

Radiation Damping as an Initial Value Problem*

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Using a simple, exactly soluble model for the interaction of one particle and a scalar field Φ , we discuss the problem of radiation reaction in terms of the initial value solution. We show that if the Cauchy data of the field fall off at spatial infinity in such a way that the field has finite energy, the particle motion is damped for $t \rightarrow \infty$. Further, we point out that no solutions with finite field energy exist for the boundary conditions $\Phi^{\text{out}} = 0$ and $\Phi^{\text{in}} + \Phi^{\text{out}} = 0$. For $\Phi^{\text{in}} = 0$, nontrivial solutions exist only if it is assumed that the system has been open in the past of some initial hypersurface.

1. INTRODUCTION

Since the days of the famous Einstein-Ritz controversy [1] it has been widely believed that classical electrodynamics, because of its symmetry under time inversion, does not necessarily predict the irreversible features of radiation phenomena observed in nature. We find essentially two roots for this opinion.

(1) There exist solutions of Maxwell's equations for given sources of the so-called (a) retarded type, (b) advanced type, or (c) certain combinations of both. To obtain a unique solution one has to impose suitable boundary conditions (e.g. Sommerfeld's condition of outgoing radiation) which cannot be derived from the theory. It is sometimes argued that boundary conditions leading to (a) imply radiation damping on the particle level, while those leading to (b) imply antidamping.

(2) In an action-at-a-distance theory, usually attributed to Fokker or Wheeler and Feynman, one tries to set up a relativistically invariant framework for electrodynamics, where the field degrees of freedom are at least formally eliminated and only particle variables appear. These particles feel no electromagnetic self-interaction, and, in the language of field theory, interact with one another via half-retarded plus half-advanced potentials. Therefore the theory

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is time-symmetrical. However, a classical theory which is time-symmetrical on the *particle level* clearly cannot describe radiation damping. Therefore people looked out for a “source of time-asymmetry” and found it within thermodynamics [2, 3] or nonstatic (expanding) cosmological models (see e.g. [4, 5, 6]).

Without judging this program, we wish to emphasize that field theory in flat spacetime, viewed as an initial value problem for *particle* and *field*, is already sufficient to deduce the dissipative features of radiation phenomena. Although this in itself should be a well-known fact (see e.g. [7]) no detailed discussion seems to exist in the literature of the connection of such an initial value problem with the boundary conditions, usually either imposed from the beginning (see [1]) or “derived” from thermodynamics or cosmology (see [2]). Usually radiation reaction is discussed on the basis of a “mixed” problem: One has a radiation reaction equation for the particle in which the incoming field plays the role of an external force e.g., the Lorentz–Dirac equation for a point charge interacting with the electromagnetic field [8]. A particular solution is determined by prescribing the *incoming field* and the *initial data for the particle*. Our approach is to discuss radiation damping as an initial value problem for *particle* and *field*. We use a simple, exactly soluble model for one particle interacting with a relativistic field. It consists of a one-dimensional, nonrelativistic, harmonic oscillator, coupled to a scalar, massless field in three space dimensions and seems to have been first considered and formally solved by Schwabl and Thirring [9]. The model parallels electrodynamics in the dipole approximation as discussed by Kramers [10] and van Kampen [11]. With slight modifications nearly all statements of this paper can probably be proved for the latter model also. The initial value solution for the electromagnetic case has been given by one of the authors [12] and more recently in [13]. Since, however, we wish to extend this work to more complex situations where there is hope to do the calculations only when the particle is shrunk to a point and since our model is more well-behaving in this limit than the corresponding electrodynamic model (due to the presence of “runaway-solutions” in electromagnetism) we preferred the scalar model. The price we have to pay for this choice is that the uniqueness property of the Cauchy problem is proved less easily because the conserved energy in this model is not manifestly positive.

In Section II the model is introduced and solved by use of Laplace-transformation (the formalism used in [9] is not suitable for our purposes). The exact solution for the coupled system of equations is given in terms of the Cauchy data on the initial hypersurface. Uniqueness of the solution is proved in Appendix A.

In Section III we study the time development of the system for $t \rightarrow \infty$. It is shown, that, if the field and its time derivative on the initial surface decreases with spatial distance $r \rightarrow \infty$ in such a manner that the finiteness of the field energy

variable. This shows that under physically reasonable conditions the particle in the course of time radiates all its energy away.

In Section IV the solution is calculated analytically in the limiting case where the interaction between field and particle is confined to a point (the "point-limit"). This limit is convenient for discussing the connection between boundary conditions and initial values. However, some care must be taken because the field becomes singular and an infinite renormalization of the spring-constant of the oscillator is required.

Section V is the main part of this work. We discuss whether solutions of our initial value problem exist which satisfy one of the following *boundary conditions*: the field is given by the

- (a) retarded ("conventional electrodynamics"),
- (b) advanced ("time-reversed electrodynamics"),

(c) half-retarded plus half-advanced ("Wheeler-Feynman electrodynamics"), solution of the wave equation. Since such boundary conditions are clearly tied to a knowledge of the particle variable for all times, assumptions have to be made concerning the *preparation of the initial values*. This leads us to the notion of a *closed system* which satisfies the field and particle equations for all times, and that of an *open system* which is subject to external influence for $t < 0$. It is shown that for the closed system (a), (b), (c) are inconsistent with the assumption of finite energy. For the open system (a) is consistent, while (b) and (c) are not.

We wish to point out that the way in which these conclusions are arrived at suggests that they are tied to a model where a particle that is bound for *all times* interacts with radiation. As typical examples of this we have in mind: an atom, an antenna, a star or perhaps a binary star. Our conclusions are definitely inapplicable to situations like scattering or capture of particles coupled to radiation. In systems of the latter kind the arrow of time enters differently.

II. THE MODEL AND ITS SOLUTION

Our starting point is a one dimensional, nonrelativistic harmonic oscillator coupled to a scalar field. This model is defined by the following coupled system of equations ($c = 1$)

$$\dot{Q}(t) + \omega_0^2 Q(t) = \lambda \int d^3x \rho(\mathbf{x}) \Phi(\mathbf{x}, t), \tag{2.1}$$

$$\square \Phi(\mathbf{x}, t) = \lambda \sigma(\mathbf{x}) O(t). \tag{2.2}$$

where $\dot{Q} = (d/dt) Q(t)$, $\square = (\partial^2/\partial t^2) - \Delta$ and λ is a coupling constant. The peculiarity of this model is that the interaction between oscillator and field is determined by a prescribed density $\rho(\mathbf{x})$ which is independent of the particle variable $Q(t)$. This corresponds to the dipole approximation in the electromagnetic case and is the reason for the system being exactly soluble.

