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SMR 281/2

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
(11 January - 5 February 1988)

THE EVOLUTION OF HARMONIC MAPS (PART I)

Heat-flow methods for harmonic maps of surfaces and applications to free boundary problems

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from: Proc. VIII ICLHM, Springer lecture notes

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Abstract

In [17] the Eells-Sampson method for constructing harmonic maps between manifolds was extended to maps from a surface to an arbitrary compact manifold. We review the results in [17] and present several applications: First a new proof of the Sacks-Uhlenbeck results is given. Then we study minimal surfaces and surfaces of constant mean curvature with free boundaries on a supporting surface in $\mathbb{R}^3$.

AMS classification code: 53 A 10, 53 C 20, 58 E 20, 58 G 11.

1. Basic definitions

Let M, N be compact Riemannian manifolds of dimensions $\dim M = 2$, $\dim N = n$, and with metrics γ, g respectively. For differentiable maps $u = M \to N$ let

$$E(u) = \int_M e(u)\,dM$$

be the energy of u with energy density $e(u)$ in local coordinates being given by

$$e(u) = \frac{1}{2} g_{ik}(u)\gamma^{\alpha\beta}(x)\partial_\alpha u^i \partial_\beta u^k .$$

Here and in the sequel $\partial_\alpha u^i = \frac{\partial}{\partial x^\alpha} u^i$, $(\gamma^{\alpha\beta})$ denotes the inverse of the coefficient matrix $(\gamma_{\alpha\beta})$ of γ, and (g_{ik}) is

a local representation of g. By convention double Greek indeces will be summed from 1 to 2, double Latin indeces from 1 to n. Of course, locally the volume element on M

$$dM = \sqrt{\gamma}\,dx$$

where $\gamma = \det(\gamma_{\alpha\beta})$.

u is harmonic, iff E is stationary at u, i.e. iff

$$\text{(1.1)}\qquad \delta E(u) := -\Delta_M u - {}^N\Gamma(u)(\nabla u,\nabla u)_M = 0$$

with

$$(-\Delta_M u)^\ell = -\frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\beta u^\ell), \quad 1 \le \ell \le n$$

denoting the Laplace-Beltrami operator on M and

$$({}^N\Gamma(u)(\nabla u,\nabla u)_M)^\ell =$$

$$= \Gamma^\ell_{ik}(u)\gamma^{\alpha\beta}\partial_\alpha u^i\,\partial_\beta u^k, \quad 1 \le \ell \le n,$$

where

$$\Gamma^\ell_{ik} = \frac{1}{2} g^{\ell j}\left(\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^k} g_{ij} - \frac{\partial}{\partial u^j} g_{ik}\right)$$

are the Christoffel symbols of the metric g.

Note that formally at a smooth map $u : M \to N$ for any smooth variation vector φ

$$\frac{d}{d\varepsilon} E(u+\varepsilon\varphi)\Big|_{\varepsilon=0} =$$

(1.2)

$$= \int_{M} g_{k\ell}(u)(-\Delta_M u - {}^N\Gamma(u)(\nabla u,\nabla u)_M)^k \varphi^\ell \, dM \quad ,$$

whence δE may be regarded as the L^2-gradient of E with respect to the metric $g(u)$.

By the Nash embedding theorem we may assume that $N \subset \mathbb{R}^N$ isometrically. A natural space on which to consider E then is the space

$$H^{1,2}(M,N) = \{u \in H^{1,2}(M;\mathbb{R}^N) \,|\, u(M) \subset N\}$$

of L^2-functions from M into N having a distributional derivative in L^2.

By a result of Schoen and Uhlenbeck [13] and since M is 2-dimensional, $H^{1,2}(M,N)$ is the closure of the space of smooth functions from M into N in $H^{1,2}(M;\mathbb{R}^N)$. Hence we may continuously extend E to $H^{1,2}(M;N)$. Moreover, E is coercive with respect to the norm induced by $H^{1,2}(M;\mathbb{R}^N)$, and therefore $H^{1,2}(M;N)$ is precisely the space of measurable functions $u : M \to N$ a.e. with $E(u) < \infty$.

However, note that $H^{1,2}(M;N)$ is not a manifold and hence E cannot be differentiable on this space.

2. Existence of harmonic maps

Given the coerciveness of E on $H^{1,2}(M,N)$ it is natural to apply the direct methods in the calculus of variations to obtain harmonic maps from M into N as (relative) minimizers

of E in this space.

Unfortunately, E trivially assumes its global minimum 0 at any constant map $u : M \to N$. Moreover (although as a consequence of the Schoen-Uhlenbeck result homotopy classes of (continuous) maps between M and N are stable in the strong $H^{1,2}(M;N)$-topology) homotopy classes in general will not be weakly closed in $H^{1,2}(M;N)$. Therefore, the existence of non-trivial harmonic maps and, in particular, the existence of harmonic representants of homotopy classes of maps between M and N does not follow from the direct methods.

Instead, in their pioneering paper [2] Eells and Sampson consider the evolution problem

$$\text{(2.1)} \qquad \partial_t u - \Delta_M u - {}^N\Gamma(u)(\nabla u,\nabla u)_M = 0$$

$$\text{(2.2)} \qquad u\big|_{t=0} = u_0$$

for harmonic maps from M into N . Their fundamental result is the following

<u>Theorem 2.1</u> (Eells-Sampson): Suppose the sectional curvature K of N is non-positive. Then for any smooth map $u_0 : M \to N$ there exists a smooth solution $u : M \times [0,\infty[\to N$ of (2.1), (2.2) which as $t \to \infty$ converges to a smooth harmonic map $u_\infty : M \to N$.

