(t)=O(t^{-\eta}).\tag{4.7}$$

The range of $\Re(t)$ considered in (1) and (2) contains two important subcases: (a) Minkowski space, where $0<\Re(t)=\text{const}$, which has been dealt with in Sec. II and (b) universes where $\Re(t)$ oscillates between finite radii. We see that the contracting phases of the universe do not in any way destroy the dissipative behavior which is displayed by the oscillator in Minkowski space.

(3) Consider now a universe which expands forever to infinite "radius," i.e.,

$$\lim_{t\to\infty}\Re(t)=\infty.$$

Then since $f_{t_0}(t)>0$ for $t>t_0$,

$$\lim_{t\to\infty}g_{t_0}(t)=\lim_{t\to\infty}\frac{[f_{t_0}(t)]^{-\eta}}{\Re(t)}=0,\tag{4.8}$$

and therefore damping occurs for *all* $\eta\geq 0$.

This is in contrast to case (1) and especially to Minkowski space. Quantitatively, damping for $\eta=0$ follows from the fact that, owing to the expansion of the universe, null rays reaching the oscillator in the remote future are not only more and more delayed but also their frequencies have an increasing red-shift.

To say something about the rate of damping, more information about the dynamics of the universe is needed: In the models falling into class (3) a future event horizon (FEH) for the oscillator can possibly exist. This will be the case if and only if[11]

$$\lim_{t\to\infty}f_{t_0}(t)<\infty.\tag{4.9}$$

(a) Models with FEH. The existence of an FEH

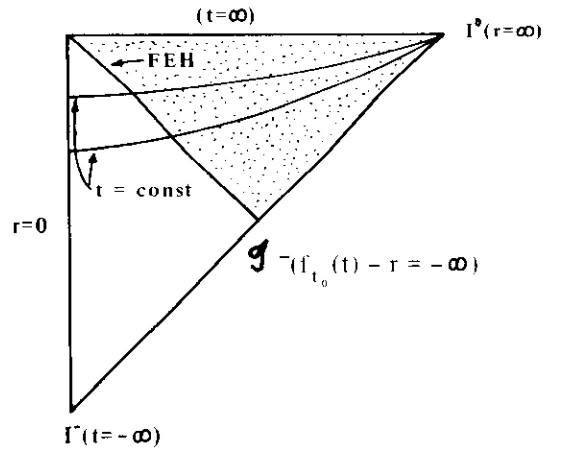

FIG. 1. Penrose diagram of expanding de Sitter space-time: $\mathcal{R}(t)=e^{t/\alpha}$ (see Ref. 13). It shows the FEH for the geodesic $r=0$; only that part of the initial surface $t=$const which lies inside the FEH influences the oscillator. In the dotted region Φ_{adv} vanishes; Φ_{ret} covers the whole space-time.

for the oscillator implies that the expansion is so fast that only signals coming from the initial surface $t=t_0$ within the radius $r=f_{t_0}(\infty)<\infty$ can ever reach the oscillator. In this case the asymptotic behavior of the field is irrelevant. The damping rate of the oscillator is independent of η. To make this clear we consider two examples.

Let $\mathcal{R}(t)=t^n$ with $n>1$ (the conformal diagram for this space is similar to that given in Fig. 1, see also Ref. 13); then the FEH of the oscillator is at

$$r=f_{t_0}(\infty)=\int_{t_0}^{\infty} t^{-n}dt=\frac{1}{n-1}\,t_0^{\,1-n}. \tag{4.10}$$

For large t the argument of ψ and $\overline{\chi}$ in Eq. (4.2)