The aim of this section is to solve the coupled system of equations in terms of the values of Q , \dot{Q} , Φ , $\dot{\Phi}$ on an initial spacelike hypersurface, say $t = 0$, for $t > 0$. A suitable way to introduce these initial values is to Laplace transform Q and Φ with respect to t :

$$q(s) = \int_0^\infty dt e^{-st} Q(t), \quad (2.3)$$

$$\phi(\mathbf{x}, s) = \int_0^\infty dt e^{-st} \Phi(\mathbf{x}, t). \quad (2.4)$$

Multiplying Eq. (2.1) with e^{-st} , integrating over $0 < t < \infty$ and performing some partial integrations we obtain

$$(s^2 + \omega_0^2) q(s) = sQ(0) + \dot{Q}(0) + \lambda \int d^3x \rho(\mathbf{x}) \phi(\mathbf{x}, s). \quad (2.5)$$

Doing the same for Eq. (2.2) gives

$$(s^2 - \Delta) \phi(\mathbf{x}, s) = s\Phi(\mathbf{x}, 0) + \dot{\Phi}(\mathbf{x}, 0) + \lambda\rho(\mathbf{x}) q(s). \quad (2.6)$$

Solving (2.6) for $\phi(\mathbf{x}, s)$ yields,

$$\phi(\mathbf{x}, s) = \int d^3x' \frac{e^{-s|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \{\lambda\rho(\mathbf{x}') q(s) + s\Phi(\mathbf{x}', 0) + \dot{\Phi}(\mathbf{x}', 0)\}. \quad (2.7)$$

Inserting for $\phi(\mathbf{x}, s)$ from Eq. (2.7) into the right-hand side of Eq. (2.5) gives

$$C(s) q(s) = sQ(0) + \dot{Q}(0) + \lambda \int d^3x d^3x' \rho(\mathbf{x}) \frac{e^{-s|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \cdot [s\Phi(\mathbf{x}', 0) + \dot{\Phi}(\mathbf{x}', 0)], \quad (2.8)$$

where $C(s)$ is defined by

$$\begin{aligned} C(s) &= s^2 + \omega_0^2 - \frac{\lambda^2}{4\pi} \int d^3x d^3x' \rho(\mathbf{x}) \rho(\mathbf{x}') \frac{e^{-s|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} \\ &= s^2 + \bar{\omega}^2 - 2\Gamma \int d^3x d^3x' \rho(\mathbf{x}) \rho(\mathbf{x}') \frac{e^{-s|\mathbf{x}-\mathbf{x}'|} - 1}{|\mathbf{x}-\mathbf{x}'|}. \end{aligned} \quad (2.9)$$

Here we have defined $\Gamma = \lambda^2/8\pi$ and

$$\bar{\omega}^2 = \omega_0^2 - 2\Gamma \int d^3x d^3x' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \quad (2.10)$$

plays the role of a renormalized spring constant of the oscillator due to self-interaction. We shall always assume ω_0^2 , I and $\rho(\mathbf{x})$ to be arranged in such a way that $\bar{\omega}^2 > 0$. In the "point-limit" $\rho(\mathbf{x}) \rightarrow \delta^3(\mathbf{x})$ this requires $\omega_0^2 \rightarrow \infty$. Finally inserting for $q(s)$ from Eq. (2.8) into Eq. (2.7) gives for the field

$$\begin{aligned} \phi(\mathbf{x}, s) = & \lambda \int d^3x' \frac{e^{-s|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \rho(\mathbf{x}') C^{-1}(s) \left\{ sQ(0) + \dot{Q}(0) \right. \\ & \left. + \lambda \int d^3u d^3v \rho(\mathbf{u}) \frac{e^{-s|\mathbf{u}-\mathbf{v}|}}{4\pi|\mathbf{u}-\mathbf{v}|} [s\Phi(\mathbf{v}, 0) + \dot{\Phi}(\mathbf{v}, 0)] \right\} + \phi_H(\mathbf{x}, s) \end{aligned} \quad (2.11)$$

where $\phi_H(\mathbf{x}, s) = \int d^3x' [e^{-s|\mathbf{x}-\mathbf{x}'|}/4\pi|\mathbf{x}-\mathbf{x}'|] [s\Phi(\mathbf{x}; 0) + \dot{\Phi}(\mathbf{x}', 0)]$ is the homogeneous solution of Eq. (2.2) with initial values $\Phi(\mathbf{x}, 0)$, $\dot{\Phi}(\mathbf{x}, 0)$. Equations (2.8) and (2.11) represent the solution of our model for arbitrary Cauchy data in the s -space. In t -space we have by the standard inversion formula [14]

$$Q(t) = \frac{1}{2\pi i} \int_{(b)} ds e^{st} q(s), \quad (2.12)$$

$$\Phi(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{(b)} ds e^{st} \phi(\mathbf{x}, s), \quad (2.13)$$

where (b) is an integration path in the complex s -plane parallel to the imaginary axis to the right of all singularities of the integrand. Now inspection of Eqs. (2.8), (2.11) shows that the distributions of zeros of $C(s)$ will be essential. In Appendix B it is shown that all of them lie to the left of the imaginary axis if $\bar{\omega}^2 > 0$.

Having constructed a solution to the Cauchy problem for Eqs. (2.1, 2) we have automatically solved the existence problem for the solutions of (2.1, 2). That this solution is unique is proved in Appendix A.

III. THE LIMIT $t \rightarrow \infty$

In this section we examine the time development of the system for $t \rightarrow \infty$. From now on we assume that $\rho(\mathbf{x})$ depends only upon the radial distance $|\mathbf{x}| = r$ and is of the special form

$$\rho(\mathbf{x}) = \rho(r) = (M^2/4\pi)(e^{-Mr}/r) \quad \left(\int d^3x \rho(\mathbf{x}) = 1 \right). \quad (3.1)$$

Due to the rotational symmetry of $\rho(r)$ only the s -wave part of the field couples to the oscillator. Moreover, the form (3.1) has the advantage that $C(s)$, defined in Eq. (2.9) can easily be evaluated to give

$$C(s) = \omega_0^2 + \bar{\omega}^2 + \frac{1}{2} M^2 (c \pm 2M)/(c \pm M)^2 \quad (3.2)$$

which has only a finite number of zeros. We define

$$\psi(r) = \frac{r}{4\pi} \int d\Omega \Phi(\mathbf{x}, 0), \quad \chi(r) = \frac{r}{4\pi} \int d\Omega \dot{\Phi}(\mathbf{x}, 0). \quad (3.3)$$

Inserting Eq. (3.3) into Eq. (2.8) and integrating by standard methods gives

$$q(s) = C^{-1}(s) \left\{ sQ(0) + \dot{Q}(0) + \lambda M^2 \int_0^\infty dr \frac{e^{-sr} - e^{-Mr}}{M^2 - s^2} [s\psi(r) + \chi(r)] \right\}. \quad (3.4)$$