Remark that by (1.2) for any solution u of (2.1), (2.2) $E(u(\cdot,t))$ is non-increasing in t , so that in particular the result of Eells and Sampson guarantees the existence of a smooth harmonic map which minimizes E (relatively) in any homotopy class of maps $u : M \to N$, if N is non-positively curved.

Theorem 2.1 was later generalized to arbitrary target manifolds assuming the range $u(M \times [0,\infty[)$ to be a-priori bounded in a geodesic ball of radius $R < \frac{\pi}{2\sqrt{K}}$ where $K > 0$ is an upper bound for the sectional curvature of N, cp. [8], [19]. For complete N, however, only constant maps can be harmonic and have range in a ball of radius $< \frac{\pi}{2\sqrt{K}}$, and the above results are useful only for harmonic maps of manifolds with boundaries.

While Theorem 2.1 is valid for all dimensions, Sacks and Uhlenbeck in 1981 obtained significantly new results for $\dim(M) = 2$, cp. [12]. In particular, they proved [1)]

Theorem 2.2 (Sacks-Uhlenbeck): If N is compact, and $\pi_2(N) = 0$, there exists an energy-minimizing harmonic map in every homotopy class of maps from $M \to N$.

For arbitrary targets N they can assert:

Theorem 2.3 (Sacks-Uhlenbeck): Every conjugacy class of homomorphisms from $\pi_1(M)$ into $\pi_1(N)$ is induced by an energy-minimizing harmonic map from M into N.

If $M = S^2$ moreover they obtain

Theorem 2.4 (Sacks-Uhlenbeck): i) If the universal covering space of N is not contractible, there exists a non-trivial harmonic map of S^2 into N.

ii) There is a set Λ_i of free homotopy classes of maps from S^2

1) Independently, related results were obtained by Luc Lemaire.

into N generating $\pi_2(N)$ acted on by $\pi_1(N)$ and represented by relatively energy-minimizing harmonic maps $u_i : S^2 \to N$.

Remarks: i) By results of Gulliver, Osserman, and Royden a non-trivial harmonic map $u : S^2 \to N$ is a branched minimal immersion [5].

ii) Note that in general not every homotopy class of maps from M into N has a harmonic representant, cp. [3].

In order to obtain their results, Sacks and Uhlenbeck construct approximate solutions as relative minimizers of a family of perturbed functionals E_α, $\alpha > 1$. As $\alpha \to 1$ approximate solutions will either converge towards a smooth harmonic map or harmonic spheres will tend to separate at finitely many points of the domain M in the sense that a sequence of blown-up solutions will tend to a smooth, non-constant harmonic map $u : \overline{\mathbb{R}^2} \cong S^2 \to N$, locally uniformly in the tangent space $T_xM \cong \mathbb{R}^2$ at a point $x \in M$.

In [17] we have extended the heat flow method to maps from a compact surface into an arbitrary compact Riemannian manifold. In the next section we review these results and indicate how the theorems of Sacks and Uhlenbeck may be derived from them.

3. The evolution of harmonic maps of surfaces

Our basic result is the following [17, Section 4]:

Theorem 3.1: Let M be a smooth, compact Riemannian surface, and let N be a smooth, compact Riemannian manifold of dimension n . For any smooth $u_o : M \to N$ there exists a global (distribution) solution $u : M \times [0,\infty[\to N$ to (2.1), (2.2) which is regular on $M \times [0,\infty[$ with exception of at most finitely many points $(x_1,t_1), \dots (x_k,t_k)$, $0 < t_j \leq \infty$, and which is unique in this class.

At a singularity $(\bar{x},\bar{t})$ of u a smooth non-constant harmonic map $\bar{u} : S^2 \to N$ separates in the sense that for suitable $r_m \to 0$, $t_m \to \bar{t}$, $x_m \to \bar{x}$ in local coordinates

$$u(x_m+r_m x,t_m) \to \tilde{u} \quad \text{in} \quad H^{2,2}_{loc}(\mathbb{R}^2;N) .$$

$\tilde{u}$ has finite energy and may be extended to a smooth, non-constant harmonic map $\bar{u} : S^2 \cong \mathbb{R}^2 \to N$. Finally, as $t \to \infty$, $u(\cdot,t)$ converges weakly in $H^{1,2}(M,N)$ to a smooth harmonic map $u_\infty : M \to N$. If all $t_j < \infty$, $1 \leq j \leq k$, $u(\cdot,t) \to u_\infty$ even strongly in $H^{2,2}(M,N)$.

<u>Remarks:</u> [5] also applies in the case of Theorem 3.1 and we conclude that the harmonic spheres separating at singular points of the flow must be C^∞ conformal branched immersions of S^2 into N .

We also remark that the time needed to create a singularity of the solution u to (2.1), (2.2) can be uniformly estimated from below on an $H^{1,2}$-neighborhood of the initial value u_o . This permits to extend the existence result stated in Theorem 3.1 to initial values of class $H^{1,2}(M,N)$, cp. [17, Theorem 4.1]. (Now, of course, we only obtain "interior" regularity on $M \times]0,\infty[\setminus\{(x_j,t_j) | 1 \leq j \leq k\}$.)

By the non-existence result of Eells and Wood [3] e.g. for initial maps $u_o : T^2 \to S^2$ of degree 1 the solution u to (2.1), (2.2) has to become singular at some point $(\bar{x},\bar{t})$. However, it is an open question if singularities of the flow (2.1), (2.2) appear in finite time.

Let us sketch for example how Theorem 2.2 may be deduced from Theorem 3.1.

<u>Proof of Theorem 2.2:</u> Let

$$\varepsilon_o = \{\inf E(u) | u : S^2 \to N \text{ is a smooth, non-constant harmonic map}\} .$$

It is well-known that $\varepsilon_o > 0$; this also follows from Lemma 4.2 below.

