INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR 281/11

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

(11 January - 5 February 1988)

AN INTRODUCTION TO CRITICAL POINT THEORY
(PART II)

M. WILLEM
Universite Catholique de Louvain
Institut de Mathematique Pure et Applique
2 Chemin du Cyclotron
B.1348 Louvain La Neuve
BELGIUM

II – Minimax theorems.

Consider the following intuitive situation. If $\varphi \in C^1(\mathbb{R}^2, \mathbb{R})$, we can view $\varphi(x,y)$ as the altitude of the point of the graph of φ having $[x,y]$ as projection on $\mathbb{R}^2$. Assume that there exists a bounded open neighborhood Ω of $u_0 \in \mathbb{R}^2$ and $u_1 \in \mathbb{R}^2 \setminus \Omega$ such that $\varphi(u) > \max\{\varphi(u_0), \varphi(u_1)\}$ whenever $u \in \partial\Omega$. We can thus consider the point $[u_0, \varphi(u_0)]$ as located in a valley surrounded by a ring of mountains pictured by the set $\{[u, \varphi(u)] : u \in \partial\Omega\}$, the point $[u_1, \varphi(u_1)]$ being located outside the ring. To go from $[u_0, \varphi(u_0)]$ to $[u_1, \varphi(u_1)]$ in a way which minimizes the highest altitude of the path, we must cross the mountain ring through the lowest mountain pass. The projection on $\mathbb{R}^2$ of the top of this mountain pass will provide a critical point of φ.

More generally, we have the following result

<u>Theorem 3.</u> (Ambrosetti-Rabinowitz mountain pass theorem). Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies (PS). If there exist $u_0 \in X$, $u_1 \in X \setminus \Omega$, where Ω is a bounded open neighborhood of u_0, such that, for some $\alpha \in \mathbb{R}$,

$$\varphi(u) \geq \alpha > \max\{\varphi(u_0), \varphi(u_1)\}$$

whenever $u \in \partial\Omega$, then φ has a critical value $c \geq \alpha$.

Proof. Let us define

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$$

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)).$$

Since every curve $\gamma \in \Gamma$ crosses $\partial\Omega$, $\max_{t \in [0,1]} \varphi(\gamma(t)) \geq \alpha$.

Hence $c \geq \alpha$ and there exists $\bar{\varepsilon} > 0$ such that

$$c - \bar{\varepsilon} \geq \max\{\varphi(u_0), \varphi(u_1)\}.$$

Assume that $K_c = \emptyset$. Let $\varepsilon \in]0, \bar{\varepsilon}[$ and $\eta \in \mathcal{C}([0,1] \times X, X)$ be given by lemma 1 when $U = \emptyset$. The definition of c implies the existence of $\gamma \in \Gamma$ such that $\max_{t \in [0,1]} \varphi(\gamma(t)) \leq c + \varepsilon$

Let us now consider the continuous path $\tilde{\gamma}$ defined by $\tilde{\gamma}(t) = \eta(1, \gamma(t))$. Since $\varphi(u_0) \leq c - \bar{\varepsilon}$, we have

$\tilde{\gamma}(0) = \eta(1, \gamma(0)) = \eta(1, u_0) = u_0.$

Similarly $\tilde{\gamma}(1) = u_1$. Thus $\tilde{\gamma} \in \Gamma$. But $\gamma([0,1]) \subset \varphi^{c+\varepsilon}$, so that $\eta(1, \gamma([0,1]) \subset \varphi^{c-\varepsilon}$ and

$$c \leq \max_{t \in [0,1]} \varphi(\tilde{\gamma}(t)) \leq c - \varepsilon,$$

a contradiction. Hence $K_c \neq \emptyset$ and the proof is complete. $\square$

Let us now consider an other geometric situation. (Rabinowitz saddle point theorem).

Theorem 4. Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies (PS) and that $X = Y \oplus Z$ with Y finite dimensional. If there exists constants $b_1 < b_2$ and a neighborhood Ω of 0 in Y bounded such that

$$\varphi|_Z \geq b_2$$

$$\varphi|_{\partial\Omega} \leq b_1$$

then φ has a critical point.