The integral in Eq. (3.4), and therefore the behavior of $Q(t)$ for large t depends crucially on the behavior of $\psi(r)$ and $\chi(r)$ for large values of r . We assume the following behavior of the field for large values of r on the initial surface:

$$\Phi(\mathbf{x}, 0) = O(1/r), \quad \nabla\Phi(\mathbf{x}, 0) = O(1/r^2), \quad \dot{\Phi}(\mathbf{x}, 0) = O(1/r^2). \quad (3.5)$$

These conditions imply that

$$\lim_{r \rightarrow \infty} \frac{d}{dr} \psi(r) = \lim_{r \rightarrow \infty} \chi(r) = 0. \quad (3.6)$$

Also they are sufficient conditions that the energy stored in the field is finite

$$E_\Phi = \frac{1}{2} \int d^3x [\dot{\Phi}^2 + (\nabla\Phi)^2] < \infty. \quad (3.7)$$

(For the point-limit, where Φ becomes singular at $r = 0$, see the end of Appendix C.) From general theorems on Laplace transforms we know that

$$Q(t) = \frac{1}{2\pi i} \int_{(b)} ds e^{st} q(s) = \sum \text{Residues}[e^{st} q(s)], \quad t \geq 0. \quad (3.8)$$

We now insert Eq. (3.4) into (3.8). Consider first the terms containing $Q(0)$ and $\dot{Q}(0)$. According to Eq. (3.2), $C(s)$ has generally four distinct zeros s_i (if double zeros occur our requirements Eq. (3.5) have to be strengthened). By the theorem proven in Appendix B all of them have a negative real part. Therefore these terms which are proportional to $e^{s_i t}$ vanish with $t \rightarrow \infty$ together with their time derivatives. Next we examine the last term of Eq. (3.4). We obtain:

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(b)} ds \frac{e^{st}}{C(s)(M^2 - s^2)} \int_0^\infty dr (e^{-sr} - e^{-Mr}) \chi(r) \\ &= \sum_{i=1}^4 \frac{e^{s_i t}}{C'(s_i)(M^2 - s_i^2)} \left\{ \int_0^t dr e^{-s_i r} \chi(r) - \int_0^\infty dr e^{-Mr} \chi(r) \right\} \\ & \quad + \frac{e^{Mt}}{2MC(M)} \int_0^\infty dr e^{-Mr} \chi(r), \quad \text{where } C'(s) = \frac{d}{ds} C(s). \end{aligned} \quad (3.9)$$

The last term on the r.h. side of Eq. (3.9) is due to the pole of $q(s)$ at $s = M$. ($s = -M$ is not a pole of the integrand since $C(s)$ diverges there). If we let $t \rightarrow \infty$ in Eq. (3.9) and remember that the real part of $s_i < 0$, then applying l'Hospital's rule shows that each term vanishes separately if the field satisfies the conditions (3.6). The same situation holds for the terms of Eq. (3.4) which contain $\psi(r)$. Therefore we have

$$\lim_{t \rightarrow \infty} Q(t) = 0. \tag{3.10}$$

Differentiating Eq. (3.9) with respect to t and applying the above argument for the limit $t \rightarrow \infty$ shows that also

$$\lim_{t \rightarrow \infty} \dot{Q}(t) = 0. \tag{3.11}$$

Thus we conclude; as long as the field satisfies the conditions (3.6) at spatial infinity, the system displays a dissipative behavior in the sense that the energy of the oscillator E_Q goes to zero with $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} E_Q = \lim_{t \rightarrow \infty} \frac{1}{2}(\dot{Q}^2 + \omega_0^2 Q^2) = 0. \tag{3.12}$$

At this stage one might wonder if this behavior is not in contradiction to the time symmetry of our original equations (1.1), (1.2). The point, however, is the following: Up to now we did not worry about the way in which the data at $t = 0$ were arrived at. The solutions Eq. (2.8), (2.11) are defined only for $t \geq 0$. Extending these expressions formally to negative values of t will *not* lead to a solution of the equations for $t < 0$. If we want to have a solution for all t we can *use* the time-symmetry of the system to construct it from the "one-sided" solution for $t > 0$, in a straightforward way (In section V we shall construct explicitly for the point model the solutions for $t < 0$ by introducing the Laplace-transform with respect to the negative time-axis): The solution for $Q(t)$ has the general structure

$$Q(t) = a(t) Q(0) + b(t) \dot{Q}(0) + \int_0^\infty \{K(t, r) \psi(r) + L(t, r) \chi(r)\} dr \quad t \geq 0 \tag{3.13}$$

Performing in (3.13) the substitutions $t \rightarrow -t$, $\dot{Q}(0) \rightarrow -\dot{Q}(0)$, $\chi(r) \rightarrow -\chi(r)$ yields another solution of our equations (2.1, 2) defined for $t < 0$ with the same data at $t = 0$. Matching both of them together gives:

$$Q(t) = a(|t|) Q(0) + \epsilon(t) b(|t|) \dot{Q}(0) + \int_0^\infty \{K(|t|, r) \psi(r) + \epsilon(t) L(|t|, r) \chi(r)\} dr \tag{3.14}$$

where

$$\epsilon(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0. \end{cases}$$

It is then evident from Eq. (3.14) that Q not only satisfies Eqs. (3.10), (3.11) and (3.12) but similar relations in the limit $t \rightarrow -\infty$, if conditions (3.5) are fulfilled. Thus,

$$\lim_{t \rightarrow \pm\infty} Q(t) = \lim_{t \rightarrow \pm\infty} \dot{Q}(t) = \lim_{t \rightarrow \pm\infty} E_0 = 0.$$

IV. THE POINT LIMIT

For the discussion of initial values and boundary conditions in the next section it is convenient, though not necessary, to know the distribution of zeros of $C(s)$ explicitly. This is the case in the point limit $\rho(\mathbf{x}) \rightarrow \delta^3(\mathbf{x})$ which corresponds to the limit $M \rightarrow \infty$ in Eq. (3.1). In this limit the field couples to the oscillator only at the origin and Eqs. (2.8) and (2.11) become

$$C(s) q(s) = sQ(0) + \dot{Q}(0) + \int_0^\infty dr e^{-sr} [s\psi(r) + \chi(r)] \quad (4.1)$$

$$\phi(\mathbf{x}, s) = \frac{\lambda}{4\pi} \frac{e^{-sr}}{r} q(s) + \phi_H(\mathbf{x}, s). \quad (4.2)$$