Let $[u_o]$ be a homotopy class of maps from M into N , represented by u_o . We may suppose that u_o is smooth and

$$E(u_o) \leq \inf_{u\in[u_o]} E(u) + \frac{\varepsilon_o}{4} .$$

Let $u : M \times [0,\infty[\to N$ be the solution to the evolution problem (2.1), (2.2), guaranteed by Theorem 3.1. Suppose u becomes singular at a point $(\bar{x},\bar{t})$. Then for $t_m < \bar{t}$, $t_m \to \bar{t}$, $x_m \to \bar{x}$ and some $r_m \to 0$

$$u_m(x) :\equiv u(x_m+r_m x,t_m) \to \tilde{u} \quad \text{in} \quad H^{2,2}_{loc}(\mathbb{R}^2;N)$$

where $\tilde{u}$ may be extended to a smooth non-constant harmonic map $\bar{u} : S^2 \to N$. In particular, we can find a sequence of radii $R_m \to \infty$ such that

$$R_m \int_{\partial B_{R_m}} (|\nabla u_m|^2 - |\nabla \tilde{u}|^2) do \to 0$$

and

$$\int_{\mathbb{R}^2\setminus B_{R_m}} |\nabla\tilde{u}|^2 dx \to 0$$

as $m \to \infty$. Inverting $\tilde{u}$ in ∂B_{R_m} we may replace the large almost spherical surface $u_m(B_{R_m})$ having energy $\geq \frac{3\varepsilon_o}{4}$ by a small disc (essentially $\tilde{u}(\mathbb{R}^2\setminus B_{R_m})$) of energy $\leq \frac{\varepsilon_o}{4}$ to obtain a new map $v_m : M \to N$. Since $\pi_2(N) = 0$ v_m lies in the same homotopy class as $u(\cdot,t_m)$, i.e. $v_m \in [u_o]$.

On the other hand, since $E(u(\cdot,t))$ is non-increasing in t, cp. Lemma 4.3 below, we have:

$$E(v_m) \leq E(u(\cdot,t_m)) - \frac{\varepsilon_o}{2}$$

$$\leq E(u_o) - \frac{\varepsilon_o}{2} \leq \inf_{u\in[u_o]} E(u) - \frac{\varepsilon_o}{4} .$$

The contradiction proves that the solution u through u_o is globally regular and as $t \to \infty$ converges to a smooth harmonic map homotopic to u_o.

q.e.d.

Similarly, Theorem 2.3 can be derived from Theorem 3.1 and the observation that the conjugacy class of homomorphisms from $\pi_1(M)$ into $\pi_1(N)$ induced by the maps $u(\cdot,t)$ envolving from a given map u_o will not change if harmonic spheres separate. Finally, also Theorem 2.4 can be proved using Theorem 3.1 by arguing as in [12, Section 5].

4. Sketch of the proof of Theorem 3.1

It may be instructive to recall the essential features of the proof of Theorem 3.1.

Let u be a smooth solution to (2.1), (2.2) in $M \times [0,T]$. We can formulate a-priori estimates for u in terms of the quantity

$$\varepsilon(R,T) = \sup_{x\in M,\, t\leq T} \int_{B_R(x)} e(u(t))\,dM$$

where $0 < R < R_o$, R_o : the injectivity radius of M, and $B_R(x)$ is the geodesic ball of radius R around $x \in M$.

Lemma 4.1: There exists $\varepsilon > 0$, $c_o \in \mathbb{R}$ such that

$$\int_0^T \int_M |\nabla^2 u|^2 dM dt \leq c_o\, E(u_o)\, (1+TR^{-2}) ,$$

whenever $\varepsilon(R,T) \leq \varepsilon$.

The proof uses the following fundamental Sobolev-type inequality, cp. [17, Lemma 3.1]:

Lemma 4.2: For any $v \in L^\infty([0,T];L^2(M,\mathbb{R}^N))$ with $|\nabla v| \in L^2(M\times[0,T])$ we have $v \in L^4(M\times[0,T];\mathbb{R}^N)$ and for any $R < R_o$ there holds

$$\int_0^T\int_M |v|^4 dM dt \leq c^*\cdot \sup_{x\in M,\, t\leq T} \Big(\int_{B_R(x)} |v(t)|^2 dM \Big) \cdot \Big(\int_0^T\int_M |\nabla v|^2 dM dt + \frac{1}{R^2} \int_0^T\int_M |v|^2 dM dt \Big)$$

with a uniform constant $c^* = c^*(M)$.

Moreover, we need:

<u>Lemma 4.3:</u> There holds the energy estimate

$$E(u(\cdot,T)) + \int_0^T \int |\partial_t u|^2 dM dt \le E(u_o) .$$

<u>Proof:</u> Simply compute, using (1.1), (2.1)

$$\begin{aligned} \frac{d}{dt} E(u(\cdot,t)) &= \int_M (-\Delta_M u - {}^N\Gamma(u)(\nabla u,\nabla u)_M) \cdot \partial_t u \, dM \\ &= - \int_M |\partial_t u|^2 dM . \end{aligned}$$

q.e.d.

<u>Proof of Lemma 4.1:</u> Multiply (2.1) by $\Delta_M u$ and integrate to obtain for sufficiently small $\varepsilon > 0$:

$$\begin{aligned} &\int_0^T \int_M \partial_t \left(\frac{|\nabla u|^2}{2}\right) + |\Delta_M u|^2 \, dM dt \\ &\le c(\delta) \int_0^T \int_M |\nabla u|^4 dM dt + \delta \int_0^T \int_M |\nabla^2 u|^2 dM dt \\ &\le (c(\delta) c^* \varepsilon(R,T) + \delta) \int_0^T \int_M |\nabla^2 u|^2 dM dt \\ &\quad + c(\delta) c^* \varepsilon(R,T) \, R^{-2} \int_0^T \int_M |\nabla u|^2 dM dt \\ &\le 2\delta \int_0^T \int_M |\nabla^2 u|^2 dM dt + c(\delta) T R^{-2} \sup_{t \le T} E(u(\cdot,t)) , \end{aligned}$$

for any pre-assigned $\delta > 0$.