Proof. Let us define

$$\Gamma = \{\gamma \in C(\bar{\Omega}, X) : \gamma(y) = y \text{ if } y \in \partial\Omega\}.$$

$$c = \inf_{\gamma \in \Gamma} \max_{y \in \bar{\Omega}} \varphi(\gamma(y)).$$

If P denotes the projector on Y with kernel Z, then $P\gamma \in C(\bar{\Omega}, Y)$ and $P\gamma(y) = y$ whenever $\gamma \in \Gamma$ and $y \in \partial\Omega$. By a topological degree argument, $P\gamma$ has a zero inside Ω, i.e. there exists $y \in \Omega$ such that $\gamma(y) \in Z$. (In the easy case when $\dim Y = 1$, the intermediate value theorem suffices.) In particular

$$\max_{y \in \Omega} \varphi(\gamma(y)) \geq \inf_{z \in Z} \varphi(z) \geq b_2 .$$

Hence $c \geq b_2$ and there exists $\bar{\varepsilon} > 0$ such that $c - \bar{\varepsilon} \geq b_1$. Assume that $K_c = \emptyset$. Let $\varepsilon \in]0, \bar{\varepsilon}[$ and $\eta \in C([0,1] \times X, X)$ be given by Lemma 1 when $U = \emptyset$. There exists $\gamma \in \Gamma$ such that

$$\max_{y \in \bar{\Omega}} \varphi(\gamma(y)) \leq c + \varepsilon .$$

Set $\tilde{\gamma} = \eta(1, \gamma(\cdot))$. If $y \in \partial\Omega$, then $\varphi(y) \leq b_1$ and $\tilde{\gamma}(y) = \eta(1, \gamma(y)) = \eta(1, y) = y$. Thus $\tilde{\gamma} \in \Gamma$.

Since $\gamma(\bar{\Omega}) \subset \varphi^{c+\varepsilon}$, we have that
$\eta(1, \gamma(\bar{\Omega})) \subset \varphi^{c-\varepsilon}$ and

$$c \le \max_{y \in \bar{\Omega}} \varphi(\tilde{\gamma}(y)) \le c - \varepsilon,$$

a contradiction. Hence $K_c \neq \emptyset$. □

The following application is due essentially to Ahmad, Lazer and Paul.

<u>Theorem</u> 5. Assume that V and $D_x V$ are continuous on $[0,T] \times \mathbb{R}^N$ and that there exists a constant $a > 0$ such

$$|D_x V(t,x)| \le a$$

on $[0,T] \times \mathbb{R}^N$. If

$$(10) \qquad \int_0^T V(t,x)\,dt \to +\infty, \quad \text{as } |x| \to \infty$$

then problem (σ) has at least one solution.

<u>Proof</u>. We shall apply theorem 4 to
$\varphi(u) = \int_0^T \left[\frac{|\dot{u}(t)|^2}{2} - V(t, u(t))\right] dt$ with $X = H^1_T$,

$$Y = \{y : [0,T] \to \mathbb{R}^N : y \text{ is constant}\}$$
$$Z = \{z \in H^1_T : \int_0^T z(t)\,dt = 0\}.$$

1) <u>φ is bounded from below on Z.</u>

For $z \in Z$ we have, using Sobolev inequality,

$$\varphi(z) = \frac{\|\dot{z}\|^2_{L^2}}{2} - \int_0^T V(t,0)\,dt - \int_0^T \int_0^1 (D_x V(t, s z(t)), z(t))\,ds\,dt$$

$$\ge \frac{\|\dot{z}\|^2_{L^2}}{2} - c_1 - T a \|z\|_\infty$$

$$\ge \frac{\|\dot{z}\|^2_{L^2}}{2} - c_1 - c_2 \|\dot{z}\|_{L^2}.$$

Thus $b_Z = \inf_Z \varphi > -\infty$.

2) φ satisfies the (PS) condition.

Let (u_k) be a sequence in H^1_T such that $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$ as $k \to \infty$. As in theorem 2, it suffices to prove that (u_k) is bounded in H^1_T.