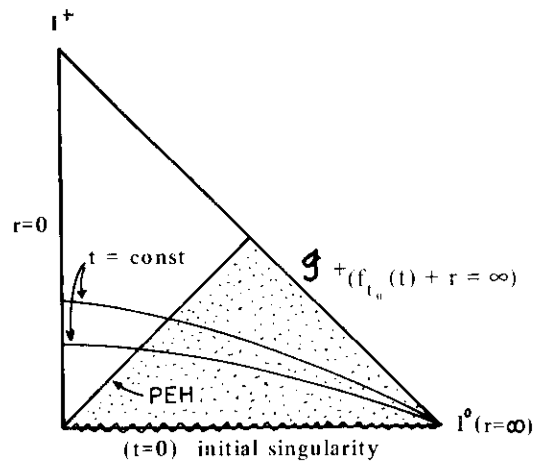

FIG. 2. Penrose diagram for $\mathcal{R}(t)=t^n$, $n<1$. It shows the existence of a PEH for the oscillator placed at $r=0$; in the dotted region Φ_{ret} is zero while Φ_{adv} covers the whole space-time.

approaches the horizon and

$$g_{t_0}(t)=O(\mathcal{R}^{-1}(t))=O(t^{-n}). \tag{4.11}$$

Again, from the Appendix, we then know (taking $\overline{g}=t^{-n}$) that

$$Q(t)=O(t^{-n}). \tag{4.12}$$

As a second example we take $\mathcal{R}(t)=e^{t/\alpha}$. This is the de Sitter space-time used in the steady-state theory[6,12] (the conformal diagram of this space is given in Fig. 1). The FEH is given by

$$r=f_{t_0}(\infty)=\alpha e^{-t_0/\alpha} \tag{4.13}$$

and therefore

$$g_{t_0}(t)=O(e^{-t/\alpha}), \tag{4.14}$$

which for $\Gamma>1/\alpha$ leads to

$$Q(t)=O(e^{-t/\alpha}). \tag{4.15}$$

(b) Models without FEH: $\lim_{t\to\infty} f_{t_0}(t)=\infty$. In this case the whole initial surface will eventually influence the oscillator, and the asymptotic behavior of the field is important. To illustrate this let $\mathcal{R}(t)=t^n$ with $n<1$ (see conformal diagram of this space, Fig. 2). For $n=\frac{2}{3}$ or $n=\frac{1}{2}$ this is the well-known Friedmann model for a dust or radiation universe. For this $\mathcal{R}(t)$ we have

$$f_{t_0}(t)=\frac{1}{1-n}\,(t^{1-n}-t_0^{\,1-n}). \tag{4.16}$$

Imposing the asymptotic conditions (4.1) on the initial fields gives

$$g_{t_0}(t)=t^{-n}O(f_{t_0}^{\,-\eta}(t))=O(t^{(n-1)\eta-n}). \tag{4.17}$$

This asymptotic behavior is also reflected in the damping rate for Q:

$$Q(t)=O(t^{(n-1)\eta-n}). \tag{4.18}$$

(4) Consider now a universe which in its late stage contracts to zero, i.e., $\lim_{t\to\infty}\mathcal{R}(t)=0$. This implies that $\lim_{t\to\infty} f_{t_0}(t)=\infty$. Damping will occur in this case only if $\eta>1$ and $\lim_{t\to\infty}(\dot{\mathcal{R}}/\mathcal{R})<\infty$, since then

$$\lim_{t\to\infty} g_{t_0}(t)=\lim_{t\to\infty}\frac{[f_{t_0}(t)]^{-\eta}}{\mathcal{R}(t)}$$

$$=\lim_{t\to\infty}\frac{\dot{\mathcal{R}}}{\mathcal{R}}\,[f_{t_0}(t)]^{1-\eta}=0. \tag{4.19}$$

As an example for a contracting universe we consider $\mathcal{R}(t)=e^{-t/\alpha}$. This is the time reverse of the de Sitter space-time. We have

$$f_{t_0}(t)=\alpha(e^{t/\alpha}-e^{t_0/\alpha}) \tag{4.20}$$

and

$$g_{t_0}(t)=O(e^{(t/\alpha)(1-\eta)}). \tag{4.21}$$

Here we see explicitly that the oscillator is damped if $\eta > 1$ independent of the rate of contraction α. This result is remarkable because even in an arbitrary fast contracting (open) universe there will be radiation damping, provided the field on the initial surface falls off with a power $\eta > 1$. In Table I we have listed the damping behavior for some important models falling into one of the discussed classes.