Note that after integrating by parts for a.e. $t \le T$

$$\int_M |\nabla^2 u|^2 dM \le c \int_M |\Delta_M u|^2 dM + c \int_M |\nabla u|^2 dM .$$

Moreover, by Lemma 4.3

$$E(u(\cdot,t)) \le E(u_o) , \quad \forall \, t \le T .$$

The claim follows.

q.e.d.

From Lemma 4.1 full regularity may be deduced. Differentiate (2.1) in t and multiply by $\partial_t u$. Integrating one obtains:

$$\begin{aligned} &\int_0^T \int_M \partial_t \left(\frac{|\partial_t u|^2}{2}\right) + |\nabla \partial_t u|^2 dM dt \\ &\le c(\delta) \int_0^T \int_M |\nabla u|^2 |\partial_t u|^2 dM dt + \delta \int_0^T \int_M |\nabla \partial_t u|^2 dM dt \\ &\le c(\delta) \left(\int_0^T \int_M |\nabla u|^4 dM dt\right)^{1/2} \left(\int_0^T \int_M |\partial_t u|^4 dM dt\right)^{1/2} \\ &\quad + \delta \int_0^T \int_M |\nabla \partial_t u|^2 dM dt , \end{aligned}$$

for any $\delta > 0$.

Let $\int_M |\partial_t u|^2 dM \Big|_t$ achieve its supremum at $t = t_o$. We may assume $t_o = T$. By Lemma 4.1 and 4.2:

$$\int_M |\partial_t u|^2 dMdt\Big|_{t=T} - \int_M |\partial_t u|^2 dM\Big|_{t=0} + \int_0^T \int_M |\nabla \partial_t u|^2 dMdt \le$$

$$\le c(\delta)\Big(\int_0^T \int_M |\nabla u|^4 dMdt\Big)^{1/2} \cdot \Big(\sup_{t\le T}\Big(\int_M |\partial_t u|^2 dM\Big) \cdot \Big(\int_0^T \int_M |\nabla\partial_t u|^2 dMdt + \int_0^T \int_M |\partial_t u|^2 dMdt\Big)\Big)^{1/2} +$$

$$+ \delta \int_0^T \int_M |\nabla \partial_t u|^2 dMdt \le$$

$$\le \Big(c(\delta)\Big(\int_0^T \int_M |\nabla u|^4 dMdt\Big)^{1/2} + \delta\Big)\Big(\int_M |\partial_t u|^2 dM\Big|_{t=T} + \int_0^T \int_M |\nabla\partial_t u|^2 dMdt + E(u_o)\Big).$$

By absolute continuity of the Lebesgue integral of $|\nabla u|^4$ upon choosing δ, $T > 0$ small enough we can achieve that

$$c(\delta)\Big(\int_0^T \int |\nabla u|^4 d\ dt\Big)^{1/2} + \delta \le 1/2$$

and obtain that

$$\sup_{t\le T} \int_M |\partial_t u|^2 dM \le c \int_M |\partial_t u|^2 dM\Big|_{t=0} + c\, E(u_o) \quad .$$

Back to (2.1) and using Lemma 4.2 for $v(x,t) \equiv \nabla u(x,T)$ this implies

$$\int_M |\nabla^2 u|^2 dM\Big|_{t=T} \le c \int_M |\nabla u|^4 dM\Big|_{t=T} + c \int_M |\partial_t u|^2 dM\Big|_{t=T}$$

$$\le c \int_M |\partial_t u|^2 dM\Big|_{t=0} + c\, E(u_o)$$

$$+ c\, \varepsilon(R,T)\Big(\int_M |\nabla^2 u|^2 dM\Big|_{t=T} + R^{-2} E(u_o)\Big) .$$

I.e.

$$\sup_{t\le T} \int_M |\nabla^2 u|^2 dM \le c \int_M |\partial_t u| dM\Big|_{t=0} + c\, E(u_o)(1+R^{-2})$$

Since $H^{2,2}(M;\mathbb{R}^N) \hookrightarrow C^\alpha(M;\mathbb{R}^N)$, $\forall\, \alpha < 1$, this estimate implies continuity of u on $M \times [0,T]$. The regularity theory of Ladyženskaya, Solonnikov and Ural'ceva [10] now is applicable, and we obtain a-priori bounds of all derivatives of u on $M \times]0,\tau]$ in terms of $E(u_o)$, R, $\int_M |\partial_t u|^2 dM\big|_{t=0}$, where $\tau > 0$ is determined by the condition that $\int_0^\tau \int_M |\nabla u|^4 dMdt$ be sufficiently small. These estimates are translation invariant. Moreover, by Fubini's theorem and Lemma 4.3

$$\inf_{\substack{t_o-\tau\le t\le t_o \\ t\ge 0}} \int_M |\partial_t u|^2 dM \le \inf(t_o,\tau)^{-1} \int_{\sup\{0,t_o-\tau\}}^{t_o} \int_M |\partial_t u|^2 dMdt \le \inf(t_o,\tau)^{-1} E(u_o) \; .$$

Hence we obtain a-priori bounds on any interval $[t_o,T]$ in terms of $E(u_o)$, R, t_o, T and τ. Using Lemma 4.1 and a compactness argument, one finally can show that the modulus τ is completely determined by $E(u_o)$ and R. This gives

Proposition 4.4: If $u : M \times [0,T] \to N$ is a smooth solution to (2.1) with smooth initial value u_o and $\varepsilon(R,T) < \varepsilon$ then u and its derivatives are uniformly bounded on $M \times [0,T]$ in terms of T, R, and u_o. On any set $M \times]t_o,T]$, $t_o > 0$, bounds will depend only on T, R, $E(u_o)$, and t_o.