There is some k_0 such that, for $k \geq k_0$,

$$|\langle \varphi'(u_k), h\rangle| \leq \|h\| \quad , \forall\, h \in H^1_T.$$

In particular, by Sobolev inequality,

$$c_1 \|\dot{u}_k\|_{L^2} \geq \|\tilde{u}_k\| \geq |\langle \varphi'(u_k), \tilde{u}_k\rangle| =$$

$$= \int_0^T [|\dot{u}_k(t)|^2 - (\nabla F(t, u_k(t)), \tilde{u}_k(t))]\, dt$$

$$\geq \|\dot{u}_k\|^2_{L^2} - aT\|\tilde{u}_k\|_\infty$$

$$\geq \|\dot{u}_k\|^2_{L^2} - c_2 \|\dot{u}_k\|_{L^2}.$$

Thus $\dot{u}_k$ is bounded in L^2 and $\tilde{u}_k$ is bounded in H^1_T. Finally

$$c_3 \leq \varphi(u_k) = \int_0^T \left[\frac{|\dot{u}_k(t)|^2}{2} + V(t, \tilde{u}_k(t)) - V(t, u(t)) - V(t, \bar{u}_k(t))\right] dt$$

$$\leq c_4 - \int_0^T V(t, \bar{u}_k)\, dt$$

and $\bar{u}_k$ is bounded by assumption (10).

3) End of the proof.

If $y \in Y$ then $\varphi(y) = -\int_0^1 V(t, y)\, dt$. By assumption (10) there exists $R > 0$ such that

$$\|y\| \geq R \quad \Rightarrow \quad \varphi(y) \leq b_2 - 1 := b_1.$$

It suffices then to apply theorem 4 to get a critical point of φ and, consequently, a solution of problem (8). $\square$

III - Lusternik-Schnirelman theory.

A basic method in critical point theory is to obtain the existence of multiple critical points by using invariance properties of the functional. In this section, we consider the case of a "periodic" functional defined on a Banach space X.

We shall need the following definitions.

A subset C of a topological space Y is <u>contractible in</u> Y if there exists $h \in \mathcal{C}([0,1]\times C, Y)$ and $y \in Y$ such that

$$h(0,u) = u, \; h(1,u) = y, \quad \forall u \in C.$$

A subset A of a topological space Y has <u>category</u> k <u>in</u> Y if k is the least integer such that A can be covered by k closed sets contractible in Y. The category of A in Y is denoted by $\mathrm{cat}_Y(A)$.

A metric space Y is an <u>absolute neighborhood extensor</u>, shortly an ANE, if for every metric space E, every closed subset F of E and every $f \in \mathcal{C}(F,Y)$ there exists a continuous extension of f defined on a neighborhood of F in E.

<u>Examples.</u> [4].

a) $\mathrm{cat}_{S^N}(S^N) = 2$, $\quad \mathrm{cat}_{T^N}(T^N) = N+1$.

b) A finite product of ANE is an ANE.
A convex subset of a normed space is an ANE.
A circle is an ANE.

<u>Proposition 4</u>. Let Y, Z be topological spaces and let $A, B \subset Y$.

i) If $A \subset B$, then $\mathrm{cat}_Y(A) \le \mathrm{cat}_Y(B)$ (monotonicity),

ii) $\mathrm{cat}_Y(A \cup B) \le \mathrm{cat}_Y(A) + \mathrm{cat}_Y(B)$ (subadditivity),

iii) If A is closed and $B = \eta(1, A)$, where $\eta \in \mathcal{C}([0,1] \times A, Y)$ is such that $\eta(0, u) = u$ for every $u \in A$, then $\mathrm{cat}_Y(A) \le \mathrm{cat}_Y(B)$.

iv) If A is a closed subset of an ANE, Y then there exist a closed neighborhood U of A such that $\mathrm{cat}_Y(A) = \mathrm{cat}_Y(U)$ (continuity).

<u>Proof</u>. See [4]. □

Let G be a discrete subgroup of a Banach space X and let $\pi : X \to X/G$ be the canonical surjection. A subset A of X is G-<u>invariant</u> if $\pi^{-1}(\pi(A)) = A$. A function f defined on X is G-<u>invariant</u> if $f(u+g) = f(u)$ for every $u \in X$ and every $g \in G$. If a differentiable functional $\varphi : X \to \mathbb{R}$ is G-invariant, then φ' is also G-invariant. Consequently if u is a critical point of such a φ, then $\pi^{-1}(\pi(u))$ is a set of critical points of φ, and is called a <u>critical orbit</u> of φ.

A G-invariant differentiable function $\varphi : X \to \mathbb{R}$ satisfies the $(PS)_G$ <u>condition</u> if, for every sequence (u_k) in X such that $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$, the sequence $(\pi(u_k))$ contains a convergent subsequence.