V. RETARDED AND ADVANCED FIELDS

In our definition of Φ_{in} and Φ_{out} as given in Sec. II, it is essential to know the range of the universal time coordinate. To find out if our system admits solutions with $\Phi_{in}=0$ ($\Phi_{out}=0$) we have to specify general properties of the dynamics of the universe in the past (future).

A. Retarded fields

We first investigate if the system admits solutions with Φ_{in} vanishing on the whole of space-time. The discussion is divided according to various types of time developments of the universe in the past.

1. "Big-bang" cosmologies

Suppose that the space-time is bounded in the past by an initial singularity, say at $t=0$, i.e.,

$$\mathcal{R}(0)=0\,. \tag{5.1}$$

Then Φ_{in} is formally defined as the homogeneous field for $t_0=0$,

$$\Phi_{in}(\vec{x},t) \equiv \lim_{t_0\to 0}\,[\,\Phi(\vec{x},t)-\Phi_I(\vec{x},t;t_0)]\,. \tag{5.2}$$

If we require this field to vanish, then the total field is given by the purely retarded field of the oscillator. By taking the limit $t_0 \to 0$ in Eq. (3.10) we have

$$\Phi(\vec{x},t)=\Phi_{ret}(\vec{x},t) = \frac{\lambda}{4\pi}\,\frac{1}{\mathcal{R}(t)r}\,Q(f^{-1}(f(t)-r))\theta(f_0(t)-r)\,. \tag{5.3}$$

From the oscillator equation for $t_0=0$ [(3.15)], which is homogeneous when $\Phi_{in}=0$, we infer the usual damped motion for all $t>0$:

$$Q(t)=e^{-\Gamma t}\left[\left(\cos\omega t-\Gamma\,\frac{\sin\omega t}{\omega}\right)Q(0)+\frac{\sin\omega t}{\omega}\,\dot{Q}(0)\right]. \tag{5.4}$$

Knowing this, we can insert Q into Eq. (4.3) thereby obtaining the total field in terms of the initial values $Q(0)$ and $\dot{Q}(0)$. The explicit form of the field will of course depend on the detailed dynamics of the metric. However, it can be inferred from rather general properties of $\mathcal{R}(t)$ whether or not the field is physically acceptable on the basis of conditions (4.1) with $\eta=0$.

(a) If the expansion of the universe near the big bang is such that

$$\lim_{t_0\to 0} f_{t_0}(t) = -\lim_{t\to 0} f_{t_0}(t)=\infty\,, \tag{5.5}$$

then the θ function in Eq. (5.3) is identical to 1. It can be inferred from the conformal structure of this space that the field produced by the oscillator at $t=0$ covers instantaneously the whole space.

TABLE I. Here it is shown how radiation damping depends on the dynamics of the universe and the asymptotic behavior of the initial field [Eq. (4.1)]. Further, one can read off whether solutions to the system with $\Phi_{in}=0$, $\Phi_{out}=0$, or $\Phi_{in}+\Phi_{out}=0$ exist for the considered cosmology. Note that the existence of purely retarded fields is tied to big-bang cosmologies.

$\mathcal{R}(t)$	Dynamics	Horizon	Damping for	Damping rate	$\Phi_{in}=0$	$\Phi_{out}=0$	$\Phi_{in}+\Phi_{out}=0$
$e^{-t/\alpha}$	Contracting de Sitter space	PEH	$\eta>1$	$e^{t(1-\eta)/\alpha}$ $\Gamma>(\eta-1)/\alpha$	No	No	No
const	Minkowski space	No	$\eta>0$	$t^{-\eta}$	No	No	No
t^n, $n<1$	Expanding Friedmann $n=\frac{2}{3}$ dust $n=\frac{1}{2}$ radiation	PEH	$\eta\geq 0$	$t^{(n-1)\eta-n}$	Yes	No	No
t^n, $n=1$	Expanding	No	$\eta\geq 0$	$(\ln t)^{-\eta}t^{-1}$	Yes	No	No
t^n, $n>1$	Expanding	FEH	Independent of η	t^{-n}	Yes	No	No
$e^{t/\alpha}$	Expanding de Sitter space	FEH	Independent of η	$e^{-t/\alpha}$ $\Gamma>1/\alpha$	No	No	No