Control of the energy density function $\varepsilon(R,T)$ therefore is crucial for Theorem 3.1:

Let

$$E_R(v;x) = \int_{B_R(x)} e(v)\, dM \; .$$

<u>Lemma 4.5:</u> For any solution u of (2.1), (2.2) in $M \times [0,T]$ and any $R < R_o$ there holds the estimate

$$E_R(u(\cdot,T);x) \leq E_{2R}(u_o;x) + c_1\, TR^{-2} E(u_o) .$$

c_1 is a constant depending only on M and N.

<u>Proof:</u> Multiply (2.1) by $\partial_t u\varphi^2$ where $\varphi \in C_o^\infty(B_{2R}(x))$ is $\equiv 1$ on $B_R(x)$ and integrate to obtain

$$\int_0^T \int_M |\partial_t u|^2 \varphi^2 + \frac{d}{dt} e(u(\cdot,t))\varphi^2 dM dt \leq c \int_0^T \int_M |\nabla u||\partial_t u||\nabla\varphi|\varphi dM dt$$

$$\leq \frac{1}{2}\int_0^T \int_M |\partial_t u|^2\varphi^2 dM dt + cTR^{-2}E(u_o) .$$

The lemma follows.

q.e.d.

<u>Remark 4.6:</u> i) In particular, if for some $R > 0$

$$\sup_x E_R(u_o;x) \leq \frac{\varepsilon}{2} ,$$

for any solution u to (2.1), (2.2) we will have

$\varepsilon(R,T) \leq \varepsilon$ for $T = \dfrac{\varepsilon R^2}{2c_1 E(u_o)}$. Thus, by Proposition 4.4 we will have uniform bounds on u in $M \times [0,T]$ in terms of u_o alone. This permits to construct local solutions to (2.1) for smooth initial data u_o. By the same token we obtain uniform a-priori bounds locally on $M \times]0,T]$ for solutions u with initial values in an $H^{1,2}$-neighborhood of u_o. Hence we may extend the flow u also to initial values u_o of class $H^{1,2}(M;N)$. By Lemma 4.3 u assumes its initial values continuously in $H^{1,2}$.

ii) By Proposition 4.4 and Lemma 4.5 a local solution to (2.1) as above may be continued to an interval $[0,T_1[$, where T_1 is characterized by the condition that

$$\liminf_{t \nearrow T_1} \left(\sup_{x\in M} E_R(u(\cdot,t);x)\right) \geq \varepsilon$$

for all $R > 0$. If $(x_1,\dots,x_k)$ are points in M where

$$\limsup_{t\nearrow T_1} (E_R(u(\cdot,t),x_j) \geq \varepsilon , \quad 1 \leq j \leq k ,$$

let $R < \frac{1}{2} \inf_{i\neq j} \operatorname{dist}(x_i,x_j)$. Then for all t close to T_1

$$E_{2R}(u(\cdot,t);x_j) \geq E_R(u(\cdot,t_j);x_j) - c_1 \frac{t_j - t}{R^2} E(u_o)$$

and by suitable choice of t_j the right hand side can be made $\geq \varepsilon/2$ for all $j = 1,\dots,k$.

But by choice of R

$$k \cdot \frac{\varepsilon}{2} \leq \sum_{j=1}^{k} E_{2R}(u(\cdot,t);x_j) \leq E(u(\cdot,t)) \leq E(u_o)$$

and $k \leq \dfrac{2E(u_o)}{\varepsilon}$ is uniformly a-priori bounded.

Moreover, $u(\cdot,t) \rightharpoonup: u_1 \in H^{1,2}(M;N)$ weakly, and we may simply continue u to some larger interval $[0,T_2[$ by letting u solve (2.1) on $[T_1,T_2[$ with initial value u_1, etc.

Since $u(\cdot,t) \to u_1$ in $L^2(M,N)$ as $t \to T_1$, the extended function u will also be a distribution solution to (2.1) on $M \times [0,T_2]$.

Note that $E(u_1) \leq \lim_{R\to 0} \int_{M\setminus \cup_j B_{2R}(x_j)} e(u_1)dM$

$$\leq \lim_{R\to 0} \lim_{t\to t_1} \int_{M\setminus \cup_j B_{2R}(x_j)} e(u(\cdot,t))dM \leq E(u_o) - \frac{k\varepsilon}{2} .$$

In particular, by iteration of our above argument there can be at most finitely many singularities (x_j,t_j), $t_j \leq \infty$, satisfying the condition

$$\limsup_{t\nearrow t_j} E_R(u(\cdot,t);x_j) \geq \varepsilon$$

for any R.

A more detailed analysis as in [17] shows that if for $(\bar{x},\bar{t}) \in M \times]0,\infty[$

$$\limsup_{t\nearrow \bar{t}} E_R(u(\cdot,t);\bar{x}) < \varepsilon$$

for some $R > 0$, then u is regular in a neighborhood of $(\bar{x},\bar{t})$.

iii) As $t_m \to \infty$ suitably, by Lemma 4.3

$$\int_M |\partial_t u|^2 dM \Big|_{t=t_m} \to 0 .$$

Suppose that $T = \infty$ is non-singular in the sense that

$$\limsup_{t\to\infty} (\sup_{x\in M} E_R(u(\cdot,t),x)) < \varepsilon$$

for some $R > 0$.