$C(s)$ is then simply given by

$$C(s) = s^2 + 2\Gamma s + \bar{\omega}^2 \quad (4.3)$$

where the spring constant renormalization as given by Eq. (2.10) becomes infinite. $Q(t)$ is essentially determined by the zeros of $C(s)$

$$s_{1,2} = -\Gamma \pm i\omega \quad (4.4)$$

where $\omega = (\bar{\omega}^2 - \Gamma^2)^{1/2}$. The s integration can then be evaluated for Eqs. (4.1) and (4.2). Consider e.g., the coefficient of $Q(0)$:

$$G(t) = \frac{1}{2\pi i} \int_{(b)} ds \frac{e^{st}}{C(s)} = \sum_{i=1}^2 \frac{e^{s_i t}}{C'(s_i)} = e^{-\Gamma t} \frac{\sin \omega t}{\omega}. \quad (4.5)$$

The integration of the terms containing the spherical part of the field follows along the same lines e.g.

$$\begin{aligned} \frac{1}{2\pi i} \int_{(b)} ds e^{st} \int_0^\infty dr \frac{e^{-sr}}{C(s)} \chi(r) &= \int_0^t dr e^{-\Gamma(t-r)} \frac{\sin \omega(t-r)}{\omega} \chi(r) \\ &= \int_\alpha^t dr G(t-r) \chi(r). \end{aligned} \quad (4.6)$$

Collecting these results we obtain for the integrated oscillator equation

$$Q(t) = e^{-\Gamma t} \left\{ \left[\cos \omega t - \frac{\Gamma}{\omega} \sin \omega t \right] Q(0) + \frac{\sin \omega t}{\omega} \dot{Q}(0) \right\} + \lambda \int_0^t dr \{ \dot{G}(t-r) \psi(r) + G(t-r) \chi(r) \}. \tag{4.7}$$

From Eq. (4.2) we deduce

$$\Phi(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{(b)} ds e^{st} \left\{ \lambda \frac{e^{-st}}{4\pi r} q(s) + \phi_H(\mathbf{x}, s) \right\} \tag{4.8}$$

which again can be integrated by standard complex integration to yield

$$\Phi(\mathbf{x}, t) = \frac{\lambda}{4\pi} \theta(t-r) \frac{Q(t-r)}{r} + \Phi_H(\mathbf{x}, t) \tag{4.9}$$

here $Q(t-r)$ has to be taken from Eq. (4.7) and

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \text{ is the step function.}$$

Eqs. (4.7) and (4.9) give the time development of the system in the point limit for $t > 0$. It is straightforward to check for this explicit solution that, if the field satisfies Eq. (3.5) on the initial surface, the energy of the oscillator goes to zero for $t \rightarrow \infty$. However, because our starting equations (1.1) and (1.2) are not well defined in the point limit the solutions display some unpleasant features: they become discontinuous for a certain class of initial values at $t = 0$. (For details see Appendix C.) This, however, is irrelevant for our discussion.

V. BOUNDARY CONDITIONS

When discussing problems of radiation the field equations are usually supplemented with boundary conditions instead of prescribing the field and particle variables on a spacelike hypersurface. We shall show in this section that prescribing boundary conditions for the field imposes constraints among the initial data of particle and field, which may be so strong that no physically acceptable solution to the system exists. In what follows we discuss the connection between boundary conditions and initial values.

The boundary condition imposed most frequently is obtained by splitting the field into retarded field of the particle plus the rest (called incoming field) and

demanding that the latter should be zero everywhere. For our model, in the point limit, Φ^{in} is defined by

$$\Phi^{\text{in}}(\mathbf{x}, t) = \bar{\Phi}(\mathbf{x}, t) - (\lambda/4\pi) [Q(t-r)/r]. \quad (5.1)$$

For $t > r$, Φ^{in} can be expressed as a function of $Q(0)$, $\dot{Q}(0)$, $\Phi(\mathbf{x}, 0)$ and $\dot{\Phi}(\mathbf{x}, 0)$ from Eqs. (4.7) and (4.9). Conversely, prescribing Φ^{in} leads to constraints among the initial values.

Outside the light cone $r > t$, however, Φ^{in} depends on the dynamics of $Q(t)$ for $t < 0$. Therefore we have to make assumptions about the motion of the oscillator and the field prior to $t = 0$. Thus we have to ask how the initial state at $t = 0$ was arrived at. Two possibilities offer themselves.

(i) The system itself prepares the state at $t = 0$. By this we mean that the equations of the system (2.1, 2) are fulfilled for $-\infty < t < \infty$. In this case which has been touched upon at the end of Section III we call the system *closed*.

(ii) For $t < 0$ at least one of the Eqs. (2.1) and (2.2) is violated. The initial state is prepared by an external force acting on the system. In this case we call the system *open*.

(i) The Closed System

We now construct the solution for all t in the point limit. Defining the Laplace transform for negative t by:

$$\tilde{q}(s) = \int_{-\infty}^0 dt e^{st} Q(t), \quad \tilde{\Phi}(\mathbf{x}, s) = \int_{-\infty}^0 dt e^{st} \Phi(\mathbf{x}, t) \quad (5.2)$$

we obtain in a completely analogous manner to section II and IV in the point limit:

$$C(s) \tilde{q}(s) = sQ(0) - \dot{Q}(0) + \int_0^{\infty} dr e^{-sr} [s\psi(r) - \chi(r)] \quad (5.3)$$

and

$$\tilde{\Phi}(\mathbf{x}, s) = (\lambda/4\pi)(e^{-sr}/r) \tilde{q}(s) + \tilde{\Phi}_H(\mathbf{x}, s). \quad (5.4)$$

Consider as an example the case where $\Phi(\mathbf{x}, 0) = \dot{\Phi}(\mathbf{x}, 0) = 0$. Calculating the inverse Laplace transform of $\tilde{q}(s)$ and $\tilde{\Phi}(\mathbf{x}, s)$ to obtain $Q(t)$ and $\Phi(\mathbf{x}, t)$ for $t < 0$ respectively, we arrive at the solution for $-\infty < t < \infty$:

$$Q(t) = e^{-\Gamma|t|} \left\{ \left(\cos \omega t - \frac{\Gamma}{\omega} \sin \omega |t| \right) Q(0) + \frac{\sin \omega t}{\omega} \dot{Q}(0) \right\} \quad (5.5)$$

and for the field

$$\Phi(\mathbf{x}, t) = (\lambda/4\pi)(1/r) \{ \theta(t-r) O(t-r) + \theta(t-r) O(t+r) \} \quad (5.6)$$

Hence we have a contracting spherical wave for $t < 0$ pumping energy into the oscillator. At $t = 0$ all the energy of the wave is absorbed. For $t > 0$ we have the time reversed picture. The oscillator radiates its energy away, thereby producing an expanding spherical wave. We have $\lim_{t \rightarrow \pm\infty} E_Q = 0$. As shown already at the end of Section III this equation is also valid in the more general case, where $\psi(r)$ and $\chi(r)$ are nonvanishing but fulfill Eq. (3.6).