(See Fig. 1.) The field on each hypersurface t = const will be nonvanishing for $0 \leq r < \infty$. Because of Eq. (5.5),

$$\lim_{r\to\infty} f^{-1}(f(t)-r)=0. \quad (5.6)$$

Thus, since f^{-1} is monotonic,

$$0 \leq f^{-1}(f(t)-r) \leq t, \quad (5.7)$$

which shows that

$$\psi_t(r)=\frac{\lambda}{4\pi}Q(f^{-1}(f(t)-r)) \quad (5.8)$$

is regular for all r. For ψ' and $\bar{\chi}$ we have

$$\psi_t'(r)=-\frac{\lambda}{4\pi}Q'(f^{-1}(f(t)-r))f^{-1\prime}(f(t)-r), \quad (5.9)$$

$$\bar{\chi}_t(r)=-\psi_t'(r)/\mathcal{R}(t). \quad (5.10)$$

Since

$$\lim_{r\to\infty} f^{-1\prime}(f(t)-r)=\lim_{r\to\infty}\mathcal{R}(f^{-1}(f(t)-r)) =0, \quad (5.11)$$

we have the following asymptotic behavior of the field for large r:

$$\psi_t(r)=\bar{\chi}_t(r)=O(r^0), \qquad \lim_{r\to\infty}\psi_t'(r)=0. \quad (5.12)$$

Therefore the field satisfies our conditions (4.1), and thus acceptable solutions to the system with $\Phi_{in}=0$ exist.

In what follows we pursue this kind of argument for various types of time developments of $\mathcal{R}(t)$.

(b) Existence of a PEH. If the expansion of the universe is such that

$$f_0(t)<\infty, \quad (5.13)$$

a past event horizon (PEH) exists. The PEH manifests itself in the θ function appearing in Eq. (5.3) for Φ_{ret}. If r varies from $r=0$ to the horizon $r = f_0(t)$, the range of the argument of Q in Eq. (5.3) is given by

$$t \geq f^{-1}(f(t)-r) \geq 0. \quad (5.14)$$

When differentiating the field (5.3) with respect to r and t to obtain ψ' and $\bar{\chi}$, respectively, no δ functions result because of our continuity conditions (3.12) for the field at $r=f_0(t)$. Therefore ψ, ψ', and $\bar{\chi}$ are finite inside and vanish outside the horizon. Thus, also in this case solutions to the system with $\Phi_{in}=0$ exist.

2. *Another possibility for a "finite-age" universe*

The other possibility for a finite-age universe is

$$\lim_{t\to 0}\mathcal{R}(t)=\infty. \quad (5.15)$$

This implies that the universe started with a contracting phase from infinity at $t=0$. From Eq. (5.15) we see that $f_0(t)<\infty$. Hence, a PEH always exists and the arguments used in the previous case can be applied. $\psi_t(r)$ is again given by Eq. (5.3) and is regular everywhere inside the horizon. The difference comes about when we look at ψ' and $\bar{\chi}$. Since now

$$\lim_{r\to f_0(t)} f^{-1\prime}(f(t)-r)=\lim_{r\to f_0(t)}\mathcal{R}(f^{-1}(f(t)-r)) =\infty, \quad (5.16)$$

ψ' and $\bar{\chi}$ diverge on the PEH. Roughly speaking, the infinite $\mathcal{R}$ at $t=0$ gives rise to an infinite blueshift on the horizon. We conclude that no acceptable solutions with $\Phi_{in}=0$ exist in this case.

Similarly to the above cases one can consider space-times which are not bounded in the past. From symmetry arguments we infer that this is equivalent to universes unbounded in the future together with the condition $\Phi_{out}=0$. We shall discuss these cases in the next subsection.