<u>Theorem 6.</u> Let $\varphi \in C^1(X,\mathbb{R})$ be a G-invariant functional satisfying the $(PS)_G$ condition. If φ is bounded from below and if the dimension N of the space generated by G is finite, then φ has at least $N+1$ critical orbits.

<u>Lemma 2.</u> Under the assumption of theorem 6, if U is an open invariant neighborhood of K_c then, for every $\bar{\varepsilon}>0$ there exists an $\varepsilon \in]0,\bar{\varepsilon}[$ and $\eta \in C([0,1]\times X, X)$ satisfying properties (a), (b), (c) of lemma 1. Moreover

(d) $\eta(t,u+g)=\eta(t,u)+g$, $\forall t\in[0,1]$, $\forall u\in X$, $\forall g\in G$.

<u>Proof.</u> We consider the special case of X Hilbert space and $\varphi\in C^2(X,\mathbb{R})$. It is easy to construct the vector field f as in lemma 1. Moreover the invariance of U and $\nabla\varphi$ implies the invariance of f. A solution to the Cauchy problem

$$\dot{\sigma} = f(\sigma)$$
$$\sigma(0) = u+g \qquad (g\in G)$$

is given by $\sigma(t,u)+g$. By uniqueness $\sigma(t,u+g) = \sigma(t,u)+g$. Since $\eta(t,u) = \sigma(\sqrt{\varepsilon}\,t, u)$ the proof is complete. □

<u>Lemma 3.</u> Under the assumption of theorem 6, there exists a closed invariant subset A of X such that $\mathrm{cat}_{\pi(X)}\pi(A) = N+1$.

<u>Proof.</u> Let V be the space generated by G. Then X is isomorphic to $\mathbb{R}^N\times Z$, where Z is a complement of V, G is isomorphic to $\mathbb{Z}^N$ and $\pi(X)$ is isomorphic to $T^N\times Z$.

Setting $A = \mathbb{R}^N \times \{o\}$, we obtain
$$\mathrm{cat}_{\pi(X)}(\pi(A)) = \mathrm{cat}_{T^N \times X}(T^N \times \{o\}) = \mathrm{cat}_{T^N}(T^N) = N+1. \square$$

By lemma 3, for $1 \le j \le N+1$, the following set is nonempty:
$$\mathcal{A}_j = \{A \subset X : A \text{ is closed, invariant and } \mathrm{cat}_{\pi(X)} \pi(A) \ge j\}.$$
Define
$$c_j = \inf_{A \in \mathcal{A}_j} \sup_A \varphi.$$
Since $\mathcal{A}_j \subset \mathcal{A}_{j-1}$, we have
$$-\infty < \inf_X \varphi \le c_1 \le c_2 \le \dots \le c_{N+1} < +\infty.$$

Lemma 4. Under the assumptions of theorem 6, if $c_k = c_j = c$ for some $1 \le j \le k \le N+1$ then
$$\mathrm{cat}_{\pi(X)}(\pi(K_c)) \ge k-j+1.$$

Proof. Since $\pi(X) = T^N \times X$ is an ANE and since $\pi(K_c)$ is compact by $(PS)_c$, there exists, by proposition 4, a closed neighborhood N of $\pi(K_c)$ such that $\mathrm{cat}_{\pi(X)}(N) = \mathrm{cat}_{\pi(X)}(\pi(K_c))$. Let $\varepsilon > 0$ be given by lemma 2 applied to $\bar{\varepsilon} = 1$ and $U = \mathrm{int}\, \pi^{-1}(N)$. By definition of c_k, there exists $A \in \mathcal{A}_k$ such that
$$\max_A \varphi \le c_k + \varepsilon = c + \varepsilon.$$

Proposition 4 implies that

(11) $k \le \mathrm{cat}_{\pi(X)}(\pi(A)) \le \mathrm{cat}_{\pi(X)}(\pi(A \setminus U) \cup \pi(U))$

$$\le \mathrm{cat}_{\pi(X)}(\pi(A \setminus U)) + \mathrm{cat}_{\pi(X)}(N)$$

$$= \mathrm{cat}_{\pi(X)}(\pi(A \setminus U)) + \mathrm{cat}_{\pi(X)}(\pi(K_c)).$$

By lemma 2, $\eta(1, A \setminus U) \subset \varphi^{c-\varepsilon}$ and η induces a deformation from $\pi(A\setminus U)$ to $\pi(C)$. But $\max_C \varphi \le c-\varepsilon = c_j - \varepsilon$. By the definition

of c_j, $\mathrm{cat}_{\pi(X)}(\pi(C)) \le j-1$, so that, by proposition 4, $\mathrm{cat}_{\pi(X)}(\pi(A\setminus U)) \le j-1$.