Let us now discuss the boundary conditions. If the system is closed, the asymptotic fields defined by

$$\Phi^{\text{out}}_{\text{in}}(\mathbf{x}, t) = \Phi(\mathbf{x}, t) - (\lambda/4\pi)[Q(t \mp r)/r] \tag{5.7}$$

can be expressed for all t as a function of $\{Q(0), \dot{Q}(0), \Phi(\mathbf{x}, 0), \dot{\Phi}(\mathbf{x}, 0)\}$. We can rewrite the equation for the oscillator by introducing $\Phi^{\text{out}}_{\text{in}}$ at $t = 0$

$$\begin{aligned} \Phi(\mathbf{x}, 0) &= \frac{\lambda}{4\pi} \frac{Q(\mp r)}{r} + \Phi^{\text{out}}_{\text{in}}(\mathbf{x}, 0), \\ \dot{\Phi}(\mathbf{x}, 0) &= \frac{\lambda}{4\pi} \frac{\dot{Q}(\mp r)}{r} + \dot{\Phi}^{\text{out}}_{\text{in}}(\mathbf{x}, 0). \end{aligned} \tag{5.8}$$

Inserting this decomposition into Eq. (4.1) we have for $t > 0$

$$\begin{aligned} C(s) q(s) &= sQ(0) + \dot{Q}(0) + \int_0^\infty dr e^{-sr} \{sQ(\mp r) + \dot{Q}(\mp r)\} \\ &+ \lambda \int d^3x \frac{e^{-sr}}{r} \{s\Phi^{\text{out}}_{\text{in}}(\mathbf{x}, 0) + \dot{\Phi}^{\text{out}}_{\text{in}}(\mathbf{x}, 0)\}. \end{aligned} \tag{5.9}$$

To study the connection to the boundary conditions usually imposed in classical electrodynamics we consider the following ‘‘mixed’’ problem: Let $Q(0)$ and $\dot{Q}(0)$ be given arbitrarily. Is it possible to choose $\Phi(\mathbf{x}, 0)$ and $\dot{\Phi}(\mathbf{x}, 0)$ in such a way that $\Phi^{\text{in}}(\mathbf{x}, t) = 0$, in the whole space time? If $\Phi^{\text{in}}(\mathbf{x}, t) = 0$ then Eq. (5.9) yields after partial integration

$$C(s) q(s) = (s + 2\Gamma) Q(0) + \dot{Q}(0). \tag{5.10}$$

Note that the equation for $q(s)$ does not contain $\tilde{q}(s)$ which means that the dependence of $Q(t)$ on times $t < 0$ has dropped out. Similarly, from Eq. (5.3) we have for times $t < 0$

$$C(s) \tilde{q}(s) = sQ(0) - \dot{Q}(0) + 2\Gamma \int_0^\infty dr e^{-sr} [sQ(-r) - \dot{Q}(-r)]$$

Here the term $Q(-r)$ is not cancelled after partial integration but gives $4\Gamma s\tilde{q}(s)$. Hence

$$\tilde{C}(s)\tilde{q}(s) \equiv (s^2 - 2\Gamma s + \bar{\omega}^2)\tilde{q}(s) = (s + 2\Gamma)Q(0) - \dot{Q}(0). \quad (5.12)$$

$\tilde{C}(s)$ has zeros with positive real part $s_{\pm} = \Gamma \pm i\omega$.

By making use of inverse Laplace transformation we find for $q(s)$ and $\tilde{q}(s)$ from Eqs. (5.11) and (5.12) that

$$Q(t) = e^{-\Gamma t} \{ [\cos \omega t + (\Gamma/\omega) \sin \omega t] Q(0) + [\sin \omega t/\omega] \dot{Q}(0) \} \quad (5.13)$$

for all t . The field is simply given by

$$\begin{aligned} \Phi(\mathbf{x}, t) = \frac{\lambda}{4\pi} \frac{e^{-\Gamma(t-r)}}{r} \left\{ \left(\cos \omega(t-r) + \frac{\Gamma}{\omega} \sin \omega(t-r) \right) Q(0) \right. \\ \left. + \frac{\sin \omega(t-r)}{\omega} \dot{Q}(0) \right\}. \end{aligned} \quad (5.14)$$

For every t , Φ grows exponentially as $r \rightarrow \infty$. This is due to the fact that the oscillator has been radiating for an infinite time. This solution violates our conditions (3.6) and is physically unacceptable.

Using the advanced fields and the condition $\Phi^{\text{out}} = 0$ should give us the time reversed solution of Eq. (5.13). It is illuminating to see how by a change of initial conditions, *damping* is turned into *antidamping*. The calculation is analogous to the one for $\Phi^{\text{in}} = 0$. If $\Phi^{\text{out}} = 0$, the constraints on the initial variables read

$$\Phi(\mathbf{x}, 0) = \frac{\lambda}{4\pi} \frac{Q(r)}{r}, \quad \dot{\Phi}(x, 0) = \frac{\lambda}{4\pi} \frac{\dot{Q}(r)}{r}. \quad (5.15)$$

With this condition Eq. (5.9) gives for $t > 0$

$$\tilde{C}(s)q(s) = (s - 2\Gamma)Q(0) + \dot{Q}(0) \quad (5.16)$$

and for times $t < 0$

$$C(s)\tilde{q}(s) = (s - 2\Gamma)Q(0) - \dot{Q}(0). \quad (5.17)$$

Because of the exchange of $\tilde{C}(s)$ and $C(s)$ in Eqs. (5.16) and (5.17) in comparison to Eqs. (5.10) and (5.11) the oscillator shows now an exponentially growing amplitude!

$$Q(t) = e^{+\Gamma t} \left\{ \left(\cos \omega t - \frac{\Gamma}{\omega} \sin \omega t \right) Q(0) + \frac{\sin \omega t}{\omega} \dot{Q}(0) \right\} \quad (5.18)$$

for all t . Since Q does not fulfill Eq. (3.12) the initial fields $\Phi(\mathbf{x}, 0)$ and $\dot{\Phi}(\mathbf{x}, 0)$ cannot satisfy our conditions (3.5) on the initial surface as can be checked using Eq. (5.15).