B. Advanced fields

We wish to impose the condition

$$\Phi_{out}=0. \quad (5.17)$$

In order to study the properties of these solutions we have to make assumptions about the dynamics of the universe in the distant future. Suppose that the universe is infinite in time, i.e., $\mathcal{R}(t)$ does not go to zero nor does it become infinite at a finite time in the future. Then the outgoing field is given by

$$\Phi_{out}(\vec{x},t)=\lim_{t_0\to\infty}[\Phi(\vec{x},t)-\Phi_I(\vec{x},t;t_0)]. \quad (5.18)$$

Imposing condition (5.17), the total field is obtained from Eq. (3.10) for $t_0\to\infty$:

$$\Phi(\vec{x},t)=\Phi_{adv}(\vec{x},t) =\frac{\lambda}{4\pi}Q(f^{-1}(f(t)+r))\theta(-f_\infty(t)-r). \quad (5.19)$$

The oscillator equation (3.15) for this case gives an "antidamped" motion for all values of t:

$$Q(t)=e^{\Gamma t}\left[\left(\cos\omega t-\frac{\Gamma}{\omega}\sin\omega t\right)Q(0)+\frac{\sin\omega t}{\omega}\dot{Q}(0)\right]. \quad (5.20)$$

Again we have to distinguish between $f_\infty(t)>-\infty$ and $f_\infty(t)=-\infty$. If

$$f_\infty(t)=-f_t(\infty)=-\infty, \quad (5.21)$$

no FEH exists. This is certainly the case if $\lim_{t\to\infty}\mathcal{R}(t)<\infty$, but can happen also if $\lim_{t\to\infty}\mathcal{R}(t)=\infty$. In this case the expression for the field contains no θ function, and $\psi_t(r)$ is given by

$$\psi_t(r)=\frac{\lambda}{4\pi}Q(f^{-1}(f(t)+r))\propto e^{\Gamma f^{-1}(f(t)+r)}. \quad (5.22)$$

However,

$$\lim_{r\to\infty} f^{-1}(f(t)+r)=\infty\,. \quad (5.23)$$

Thus on every hypersurface $t=\text{const}$ the field blows up with $r\to\infty$. If, on the contrary,

$$f_\infty(t)>-\infty\,, \quad (5.24)$$

an FEH exists. Now the θ function in Eq. (5.19) cuts the fields off at $r=f_t(\infty)$. Nevertheless,

$$\lim_{r\to f_t(\infty)} f^{-1}(f(t)+r)=\infty\,. \quad (5.25)$$

Therefore the field diverges at the FEH. We conclude that no acceptable solutions with $\Phi_{\text{out}}=0$ exist if the universe is not bounded in time in the future. Note that this includes not only ever-expanding models but also contracting universes such as the contracting de Sitter model.

We still have to discuss universes which are finite in time in the future. This happens if

$$\lim_{t\to T}\mathcal{R}(t)=0 \text{ or } \lim_{t\to T}\mathcal{R}(t)=\infty\,. \quad (5.26)$$

However, they are just the time reverse of cases where

$$\lim_{t\to 0}\mathcal{R}(t)=0 \text{ or } \lim_{t\to 0}\mathcal{R}(t)=\infty\,,$$

with $\Phi_{\text{in}}=0$, which have been treated at the beginning of this section. Therefore the same results hold for the present case.