Hence (11) implies that $k \le j-1 + \mathrm{cat}_{\pi(X)}\, \pi(K_c)$. □

Proof of theorem 6. If $c_j < c_k$ whenever $1 \le j < k \le N+1$, then φ has at least $N+1$ critical values.

If $c_j = c_k = c$ for some $1 \le j < k \le N+1$ then $\mathrm{cat}_{\pi(X)}(\pi(K_c)) \ge 2$ and K_c

contains infinitely many critical orbits. □

Theorem 6 generalizes a recent result of S. Li (ICTP Rep. IC-86-191) and was also motivated by a recent result due independently to J. Mawhin and P. Rabinowitz which is the following: Consider the classical Lagrangian

$$L(t,x,y) = \frac{1}{2}(M(t,x)y,y) - V(t,x) + (f(t),x).$$

Assume that

(A_1) $M \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N))$ is a symmetric matrix which is T-periodic in t and x_i, $1 \le i \le N$, (for simplicity).

(A_2) There exists $\alpha > 0$ such that
$(M(t,x)y,y) \ge \alpha|y|^2$, $\forall [t,x,y] \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$.

(R3) $V \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t and x_i, $1 \leq i \leq N$.

(R4) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R}^N)$ is T-periodic in t and $\int_0^T f(t)\,dt = 0$.

Then the Lagrangian system

$$\frac{d}{dt} D_y L(t, u(t), \dot{u}(t)) = D_x L(t, u(t), \dot{u}(t))$$

has at least $N+1$ geometrically distinct periodic solutions.

In particular, when the mean value of the forcing term is 0, a simple forced pendulum has at least 2 geometrically distinct periodic solutions and a double forced pendulum has at least 3 geometrically distinct periodic solutions.

The above result follows easily from theorem 6. In particular, if (e_i) denotes the canonical basis of $\mathbb{R}^N$, the functional

$$\varphi(u) = \int_0^T L(t, u(t), \dot{u}(t))\,dt$$

satisfies the following periodicity property on H^1_T:

$$\varphi(u + Te_i) = \varphi(u), \quad 1 \leq i \leq N.$$

Remark. Using Ekeland principle it is possible to extend Lusternik-Schnirelman theory to $\mathcal{C}^1$ Finsler-manifolds (Szulkin, 1987).

IV - Borsuk-Ulam theorems and index theories.

In this section we consider functionals which are invariant under the action of a compact topological group.

Let G be a topological group. A representation of G over a Banach space X is a family $\{T(g)\}_{g\in G}$ of linear operators $T(g): X \to X$ such that

$$T(0) = id$$
$$T(g_1 + g_2) = T(g_1) \circ T(g_2)$$
$$(g,u) \mapsto T(g)u \text{ is continuous.}$$

A subset A of X is invariant (under the representation) if $T(g)A = A$ for all $g \in G$.
A representation is isometric if $\|T(g)u\| = \|u\|$ for all $g \in G$ and all $u \in X$.

Examples. 1) $G = \mathbb{Z}_2$, $T(0) = id$, $T(1) = -id$.
2) $G = S^1 \simeq \mathbb{R}/T\mathbb{Z}$, $X = W^{1,2}_T$, $T(\theta)$ is given by
$$(T(\theta)u)(t) = u(t+\theta).$$

Let G be a compact topological group and let $\{T(g)\}_{g\in G}$ be an isometric representation of G over a Banach space X.

A mapping R between two invariant subsets of X is equivariant if
$$R \circ T(g) = T(g) \circ R, \quad \forall g \in G.$$

A real function φ defined on an invariant subset of X is invariant if
$$\varphi \circ T(g) = \varphi, \quad \forall g \in G.$$

An index (for $\{T(g)\}_{g\in G}$) is a mapping from the closed invariant subsets of X into $\mathbb{N}\cup\{\infty\}$ such that

(i) $\operatorname{ind} A = 0 \Leftrightarrow A = \emptyset$

(ii) if $R: A_1 \to A_2$ is equivariant and continuous, then
$$\operatorname{ind} A_1 \leq \operatorname{ind} A_2.$$

(iii) if A is compact and invariant, there exists a closed invariant neighborhood N of A such that
$$\operatorname{ind} N = \operatorname{ind} A.$$

(iv) $\operatorname{ind}(A_1 \cup A_2) \leq \operatorname{ind} A_1 + \operatorname{ind} A_2$ for invariant subsets A_1 and A_2.