Finally let us examine the Wheeler–Feynman condition

$$\Phi^{\text{in}} + \Phi^{\text{out}} = 0$$

which implies that the field is half retarded plus half advanced. For $t = 0$

$$\begin{aligned} \Phi(\mathbf{x}, 0) &= \frac{\lambda}{4\pi} \frac{1}{2} \left\{ \frac{Q(-r) + Q(r)}{r} \right\}, \\ \dot{\Phi}(\mathbf{x}, 0) &= \frac{\lambda}{4\pi} \frac{1}{2} \left\{ \frac{\dot{Q}(-r) + \dot{Q}(r)}{r} \right\}. \end{aligned} \tag{5.19}$$

Inserting this into our initial value solution Eqs. (5.3) and (5.9) and, again, partial integration gives

$$\begin{aligned} (s^2 + \bar{\omega}^2) q(s) &= sQ(0) + \dot{Q}(0), \\ (s^2 + \bar{\omega}^2) \tilde{q}(s) &= sQ(0) - \dot{Q}(0). \end{aligned} \tag{5.20}$$

Hence,

$$Q(t) = \cos \bar{\omega}t Q(0) + [\sin \bar{\omega}t / \bar{\omega}] \dot{Q}(0) \tag{5.21}$$

for all t , and the oscillator oscillates undamped with the renormalized frequency $\bar{\omega}$. Again the field on any surface $t = \text{const}$ decreases too slowly in order to satisfy conditions (3.5).

We remark that one can treat the system within the S -matrix formalism. Then, in fact it can be shown that for the closed system in some sense [9]

$$\lim_{t \rightarrow \infty} \Phi(\mathbf{x}, t) = \Phi^{\text{out}}(\mathbf{x}, t)$$

and

$$\lim_{t \rightarrow -\infty} \Phi(\mathbf{x}, t) = \Phi^{\text{in}}(\mathbf{x}, t).$$

This means that one can think of the system as scattering of the incoming field (the s -wave part) at the oscillator to give the outgoing field in the asymptotic future. However, treating the system in this way would eliminate solutions with $\Phi^{\text{out}} = 0$ from the beginning.

Closed System for extended ρ

In the last subsection we have studied solutions of our system in the “point-limit” $\rho(x) \rightarrow \delta^3(\mathbf{x})$ for which $\Phi(\mathbf{x}, t)$ and $Q(t)$ are arranged in such a way that Φ is given by the retarded (advanced) solution of Eq. (2.2). We have shown that these solutions correspond to $Q(t)$ performing damped (antidamped) oscillations for all times. Expressed in terms of data on some spacelike hypersurface $t = \text{const}$ both solutions were seen to violate our assumption of finite energy stored in the field. Hence $\Phi^{\text{in}} = \Phi^{\text{ret}} = 0$ were ruled out.

In this subsection we intend to give a rather different proof of the nonexistence of solutions with $\Phi^{\text{in}} = 0$, which is a little more "a priori" in the sense that it does not use the solution to (2.1, 2) explicitly, but only the fact that $\lim_{t \rightarrow \pm\infty} Q(t) = \lim_{t \rightarrow \pm\infty} \dot{Q}(t) = 0$. Furthermore, it works also for our extended $\rho(\mathbf{x})$.

At first we give the argument when $\Phi^{\text{in}} = 0$. In this case

$$\Phi(\mathbf{x}, t) = \frac{\lambda}{4\pi} \int d^3x' \frac{\rho(x') Q(t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}. \quad (5.22)$$

If the field satisfies our conditions (3.5) at the initial surface then

$$\lim_{t \rightarrow -\infty} Q(t) = 0, \quad (5.23)$$

and it is easy to see that $\Phi(\mathbf{x}, t)$ tends to zero

$$\lim_{t \rightarrow -\infty} \Phi(\mathbf{x}, t) = 0 \quad \text{uniformly in } \mathbf{x}. \quad (5.24)$$

In order to show that this implies only the trivial solution we look at the total energy of the system which is given by

$$E = E_Q + E_\Phi + E_{Q\Phi} \quad \text{where} \quad E_{Q\Phi} = -\lambda Q \int d^3x \rho \Phi. \quad (5.25)$$

The total energy is conserved if conditions (3.5) are satisfied. Using Eqs. (5.23) and (5.24) it is straightforward to verify that

$$\lim_{t \rightarrow -\infty} E_\Phi = \lim_{t \rightarrow -\infty} E_{Q\Phi} = 0. \quad (5.26)$$

Hence $\lim_{t \rightarrow -\infty} E = 0$ and therefore E vanishes identically. We now write E as a sum of quadratic terms. This is achieved by the substitution

$$\Phi(\mathbf{x}, t) = \bar{\Phi}(\mathbf{x}, t) + \frac{\lambda Q(t)}{4\pi} \int d^3x' \frac{\rho(x')}{|\mathbf{x} - \mathbf{x}'|}. \quad (5.27)$$

Inserting for $\bar{\Phi}$ and $\nabla \bar{\Phi}$ into Eq. (5.25) leads to

$$E = \frac{1}{2} \dot{Q}^2 + \frac{1}{2} \int d^3x \dot{\bar{\Phi}}^2 + \frac{\bar{\omega}^2 Q^2}{2} + \frac{1}{2} \int d^3x (\nabla \bar{\Phi})^2 \quad (5.28)$$

where $\bar{\omega}^2$ is again the renormalized spring constant given by Eq. (2.10). Therefore $E \geq 0$, and $E = 0$ implies $\dot{Q} = \dot{\bar{\Phi}} = Q = \nabla \bar{\Phi} = 0 \rightarrow \nabla \Phi = 0$. Since $\Phi = O(1/r)$, also $\bar{\Phi} = 0$

This proves our assertion and moreover shows the positive definiteness of the energy of our system. To rule out the case where $\Phi^{\text{out}} = 0$, a completely analogous argument may be used by studying the energy at $t = \infty$ instead of $t = -\infty$.

It is also possible to rule out $\Phi^{\text{in}} + \Phi^{\text{out}} = 0$ for extended ρ . This, however, cannot be done by an a priori argument of the type employed above. We mention without proof that $Q(t)$ in this case is a superposition of undamped oscillations for all times, which is therefore excluded.

(ii) *The Open System*

In the last two subsections we were able to rule out for the closed system solutions with $\Phi^{\text{in}} = 0$, $\Phi^{\text{out}} = 0$ and, at least in the point-limit, with $\Phi^{\text{in}} + \Phi^{\text{out}} = 0$. Now we want to investigate the existence of these solution for the open system. Going through the proofs once more we find that the exclusion of the latter two situations is completely independent of the system being closed for $t < 0$. Therefore they cannot be realized for the open system either. We are left with $\Phi^{\text{in}} = 0$. This suggests that we should consider the following class of open systems: $Q(t)$ for $t \leq 0$ is arbitrary except that it should satisfy

$$\lim_{t \rightarrow -\infty} Q(t) = \lim_{t \rightarrow -\infty} \dot{Q}(t) = 0, \tag{5.29}$$

and, of course,

$$Q(0_-) = Q(0) \quad \text{and} \quad \dot{Q}(0_-) = \dot{Q}(0)$$

while Φ is given by the retarded solution of Eq. (2.2). This means that we imagine the oscillator to be driven “by hand” and the system to be isolated in the sense that there is no incoming radiation for $t < 0$. These conditions determine $\Phi(\mathbf{x}, 0)$, $\dot{\Phi}(\mathbf{x}, 0)$. Condition (5.29) guarantees that (3.6) is satisfied. Then section II tells us that $\lim_{t \rightarrow \infty} Q(t) = \lim_{t \rightarrow \infty} \dot{Q}(t) = 0$. Furthermore it is easy to see that $\Phi^{\text{in}} = 0$ is automatically satisfied for $t > 0$ also.