C. The Wheeler-Feynman condition

Finally we require the Wheeler-Feynman condition

$$\Phi_{\text{in}}+\Phi_{\text{out}}=0 \quad (5.27)$$

to be satisfied. This means that the field can be written as

$$\Phi=\tfrac{1}{2}(\Phi_{\text{ret}}+\Phi_{\text{adv}}). \quad (5.28)$$

Here the damping term cancels an antidamping term, so that we have a pure periodic motion for the oscillator

$$Q(t)=\cos\omega t\,Q(0)+\frac{\sin\omega t}{\omega}\dot{Q}(0)\,. \quad (5.29)$$

As an example we consider a universe for which the range of t is $0\le t<\infty$. Then the total field is

$$\Phi(\vec{x},t)=\frac{\lambda}{8\pi}\frac{1}{\mathcal{R}(t)r}[Q(f^{-1}(f(t)-r))\theta(f_0(t)-r)+Q(f^{-1}(f(t)+r))\theta(-f_\infty(t)-r)]\,. \quad (5.30)$$

Let us assume that the universe has a PEH but no FEH (e.g. Friedmann model with $k=0$), which means that

$$f_0(t)<\infty \text{ but } f_\infty(t)=-\infty\,.$$

Then

$$\psi_t(r)=\frac{\lambda}{8\pi}[Q(f^{-1}(f(t)-r))\theta(f_0(t)-r)+Q(f^{-1}(f(t)+r))] \quad (5.31)$$

and

$$\lim_{r\to\infty}\psi_t(r)=0+Q(\infty)\,.$$

Since $Q(t)$ is periodic, ψ remains finite. Taking the derivative with respect to r gives (omitting the δ terms)

$$\psi_t'(r)=\frac{\lambda}{8\pi}[-Q'(f^{-1}(f(t)-r))\mathcal{R}(f^{-1}(f(t)-r))\times\theta(f_0(t)-r)+Q'(f^{-1}(f(t)+r))\mathcal{R}(f^{-1}(f(t)+r))] \quad (5.32)$$

and

$$\lim_{r\to\infty}\psi_t'(r)=0+Q'(\infty)\mathcal{R}(\infty)\,. \quad (5.33)$$

From this we learn that ψ' will be finite only if $\mathcal{R}(t)$ remains bounded as $t\to\infty$. The same condition must hold for $\bar{\chi}$ to be finite. It is straightforward to show that this result is valid for all cases. Thus, solutions with condition (5.27) are possible for universes where $\mathcal{R}(t)$ remains finite for all times.

An overview of the results of this section is given in Table II. Applying these results to the cosmological models considered in Table I, one can deduce the existence or nonexistence of purely retarded or purely advanced fields listed there. For two of the cases we give the explicit form for Φ_{ret} and Φ_{adv}.

(a) Expanding de Sitter model: $\mathcal{R}(t)=e^{t/\alpha}$, $-\infty<t<\infty$, and an FEH exists, $f_t(\infty)=\alpha e^{-t/\alpha}$ but no PEH, $f_{-\infty}(t)=\infty$. Retarded and advanced fields are given by

$$\Phi_{\text{ret}}(\vec{x},t)=\frac{\lambda}{4\pi}\frac{e^{-t/\alpha}}{r}Q(-\alpha\ln(e^{-t/\alpha}+r/\alpha))\,,$$

$$\Phi_{\text{adv}}(\vec{x},t)=\frac{\lambda}{4\pi}\frac{e^{-t/\alpha}}{r}Q(-\alpha\ln(e^{-t/\alpha}-r/\alpha))\times\theta(\alpha e^{-t/\alpha}-r)\,,$$

TABLE II. Here it is shown whether purely retarded or advanced solutions exist in our system. Note that the existence of retarded (advanced) fields depends only on $\mathfrak{R}(t)$ in the remote past (future). For $\Phi_{in}+\Phi_{out}=0$ solutions exist only if $\mathfrak{R}(t)<\infty$ for all values of t.

Restriction on t	$\mathfrak{R}(t)$ past	Horizon	Solutions with $\Phi_{in}=0$
$0\le t$	$\lim_{t\to 0}\mathfrak{R}(t)=0$	PEH	yes
	$=0$	no PEH	yes
	$=\infty$	no PEH	no
$-\infty<t$	$\lim_{t\to-\infty}\mathfrak{R}(t)<\infty$	no PEH	no
	$=\infty$	PEH	no
	$=\infty$	no PEH	no