Examples.

1) (Krasnoselski) Let $G = \mathbb{Z}_2$, $T(0) = id$, $T(1) = -id$. The $\mathbb{Z}_2$-index of a closed invariant subset A of X is the smallest integer k such that there exists an odd mapping $\Phi \in \mathcal{C}(A, \mathbb{R}^k\setminus\{0\})$. If such a mapping does not exist, one defines $\operatorname{ind} A = +\infty$. Finally $\operatorname{ind}\emptyset = 0$.

2) (Benci). Let $\{T(\theta)\}_{\theta\in S^1}$ be an isometric representation of S^1 over X.

The S^1-index of a closed invariant subset A of X is the smallest integer k such that there exists a $n \in \mathbb{N}\setminus\{0\}$ and a $\Phi \in \mathcal{C}(A, \mathbb{C}^k\setminus\{0\})$ satisfying the following equivariance property:
$$\Phi(T(\theta)u) = e^{in\theta}\Phi(u), \quad \theta\in S^1, u\in A.$$
If such a mapping does not exist, we define $\operatorname{ind} A = +\infty$. Finally $\operatorname{ind}\emptyset = 0$.

The verification of properties (i) to (iv) is left to the reader (see [4], [5]).

The following deep results are formulation of Borsuk-Ulam theorems.

Let us recall that

$$\mathrm{Fix}(G) = \{u \in X : T(g)u = u, \ \forall u \in X\}.$$

<u>Theorem 7</u>. The Z_2-index of the sphere S^{k-1} is k.

<u>Theorem 8.</u> Let $\{T(\theta)\}_{\theta \in S^1}$ be an isometric representation of S^1 over $\mathbb{R}^{2k}$ such that $\mathrm{Fix}(\theta) = \{0\}$. Then the S^1-index of the sphere S^{2k-1} is k.

A proof using transversality and degree theory is contained in [4].

Let us now consider the nonlinear eigenvalue problem

$$(12) \qquad \varphi'(u) = \mu \chi'(u), \qquad \chi(u) = a,$$

where $\varphi, \chi \in C^1(\Omega, \mathbb{R})$, Ω open subset of $\mathbb{R}^N$. Assume that

$$\chi(u) = a \Rightarrow \chi'(u) \neq 0$$

so that

$$Z_a = \{u \in \Omega : \chi(u) = a\}$$

is a submanifold of $\mathbb{R}^N$ of codimension 1. Equation (12) is equivalent to

$$\mathrm{Ker}\, \chi'(u) \subset \mathrm{Ker}\, \varphi'(u).$$

Since the tangent space to Z_a at u, $T_u Z_a$ is given by $\mathrm{Ker}\, \chi'(u)$, equation (12) is equivalent to

$$T_u Z_a \subset \mathrm{Ker}\, \varphi'(u)$$

so that u is a solution of (12) if and only if u is a critical point of φ restricted to Z_a.

By the Lagrange multiplier rule, a maximum or a minimum of $\varphi|_{Z_a}$ is a critical point of $\varphi|_{Z_a}$.

Assume now that Ω and φ are invariant under an isometric representation of the compact group G over $\mathbb{R}^N$. It is then easy to verify that $\nabla\varphi$ is equivariant, i.e.

$$\nabla\varphi \circ T(g) = T(g) \circ \nabla\varphi, \quad \forall g \in G$$

Consider the nonlinear eigenvalue problem

$$\varphi'(u) = \mu u \qquad |u|^2 = 1$$

If u is a critical point of $\varphi|_{S^{N-1}}$ then $\{T(g)u : g \in G\}$ is a set of critical points of $\varphi|_{S^{N-1}}$ and is called

a <u>critical orbit</u> of $\varphi|_{S^{N-1}}$.

Lusternik and Schnirelman showed that if φ is $\mathbb{Z}_2$-invariant (i.e. even), then $\varphi|_{S^{N-1}}$ has at least N distinct pairs

$\{u, -u\}$ of critical points. This generalizes the classical result on the existence of N linearly independent eigenvectors for every linear symmetric mapping $L : \mathbb{R}^N \to \mathbb{R}^N$.