APPENDIX A

We prove uniqueness of the initial value solution for the system given by Eqs. (2.1, 2). Consider Fig. 1. We want to show that if $Q(0) = \dot{Q}(0) = 0$ and if Φ and $\dot{\Phi}$ vanish in V_0 , then Q and Φ vanish in the region G_k bounded by V_0 , V_k and the null hypersurface Σ . We define

$$E_\Phi(\tau) = \frac{1}{2} \int_{V_\tau} d^3x \{ \dot{\Phi}^2 + (\nabla\Phi)^2 \} \Big|_{t=\tau}, \tag{A.1}$$

$$E_Q(\tau) = \frac{1}{2} \dot{Q}^2 + \dots + 2Q\dot{Q} \tag{A.2}$$

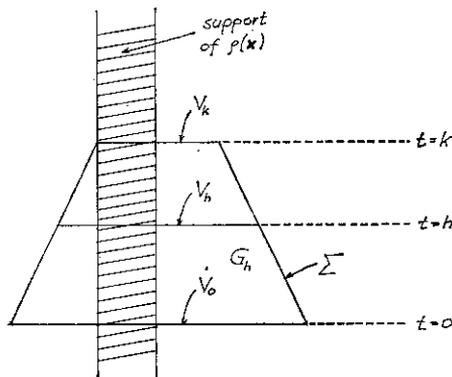


FIG. 1. A diagram with two space dimensions suppressed showing the "domain of dependence" bounded by $t = 0$, the null hypersurface Σ and $t = k$.

Multiplication of Eq. (2.2) with $\dot{\Phi}$ yields

$$\dot{\Phi} \square \Phi = \{\Phi_{,0} \dot{\Phi}_{, \nu} - \frac{1}{2} \eta_{\mu\nu} \dot{\Phi}_{, \mu} \Phi^{, \nu}\}_{, \nu} \equiv T_{0\nu}{}^{, \nu} = 0 \tag{A.3}$$

(with $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $\dot{\Phi}_{, \nu} = (\dot{\Phi}, \nabla \Phi)$. Summation runs over 0, 1, 2, 3.) Hence

$$\int_{G_h} d^4x \dot{\Phi} \square \Phi = E_\phi(h) + \int_\Sigma d\Sigma^\mu T_{0\mu} \geq E_\phi(h). \tag{A.4}$$

Therefore

$$0 \leq E_\phi(h) \leq \lambda \int_{G_h} d^4x \rho(\mathbf{x}) \dot{\Phi}(\mathbf{x}, t) Q(t). \tag{A.5}$$

From (2.1) we obtain

$$0 \leq E_Q(h) = \lambda \int_{G_h} d^4x \rho(\mathbf{x}) \Phi(\mathbf{x}, t) \dot{Q}(t). \tag{A.6}$$

Hence

$$\begin{aligned} 0 \leq E_Q(h) + E_\phi(h) &\leq \lambda \int_{G_h} d^4x \rho(\mathbf{x}) \frac{d}{dt} [\Phi(\mathbf{x}, t) Q(t)] \\ &= \lambda \int_{V_h} d^3x \rho(\mathbf{x}) \Phi(\mathbf{x}, h) Q(h) \leq \lambda \left\{ \int_{V_h} d^3x \rho^2(\mathbf{x}) \right\}^{1/2} \left\{ \int_{V_h} d^3x \Phi^2(\mathbf{x}, h) \right\}^{1/2} Q(h). \end{aligned} \tag{A.7}$$

Using the inequality, valid for every $v(t)$ with $v(0) = 0$

$$v^2(t) \leq t \int_0^t d\tau (dv/d\tau)^2 \tag{A.8}$$

we obtain

$$\begin{aligned}
 0 \leq E_O(h) + E_\Phi(h) &\leq Ah \left\{ \int_0^h d\tau \int_{V_\tau} d^3x \Phi^2(\mathbf{x}, \tau) \right\}^{1/2} \left\{ \int_0^h d\tau Q^2(\tau) \right\}^{1/2} \\
 &\leq Ah \left\{ \int_0^h d\tau E_\Phi(\tau) \right\}^{1/2} \left\{ \int_0^h d\tau E_O(\tau) \right\}^{1/2} \leq Ak \left\{ \int_0^h d\tau E_\Phi(\tau) \right\}^{1/2} \left\{ \int_0^h d\tau E_O(\tau) \right\}^{1/2}
 \end{aligned}
 \tag{A.9}$$

where $A = \lambda \{ \int d^3x \rho^2(\mathbf{x}) \}^{1/2}$. Take an arbitrary l satisfying $h < l < k$. Then

$$0 \leq E_\Phi(h) + E_O(h) \leq Ak \left\{ \int_0^l dt E_\Phi(\tau) \right\}^{1/2} \left\{ \int_0^l d\tau E_O(\tau) \right\}^{1/2}. \tag{A.10}$$

Integrating (A.10) between 0 and l yields

$$0 \leq \int_0^l d\tau E_\Phi(\tau) + \int_0^l d\tau E_O(\tau) \leq Akl \left\{ \int_0^l d\tau E_\Phi(\tau) \right\}^{1/2} \left\{ \int_0^l d\tau E_O(\tau) \right\}^{1/2}. \tag{A.11}$$

Inequality (A.11) is valid for all $l < k$. Now assume that $E_\Phi(\tau)$ and $E_O(\tau)$ are not identically zero for $0 < l < k$. Then we choose $l < 2/kA$ which implies

$$\left(\left\{ \int_0^l d\tau E_\Phi(\tau) \right\}^{1/2} - \left\{ \int_0^l d\tau E_O(\tau) \right\}^{1/2} \right)^2 < 0 \tag{A.12}$$

except when

$$E_\Phi(\tau) = E_O(\tau) = 0 \quad \text{for } 0 < \tau < k. \tag{A.13}$$

This implies that $\Phi, \nabla\Phi$ vanish in G_k and $Q(t)$ vanishes for $0 < t < k$. Since Φ vanishes on V_0 it vanishes identically in G_k .