Then Lemma 4.2 (applied to $v(x,t) \equiv u(x,t_m)$) assures that at $t = t_m$

$$\int_M |\nabla^2 u|^2 dM \leq c \int_M |\nabla u|^4 dM + c \int_M |\partial_t u|^2 dM \leq$$

$$\leq \frac{1}{2} \int_M |\nabla^2 u|^2 dM + c\, E(u_o) + o(1)$$

where $o(1) \to 0$ $(m \to \infty)$. By Rellich's theorem we may assume that $u(\cdot,t_m) \to u_\infty$ strongly in $H^{1,p}(M,N)$, for any $p < \infty$. But then by (2.1)

$$-\Delta_M u(\cdot,t_m) + {}^N\Gamma(u_\infty)(\nabla u_\infty,\nabla u_\infty)_M = -\Delta_M u_\infty$$

in L^2. Thus, $u_\infty \in H^{2,2}(M,N)$ is harmonic (and hence regular). Moreover, $u(\cdot,t_m) \to u_\infty$ strongly in $H^{2,2}(M,N)$.

Again, a local analysis shows that also if $T = \infty$ is singular in the sense that at points $\{x_1,\dots,x_k\}$

$$\limsup_{t\to\infty} E_R(u(\cdot,t),x_j) \geq \varepsilon$$

for all $R > 0$, then for suitable numbers $t_m \to \infty$ the family $\{u(\cdot,t_m)\}$ will be equi-bounded locally in $H^{2,2}$ on $M\setminus\{x_1,\dots,x_k\}$, and hence accumulate in $H^{2,2}_{loc}(M\setminus\{x_1,\dots,x_k\},N)$ at a harmonic map $u_\infty : M\setminus\{x_1,\dots,x_k\} \to N$. Since $E(u_\infty) \leq E(u_o)$, by [12, Theorem 3.6] u_∞ may be extended to a smooth harmonic map $u_\infty : M \to N$.

iv) If at $(\bar{x},\bar{t})$ for all $R > 0$ there holds

$$\limsup_{t\nearrow \bar{t}} E_R(u(\cdot,t),\bar{x}) \geq \varepsilon$$

we can rescale $u \to u^{(m)}(x,t) \equiv u(x_m+r_mx, t_m+r_m^2t)$ such that

$$\limsup_{t\nearrow 0} \sup_{r_m|x|<R_o} E_1(u^{(m)}(\cdot,t),x) \leq E_1(u^{(m)}(\cdot,0),0) = \frac{\varepsilon}{2} ,$$

while by Lemma 4.3 and absolute continuity of the Lebesgue integral

$$\int_{-1}^{0} \int_{B_{R_o/r_m}(0)} |\partial_t u^{(m)}|^2 dxdt \to 0 \quad (m \to \infty) .$$

Note that $u^{(m)}$ solves an equation like (2.1) on $B_{R_o/r_m}(0) \times [-1,0]$. Hence a local estimate like Lemma 4.1 is valid and we obtain uniform local a-priori bounds for $u^{(m)}$. A reasoning as in iii) now shows that $u^{(m)}(\cdot,\tau_m) \equiv u_m \to \tilde{u}$ strongly locally in $H^{2,2}_{loc}(\mathbb{R}^2;N)$ for some sequence $\tau_m \in [-1,0]$, where $\tilde{u}$ is harmonic and has energy $\frac{\varepsilon}{2} \leq E(\tilde{u}) \leq E(u_o)$. By [12, Theorem 3.6] again, $\tilde{u}$ may be extended to a smooth, non-constant harmonic map $\bar{u} : S^2 \cong \overline{\mathbb{R}^2} \to N$.

This essentially proves Theorem 3.1.

We now proceed to give further applications of this method to surfaces of prescribed constant mean curvature with free boundaries on a supporting surface in $\mathbb{R}^3$.

5. Applications to H-surfaces with free boundaries

Let S be a smooth embedded compact hypersurface in $\mathbb{R}^3$. An H-surface supported by S is a surface X of mean curvature H which meets S orthogonally along its boundary. Suppose X is of the type of the disc

$$B = \{w \in \mathbb{R}^2 \,|\, |w| < 1\} ,$$

then we may introduce isothermal coordinates $w = (w^1,w^2)$, $X(w) = (X^1(w),X^2(w),X^3(w))$ over B on X . In these coordinates the following relations hold:

(5.1) $\quad \Delta X = 2H\, \partial_1 X \wedge \partial_2 X ,$

(5.2) $\quad |\partial_1 X|^2 - |\partial_2 X|^2 = 0 = \partial_1 X \cdot \partial_2 X ,$

(5.3) $\quad X|_{\partial B} : \partial B \to S ,$

(5.4) $\quad \frac{\partial}{\partial n} X(w) \perp T_{X(w)} S , \ \forall\, w \in \partial B .$

Here, "$\wedge$" is the exterior product in $\mathbb{R}^3$, "$\cdot$" denotes scalar product, n is the unit exterior normal on ∂B , "$\perp$" means orthogonal; and T_pS denotes the tangent space to S at p .

In particular, if $H = 0$ a solution X to (5.1) - (5.4) is a (parametric) minimal surface supported by S .

Minimal surfaces supported by compact hypersurfaces or Schwarz' chains consisting of a collection of Jordan arcs and plane segments have been studied as early as 1816. The famous Gergonne problem was solved by H.A. Schwarz in 1872 [14]. For smooth supporting surfaces of positive genus

Courant [1] proved the existence of minimal surfaces minimizing Dirichlet's integral

$$D(X) = \frac{1}{2} \int_B |\nabla x|^2 dw$$

among surfaces X with boundary $X(\partial B)$ in any given homology class of curves on S .

Courant's results were generalized to H-surfaces by Hildebrandt [6] for the condition that $|H|R \leq 1$ for some $R > 0$ such that $S \subset B_R(0) \subset \mathbb{R}^3$.

If $S \cong S^2$ Courant's approach fails since S^2 does not support any non-trivial 1-dimensional cycle.