In the case of S^1-invariant functionals, we have the following result due to Krasnoselski.

Theorem 9. Let $T(\vartheta)$ be an isometric representation of S^1 over $\mathbb{R}^{2k}$ such that $\mathrm{Fix}(S^1)=\{0\}$. Let $\Omega\subset\mathbb{R}^N$ be open and invariant, let $\varphi\in C^1(\Omega,\mathbb{R})$ be invariant. If $S^{2k-1}\subset\Omega$ then $\varphi|_{S^{2k-1}}$ has at least k critical orbits.

Proof. a) Let us define

$$\mathcal{A}_j=\{A\subset S^{2k-1}: A \text{ is closed, invariant and } \operatorname{ind} A\geq j\}$$

By theorem 8, $\mathcal{A}_j\neq\emptyset$ for $1\leq j\leq k$. Let

$$c_j=\inf_{A\in\mathcal{A}_j}\max_A \varphi$$

so that $\min\varphi|_{S^{2k-1}}\leq c_1\leq c_2\leq\dots\leq c_k\leq\max\varphi|_{S^{2k-1}}$.

If

$$K_c=\{u\in S^{2k-1}: u \text{ is a critical point of } \varphi|_{S^{2k-1}}\}$$

then it suffices to prove that

$c_i=c_j=c$, $1\leq i\leq j\leq k \Rightarrow \operatorname{ind} K_c\geq i-j+1$,

since the index of a finite number of S^1-orbits is 1.

b) Consider the vector field defined on S^{2k-1} by

$$v(u)=-\nabla\varphi(u)-(\nabla\varphi(u),u)\,u.$$

Clearly v is equivariant and

$$(u,v(u))=0$$

$$(\nabla\varphi(u),v(u))=-(v(u),v(u))=-|v(u)|^2\leq 0,$$

for every $u\in S^{2k-1}$. Assuming that $\varphi\in C^2(\Omega,\mathbb{R})$, it is then easy to construct, as in lemma 1, a deformation $\eta\in C([0,1]\times S^{2k-1},S^{2k-1})$ satisfying the usual properties and such that

$$\eta(t,T(\vartheta)u)=T(\vartheta)\eta(t,u)$$

for every $t\in[0,1]$, $\vartheta\in S^1$, $u\in S^{2k-1}$.

The end of the proof is similar to the proof of lemma 4 and is left to the reader. □

· Remarks. 1. In the C^1 case, it suffices to replace v by an appropriate pseudo-gradient vector field.

2. Assume that $\chi \in C^1(\Omega, \mathbb{R})$ is invariant and that

$$Z_a = \{u \in \Omega : \chi(u) = a\}$$

is diffeomorphic to S^{2k-1}. If the corresponding diffeomorphism is equivariant, then theorem 3 implies the existence of k critical orbits for $\varphi|_{Z_a}$.

3. Since S^{2k-1}/S^1 is not, in general, a manifold, it is impossible to deduce theorem 3 from Lusternik-Schnirelmann theory.

V. Bifurcation theory and periodic solutions of Hamiltonian systems near an equilibrium.

Consider the nonlinear eigenvalue problem

(13) $$\nabla\alpha(u) + \lambda\nabla\beta(u) = 0$$

where α and β are C^2 functionals defined on a Hilbert space X. We shall assume that

$$\nabla\alpha(0) = \nabla\beta(0) = 0 .$$

Thus $\mathbb{R}\times\{0\}$ is a branch of trivial solutions of (13). Using Liapunov - Schmidt method and an elementary variational argument, we shall construct two distinct one parameter families of non trivial solutions of (13). We shall also prove a stronger multiplicity result when α and β are S^1-invariant.

Our basic assumptions are

(H1) $\nabla\alpha(0) = 0$, $\nabla\beta(0) = 0$

(H2) $L = \alpha''(0)$ is Fredholm, i.e. $\dim \mathrm{Ker}\, L < +\infty$, $\mathrm{codim}\, R(L) < +\infty$.

(H3) $\dim \mathrm{Ker}\, L \geq 2$ and $M = \beta''(0)$ is positive definite on $\mathrm{Ker}\, L$.

Remarks. 1. Since L is Fredholm, $R(L)$ is closed. The symmetry of L implies then that X is the orthogonal sum of $R(L)$ and $\mathrm{Ker}\, L$.