APPENDIX B

We intend to show that

$$C(s) = s^2 + \tilde{\omega}^2 - 2\Gamma \int d^3x d^3x' \frac{e^{-s|\mathbf{x}-\mathbf{x}'|} - 1}{|\mathbf{x} - \mathbf{x}'|} \rho(\mathbf{x}) \rho(\mathbf{x}') \tag{B.1}$$

has only zeros with negative real part. Set $s = -i\omega$. Then (B.1) may be written as

$$C(-i\omega) = D(\omega) = -\omega^2 + \tilde{\omega}^2 - 2\Gamma \int d^3k \frac{|c(k)|^2 \omega^2}{k^2[k^2 - (\omega + i\epsilon)^2]} \tag{B.2}$$

where $c(k)$ means the Fourier transform of $\rho(\mathbf{x})$. We have to show that $D(\omega)$ has no zeros $\tilde{\omega}$ with $\text{Im } \tilde{\omega} \geq 0$. Consider first $\text{Im } \tilde{\omega} = 0$. Then, using the well-known formula $1/(x - i\epsilon) = P(1/x) + i\pi\delta(x)$, we obtain $\text{Im } D(\omega) < 0$.

We still have to exclude $\text{Im } \tilde{\omega} > 0$. In this case

$$\begin{aligned} D(\tilde{\omega}) &= -\tilde{\omega}^2 + \bar{\omega}^2 - 2\Gamma \int d^3k \frac{|c(k)|^2 \tilde{\omega}^2}{k^2 |k^2 - \tilde{\omega}^2|} \\ &= -\tilde{\omega}^2 + \bar{\omega}^2 - 2\Gamma \int d^3k \frac{|c(k)|^2 \tilde{\omega}^2 (k^2 - \tilde{\omega}^{*2})}{k^2 |k^2 - \tilde{\omega}^2|^2}. \end{aligned} \quad (\text{B.3})$$

Hence

$$\text{Im } D(\tilde{\omega}) = \text{Im } \tilde{\omega}^2 \left[-1 - 2\Gamma \int d^3k \frac{|c(k)|^2}{|k^2 - \tilde{\omega}^2|^2} \right] = 0 \quad (\text{B.4})$$

only if $\text{Im } \tilde{\omega}^2 = 0$. In this case $\text{Re } \tilde{\omega} = 0$. However, then

$$\text{Re } D(\tilde{\omega}) = (\text{Im } \tilde{\omega})^2 + 2\Gamma \int d^3k \frac{|c(k)|^2 |\tilde{\omega}^2|^2}{k^2 |k^2 - \tilde{\omega}^2|^2} + \bar{\omega}^2 > 0. \quad (\text{B.5})$$

APPENDIX C

Suppose we take as initial values $\psi(r) = \chi(r) = 0$ while $Q(0)$ and $\dot{Q}(0)$ are arbitrary. Then from Eq. (4.7) we have

$$\lim_{t \rightarrow 0} Q(t) = Q(0), \quad (\text{C.1})$$

$$\lim_{t \rightarrow 0} \dot{Q}(t) = \dot{Q}(0) - 2\Gamma Q(0). \quad (\text{C.2})$$

A careful analysis shows that Eq. (4.7) is the correct point-limit of Eq. (3.8) for $t > 0$ but that $\dot{Q}(t)$ is discontinuous at $t = 0$ the discontinuity being due to the fact that

$$\dot{Q}(0) = \lim_{M \rightarrow \infty} \lim_{t \rightarrow 0} \dot{Q}(t) \neq \lim_{t \rightarrow 0} \lim_{M \rightarrow \infty} \dot{Q}(t). \quad (\text{C.3})$$

This can be seen by considering the extended model in some detail: For large M , $C(s)$ has four zeros approximately given by

$$\begin{aligned} s_{1,2} &= -\Gamma \pm i\omega + O(1/M^{1/2}) \\ s_{3,4} &= -M \pm (\Gamma M)^{1/2} + O(M^0). \end{aligned} \quad (\text{C.4})$$

The leading terms for large M of Eq. (3.8) in t -space are

$$\begin{aligned} Q(t) &= \left\{ e^{-\Gamma t} \left[\cos \omega t - \frac{\Gamma}{\omega} \sin \omega t \right] - \frac{1}{2M} \left(\frac{\Gamma}{M} \right)^{1/2} \left[e^{-t(M+(\Gamma M)^{1/2})} (M + (\Gamma M)^{1/2}) \right. \right. \\ &\quad \left. \left. - e^{-t(M-(\Gamma M)^{1/2})} (M - (\Gamma M)^{1/2}) \right] \right\} \cdot Q(0) + \left\{ e^{-\Gamma t} \frac{\sin \omega t}{\omega} + \frac{1}{2M} \left(\frac{\Gamma}{M} \right)^{1/2} \right. \\ &\quad \left. \cdot \left[e^{-t(M+(\Gamma M)^{1/2})} - e^{-t(M-(\Gamma M)^{1/2})} \right] \right\} \dot{Q}(0). \end{aligned} \quad (\text{C.5})$$

Letting $M \rightarrow \infty$ we recover $Q(t)$ of Eq. (4.7) while $\dot{Q}(t)$ has the discussed jump at $t = 0$. A similar discontinuity occurs in $\Phi(x, t)$ for $r = t$.

The point solution is continuous if the initial values satisfy the relation:

$$\begin{aligned} \lim_{r \rightarrow 0} \psi(r) &= \lambda Q(0)/4\pi, \\ \lim_{r \rightarrow 0} \chi(r) &= \lambda \dot{Q}(0)/4\pi, \end{aligned} \quad (\text{C.6})$$

i.e., if we require that the source is "dressed" with its instantaneous Coulomb field at $t = 0$: In this case Eq. (4.7) leads to

$$\begin{aligned} Q(t) &= e^{-\Gamma t} \left[\cos \omega t + \frac{\Gamma}{\omega} \sin \omega t \right] Q(0) \\ &+ e^{-\Gamma t} \frac{\sin \omega t}{\omega} \dot{Q}(0) + \int_0^t dr G(t-r) \left[\chi(r) + \frac{d}{dr} \psi(r) \right]. \end{aligned} \quad (\text{C.7})$$

Note that now the second term of Eq. (C.7) has the opposite sign compared to Eq. (4.7) and ensures that \dot{Q} is continuous at $t = 0$. If, on the contrary, the field is nonsingular at $r = 0$, the oscillator "feels" the bare spring constant ω_0^2 at $t = 0$, but the physical one $\bar{\omega}^2$ for any time later. This can be inferred from the field equations and is the reason for the discontinuity of the solution at $t = 0$.

Of course, because of the $1/r$ -singularity of the field, E_ϕ is infinite. However, conditions (C.6) imply that the total energy, as expressed in terms of $\bar{\omega}^2$, is finite.

Note also that conditions (C.6) are fulfilled for the solutions obtained by imposing the three discussed boundary conditions.

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