In fact, if S is strictly convex (the boundary of a strictly convex body in $\mathbb{R}^3$) it is easy to see that the only relative minimizers of D among surfaces X with boundary $X(\partial B) \subset S$ are the constants $X(w) \equiv p \in S$.

In 1984, by adapting the Sacks-Uhlenbeck approximation method, the author was able to establish the following existence result [16]:

<u>Theorem 5.1:</u> Suppose S is diffeomorphic to the standard sphere S^2 in $\mathbb{R}^3$. Then S supports a non-constant minimal surface.

Simultaneously, Smyth [15] obtained existence results for the tetrahedron. His proof, however, cannot be extended to more complicated polyhedral boundaries or smooth supporting surfaces.

Later, Grüter and Jost [4], [9] improved Theorem 5.1 for <u>convex</u> surfaces S and established the existence of smoothly <u>embedded</u> minimal discs supported by S .

Recently, the author succeded in extending Theorem 5.1 to surfaces of prescribed constant mean curvature [18]:

<u>Theorem 5.2:</u> Suppose $S \subset B_R(0) \subset \mathbb{R}^3$ is diffeomorphic to S^2 . Then for almost every (in the sense of Lebesgue measure) $H \in \mathbb{R}$ satisfying the condition $|H|R < 1$ there exists an H-surface supported by S .

By absence of suitable a-priori bounds of Dirichlet's integral for solutions to (5.1) - (5.4) in the case of non-vanishing H Theorem 5.2 could not be derived by the Sacks-Uhlenbeck method.

Instead, the heat flow methods outlined in Section 3 proved applicable - as we will explain. (For simplicity we restrict ourselves to the case $H = 0$.)

Remark that by regularity of S there exists a δ-neighborhood $U_\delta(S)$ of S such that any point $p \in U_\delta(S)$ has a unique nearest neighbor $\pi(p) \in S$:

$$|p - \pi(p)| = \inf_{q \in S} (p-q) \quad .$$

This defines the reflection $R : U_\delta(S) \to U_\delta(S)$, $R(p) = 2\pi(p) - p$ in S . If $S \in C^m$, $m \geq 1$, then $R \in C^{m-1}$.

Reflecting X in S we obtain a surface

$$\tilde{X}(w) = \begin{cases} X(w), & w \in B \\ R(X(\frac{w}{|w|^2})), & w \notin B, \; X(\frac{w}{|w|^2}) \in U_\delta(S), \end{cases}$$

defined in a domain $\mathcal{D}(\tilde{X}) \supset \bar{B}$.

Note that since R is involutory there holds

$$R(\tilde{X}(w)) = X(\frac{w}{|w|^2}), \quad \forall\, w \in \mathcal{D}(\tilde{X}) \setminus B .$$

Inserting into (5.1) we see that $\tilde{X}$ satisfies an equation like (1.1) with a metric

$$\tilde{g}_{ik}(p) = \frac{\partial}{\partial p^i} R(p) \cdot \frac{\partial}{\partial p^k} R(p)$$

on $\mathcal{D}(\tilde{X}) \setminus B$ carrying the metric $\tilde{\gamma}_{\alpha\beta}$ induced by the reflection $w \to \frac{w}{|w|^2}$.

If now we let

$$g_{ik}(p,w) = \begin{cases} \delta_{ik}, & w \in B \\ \tilde{g}_{ik}(p,w), & \text{else} \end{cases}$$

$$\gamma_{\alpha\beta}(w) = \begin{cases} \delta_{\alpha\beta}, & w \in B \\ \tilde{\gamma}_{\alpha\beta}(w), & \text{else} \end{cases}$$

$\tilde{X}$ may be viewed as a (generalized) harmonic map $\tilde{X} : (\mathcal{D}(\tilde{X}), \gamma) \to (\mathbb{R}^3, g)$.

Note that g, γ are Lipschitz continuous in $w = re^{i\phi}$ and even smooth in angular direction ϕ and as functions

of p . Hence the Christoffel symbols Γ related to g will be smooth in p , bounded and measurable in $w \in \mathcal{D}(\tilde{X})$.

Similarly, a solution to the evolution problem[1)]

(5.5) $\quad \partial_t X - \Delta X = 0 \quad$ in $\; B \times [0,T]$

(5.6) $\quad X(\partial B \times [0,T]) \subset S$

(5.7) $\quad \frac{\partial}{\partial n} X(w,t) \perp T_{X(w,t)} S, \quad \forall\, (w,t) \in \partial B \times]0,T]$

(5.8) $\quad X|_{t=0} = X_0$

with $D(X_0) < \infty$, $X_0(\partial B) \subset S$, can be extended to a solution $\tilde{X} : \mathcal{D}(\tilde{X}) \supset \bar{B} \times [0,T] \to \mathbb{R}^3$ of an evolution problem

(5.9) $\quad \partial_t \tilde{X} - \Delta_\gamma \tilde{X} - {}^g\Gamma(\tilde{X})(\nabla\tilde{X}, \nabla\tilde{X}) = 0$

(5.10) $\quad \tilde{X}|_{t=0} = \tilde{X}_0 \equiv \begin{cases} X_0(w) & \text{on } B \\ R(X_0(\frac{w}{|w|^2})) & \text{else} \end{cases}$

for a harmonic map from a domain in $(\mathbb{R}^2, \gamma)$ into $(\mathbb{R}^3, g)$.

A-priori bounds for (5.5) - (5.8) may now be derived from (5.9), (5.10) in the same way as outlined for (2.1), (2.2) in Section 4. We rely on the following observations:

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1) This model of the motion of soap films with free boundaries has the best mathematical properties. Physically, the model corresponds to the assumption that boundary (adhesive) forces act instantaneously to create a vertical contact angle between the soap film and the supporting surface. Surface tension then governs the response "in the large".