Restriction on t	$\mathfrak{R}(t)$ future	Horizon	Solutions with $\Phi_{out}=0$
$t\le T$	$\lim_{t\to T}\mathfrak{R}(t)=0$	FEH	yes
	$=0$	no FEH	yes
	$=\infty$	no FEH	no
$t<\infty$	$\lim_{t\to\infty}\mathfrak{R}(t)<\infty$	no FEH	no
	$=\infty$	FEH	no
	$=\infty$	no FEH	no

which diverge for $r\to\infty$ and $r\to\alpha e^{-t/\alpha}$, respectively. Therefore no solutions exist for the boundary conditions

$$\Phi_{in}=0 \text{ or } \Phi_{out}=0 \text{ or } \Phi_{in}+\Phi_{out}=0$$

(see Fig. 1.)

(b) $\mathfrak{R}(t)=t^n,\ n<1$. $f_0(t)<\infty$ thus a PEH exists, but no FEH since $-f_\infty(t)=f_t(\infty)=\infty$. From our previous investigations we conclude immediately that solutions with $\Phi_{in}=0$ exist, but no solutions with $\Phi_{out}=0$ or $\Phi_{in}+\Phi_{out}=0$. This can also be seen by writing down Φ_{ret} and Φ_{adv} explicitly,

$$\Phi_{ret}(\vec{x},t)=\frac{\lambda}{4\pi}\frac{1}{t^n r}Q([t^{1-n}-(1-n)r]^{1/(1-n)})$$

$$\times\theta\left(\frac{1}{1-n}t^{1-n}-r\right),$$

$$\Phi_{adv}(\vec{x},t)=\frac{\lambda}{4\pi}\frac{1}{t^n r}Q([t^{1-n}+(1-n)r]^{1/(1-n)}),$$

where in the first case $Q\sim e^{-\Gamma t}$, and in the second case $Q\sim e^{\Gamma t}$ (see Fig. 2.)

VI. ROBERTSON-WALKER MODELS WITH $k=-1$

Here we take for the background metric

$$ds^2=dt^2-\mathfrak{R}^2(t)\left(\frac{dr^2}{1+r^2}+r^2d\Omega^2\right). \tag{6.1}$$

The dynamical equations for our model are again given by Eqs. (3.4a) and (3.4b), where $\Box_g$ refers now to the metric Eq. (6.1), and therefore the field equations for the spherical part of the field read

$$\left\{\frac{\partial^2}{\partial t^2}+\frac{3\dot{\mathfrak{R}}(t)}{\mathfrak{R}(t)}\frac{\partial}{\partial t}-\frac{1}{\mathfrak{R}^2(t)}\frac{(1+r^2)^{1/2}}{r^2}\frac{\partial}{\partial r}\left[(1+r^2)^{1/2}r^2\frac{\partial}{\partial r}\right]+\frac{\mathfrak{R}(t)\ddot{\mathfrak{R}}(t)+\dot{\mathfrak{R}}^2(t)-1}{\mathfrak{R}^2(t)}\right\}\Phi(r,t)=\frac{\lambda}{4\pi}\frac{\delta(r)}{r^2\mathfrak{R}^3(t)}Q(t). \tag{6.2}$$

Defining ψ_{t_0} and $\bar{\chi}_{t_0}$ as in Sec. III, Eq. (3.9), it may be checked that Φ_I and Φ_H are given by

$$\Phi_I(r,t;t_0)=\frac{\lambda}{4\pi\mathfrak{R}(t)r}\left[Q(f^{-1}(f(t)-\sinh^{-1}r))\,\theta(f_{t_0}(t)-\sinh^{-1}r)\right.$$

$$\left.+Q(f^{-1}(f(t)+\sinh^{-1}r))\,\theta(f_{t_0}(t)+\sinh^{-1}r)\right], \tag{6.3}$$

$$\Phi_H(r,t;t_0)=\frac{1}{2r\mathfrak{R}(t)}\left(\epsilon(t-t_0)\int_{||f_{t_0}(t)|-\sinh^{-1}r|}^{f_{t_0}(t)+\sinh^{-1}r} dr'\mathfrak{R}(t_0)\bar{\chi}_{t_0}(\sinh r)\right.$$
$$\left.+\psi_{t_0}(\sinh||f_{t_0}(t)|+\sinh^{-1}r|)-\epsilon(|f_{t_0}(t)|-\sinh^{-1}r)\psi_{t_0}(\sinh||f_{t_0}(t)|-\sinh^{-1}r|)\right). \quad (6.4)$$