Observation 5.3: For any smooth solution X to (5.5) - (5.8)

$$\int_0^T \int_B |\partial_t X|^2 dwdt + D(X(\cdot,T)) \leq D(X_o) .$$

Proof: Multiply (5.5) by $\partial_t X$ and integrate by parts using that $\partial_t X \cdot \frac{\partial}{\partial n} X \equiv 0$ as a consequence of (5.6), (5.7).

q.e.d.

Observation 5.4: Suppose (as we may) that $\mathcal{D}(\tilde{X}) \subset B_2(0) \times [0,T]$. Then any pointwise or integral estimate for X on $B \times [0,T]$ entrains a corresponding estimate for $\tilde{X}$ on $\mathcal{D}(\tilde{X})$ and vice versa.

Observation 5.5: The analysis of Section 4 can be localized to the set

$$\{(w,t) \in \mathcal{D}(\tilde{X}) \,|\, w \in B \quad \text{or} \quad \tilde{X}(w,t) \in U_{\delta/2}(S)\}$$

by means of the following cut-off function:

$$\tilde{\varphi}(w,t) = \begin{cases} 1 \,, & w \in B \\ \psi(\text{dist}(\tilde{X}(w,t),S)) \,, & \text{else} \end{cases}$$

where $\psi \in C_o^\infty$ equals 1 in a neighborhood of 0 and vanishes for arguments $\geq \delta/2$.

Note that $|\nabla\psi| \leq c \, |\nabla\tilde{X}|$, etc. and partial derivatives of ψ

will contribute "error terms" of the same order as have to be handled in Lemma 4.1.

Lemma 4.1 now conveys to our flow (5.9), (5.10), and we obtain

Theorem 5.6: For any $X_o \in H^{1,2}(B;\mathbb{R}^3)$ with $X_o(\partial B) \subset S$ there exists a unique distribution solution $X : B \times [0,\infty[\to \mathbb{R}^3$ of (5.5) - (5.8) which is regular on $\bar{B} \times]0,\infty[$ with exception of finitely many points $\{(w_j,t_j)\}$, $1 \leq j \leq k$, and unique in this class.

X assumes its initial value X_o continuously in $H^{1,2}$.

At a singularity $(\bar{w},\bar{t})$ a smooth, non-constant minimal surface $\tilde{X}$ supported by S separates in the sense that for suitable $r_m \to 0$, $t_m \to \bar{t}$, $w_m \to \bar{w}$ after a possible rotation of coordinates

$$X(w_m + r_m w, t_m) \to \tilde{X} \qquad (m \to \infty)$$

strongly in $H^{2,2}$ on relatively compact subdomains of the upper half-plane $\mathbb{R}^2_+$.

$\tilde{X}$ has finite Dirichlet integral and may be extended to a smooth, non-constant minimal surface $\tilde{X} : B \cong \mathbb{R}^2_+ \to \mathbb{R}^3$.

As $t \to \infty$, $X(\cdot,t)$ converges weakly in $H^{1,2}$ to a minimal surface X_∞ supported by S. If all $t_j < \infty$, $1 \leq j \leq k$, $X(\cdot,t) \to X_\infty$ even strongly in $H^{2,2}$.

We caution the reader that inspite of "good" a-priori estimates local existence for (5.5) - (5.8) cannot be established easily. Incead, it is a consequence of a rather delicate fixed point argument, [18, Lemma 3.16].

Now we can give a proof of Theorem 5.1 as follows: Let S be diffeomorphic S^2, and let

$$C(S) = \{X \in H^{1,2}(B;\mathbb{R}^3) \,|\, X(\partial B) \subset S\}$$

be the class of $H^{1,2}$-surfaces with boundary on S.

If for some $X_o \in C(S)$ the flow X through X_o becomes singular at some $(\bar{w},\bar{t})$, by Theorem 5.6 there exists a smooth non-constant minimal surface supported by S.

Otherwise, the flow induces a continuous deformation of $C(S)$. Now argue indirectly: Suppose S supports only constant (trivial) solutions to (5.1) - (5.4). Then the flow defines a homotopy equivalence of $C(S)$ with the space of constant maps $X \equiv p \in S$. The space $C(S)$ contracts to the space $C_o(S) = \{X \in C(S) \,|\, \Delta X = 0\}$. This latter space (by harmonic extension) is topologically equivalent to the space of closed curves $H^{\frac{1}{2},2}(\partial B;S)$ on S which relative to the constants supports non-trivial 1-dimensional cycles. In particular $C(S)$ cannot be homotopically equivalent to the space of constant maps. The contradiction proves Theorem 5.1.

q.e.d.

Open Problem: Taking account of the $O(2)$-action on Dirichlet's integral induced by rotation $\phi \mapsto \phi + \phi_o$ and reflection $\phi \mapsto 2\pi - \phi$ of the angular coordinates of points $w = re^{i\phi} \in B$ and the rich topological structure of the quotient space of the space of curves on S by this $O(2)$-action, cp. [7], [11], we expect higher multiplicity results for minimal surfaces supported by a surface $S \cong S^2$. However, to this moment only partial results are known [9], [15].

A possible approach to this question would be to prove global regularity of the flow (5.5) - (5.8) (at least for convex supporting surfaces) and thus to make problem (5.1) - (5.4) accessible by Ljusternik-Schnirelman or Morse theory.

Proof of Theorem 5.2: As stated earlier, by lack of suitable a-priori bounds for H-surfaces, $H \neq 0$, Theorem 5.2 is much harder to prove. Employing also variations of the parameter H, however, we do succeed in finding $H^{1,2}$-bounds for the H-surface flow for certain values of H corresponding to the points of differentiability of a monotone function and obtain Theorem 5.2. For details of the proof we refer the interested reader to [18].

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