Continuity at $r=|f_{t_0}(t)|$ again demands

$$\psi_{t_0}(0)=\frac{\lambda}{4\pi}Q(t_0),$$
$$\bar{\chi}_{t_0}(0)=\frac{\lambda}{4\pi}\dot{Q}(t_0). \quad (6.5)$$

By inserting for Φ from Eqs. (6.3) and (6.4) into Eq. (3.4b) and using Eq. (6.5) we obtain the radiation-reaction equation for the oscillator,

$$\ddot{Q}(t)+\epsilon(t-t_0)\dot{Q}(t)+\bar{\omega}^2Q(t)=\frac{\lambda}{\mathfrak{R}(t)}[\psi'_{t_0}(|\sinh f_{t_0}(t)|)\cosh f_{t_0}(t)+\epsilon(t-t_0)\mathfrak{R}(t_0)\bar{\chi}(|\sinh f_{t_0}(t)|)]. \quad (6.6)$$

In this space suitable (i.e., consistent with the time development) conditions for ψ and $\bar{\chi}$ are

$$\left.\begin{array}{l}\psi_{t_0}(r)=O(r^{-\eta}),\\ \psi'_{t_0}(r)=O(r^{-\eta-1}),\\ \bar{\chi}_t(r)=O(r^{-\eta}).\end{array}\right\}\ \eta\geq 0 \quad (6.7)$$

We state without proof that if the asymptotic conditions (6.7) are used to analyze the solutions of Eq. (6.6), all the results of Secs. IV and V remain valid.

ACKNOWLEDGMENTS

We thank Dr. A. Pflug and Dr. H. Urbantke for helpful discussions.

APPENDIX

Set

$$h(t)\equiv\int_0^t dt'e^{-\Gamma(t-t')}\frac{\sin\omega(t-t')}{\omega}g(t'). \quad (A1)$$

Obviously

$$|h(t)|\leq\frac{1}{\omega}e^{-\Gamma t}\int_0^t dt'e^{\Gamma t'}|g(t')|. \quad (A2)$$

We assume that

$$\lim_{t\to\infty}g(t)=0. \quad (A3)$$

Then there is a t_0 for all $\epsilon>0$ such that

$$|h(t)|<e^{-\Gamma t}\int_0^{t_0}dt'e^{\Gamma t'}|g(t')| + e^{-\Gamma t}\int_{t_0}^t dt'e^{\Gamma t'}\epsilon \quad \text{for } t>t_0. \quad (A4)$$

The first term in (A4) vanishes with $t\to\infty$, whereas the second term is given by

$$e^{-\Gamma t}\frac{e^{\Gamma t}-e^{\Gamma t_0}}{\Gamma}\epsilon<\frac{\epsilon}{\Gamma}.$$

Hence

$$\lim_{t\to\infty}h(t)=0 \quad (A5)$$

follows from (A3).

Now assume that there is a function $\bar{g}(t)$ with $\lim_{t\to\infty}\bar{g}(t)=0$ and

$$\lim_{t\to\infty}\frac{\bar{g}'(t)}{\bar{g}(t)}=0$$

such that $\lim_{t\to\infty}\bar{g}(t)e^{\Gamma t}=\infty$ and

$$g(t)=O(\bar{g}(t)). \quad (A6)$$

Then use of l' Hospital's rule quickly shows that also

$$h(t)=O(\bar{g}(t)). \quad (A7)$$

*Work supported in part by Fonds zur Förderung der wissenschaftlichen Forschung in Österreich, Projekt Nr. 2911.

†Publication charge supported by Fundacion Federico S.A. AR76/1.

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