2. Without loss of generality, we can assume that $\alpha(0) = \beta(0) = 0$.

3. Let P (resp. Q) be the orthogonal projector on $\mathrm{Ker}\, L$ (resp. $R(L)$). Clearly $Q = I - P$.

We shall use the following notations:

$$A = \nabla\alpha,\ B = \nabla\beta,\ R = A - L,\ S = B - M .$$

Theorem 10. (C. Stuart). Under the assumptions $(H_{1,2,3})$, for every $\varepsilon > 0$ small enough, there exists at least two solutions $[\lambda(\varepsilon), u(\varepsilon)]$ of (13) such that $\beta(u_\varepsilon) = \varepsilon$. Moreover $\lambda(\varepsilon) \to 0$, $\varepsilon \to 0$.

Proof. 1) Liapunov-Schmidt reduction. Equation (13) is equivalent to the system

$$P[R(v+w) + \lambda B(v+w)] = 0$$

$$(14) \quad Lw + Q[R(v+w) + \lambda B(v+w)] = 0,$$

where $v = Pu$, $w = Qu$.

Since $L : R(L) \to R(L)$ is invertible, it follows from the implicit function theorem that equation (14) defines near $\lambda = 0$, $v = 0$, $w = 0$ a C^1 function $w = w^*(\lambda, v)$. Since $\mathbb{R} \times \{0\}$ is a branch of solutions, $w^*(\lambda, 0) = 0$. Differentiating

$$L w^*(\lambda, v) + Q[R(v + w^*(\lambda, v)) + \lambda B(v + w^*(\lambda, v))] = 0$$

with respect to v at $[0, 0]$ and using the fact that $R'(0) = 0$, we obtain

$$L D_v w^*(0,0) = 0,$$

i.e. $D_v w^*(0,0) = 0$. Thus

$$\|w^*(\lambda, v)\| / \|v\| \to 0, \quad \|v\| \to 0$$

uniformly for λ near 0.

Near $\lambda = 0$, $v = 0$, $w = 0$, equation (13) is now equivalent to

$$(14) \quad P[R(v + w^*(\lambda, v)) + \lambda B(v + w^*(\lambda, v))] = 0.$$

2) Deparametrization. Taking the inner product of (14) with v, we obtain

$$(15) \quad (R(v + w^*(\lambda, v)), v) + \lambda (B(v + w^*(\lambda, v)), v) = 0.$$

It follows from a careful application of the implicit function theorem that equation (15) defines, near 0, for $v \neq 0$, a C^1 function

$\lambda = \lambda^*(v)$. Moreover λ^* is extended continuously at 0 by setting $\lambda^*(0) = 0$. Let us define, near 0,

$$f(v) = w^*(\lambda^*(v), v)$$

so that

$$\|f(v)\| / \|v\| \to 0, \quad \|v\| \to 0.$$

Equation (13) is now equivalent to

(16) $P\,[R(v + f(v)) + \lambda^*(v)\, B(v + f(v))] = 0.$

3) <u>Constrained extremization</u>. For $\rho > 0$ small enough, the function

$$\mathcal{K}(v) = \mathcal{B}(v + f(v))$$

is continuous on $B(0,\rho)$ and C^1 on $B(0,\rho)\setminus\{0\}$. It follows from (H_3) and some careful estimates (see [4]), that for $\rho > 0$ small enough

(17) $$\mathcal{K}(v) \geq c\|v\|^2, \quad \forall\, v \in B(0,\rho)$$
$$(\nabla\mathcal{K}(v), v) \geq c\|v\|^2, \quad \forall\, v \in B(0,\rho)\setminus\{0\}$$

where $c > 0$ is a constant. Hence, for $\varepsilon > 0$ small enough,

$$Z_\varepsilon = \{ v \in \operatorname{Ker} L : \mathcal{K}(v) = \varepsilon \}$$

is a compact subset of $B(0,\rho)\setminus\{0\}$ and

(18) $v \in Z_\varepsilon \Rightarrow (\nabla\mathcal{K}(v), v) > 0.$

The function

$$\varphi(v) = \alpha(v + f(v)) + \lambda^*(v)(\mathcal{K}(v) - \varepsilon)$$

restricted to Z_ε achieves his minimum at a point v_ε. Since $\nabla\mathcal{K}(v_\varepsilon) \neq 0$, the Lagrange multiplier rule implies the existence of μ such that

$$\nabla\varphi(v_\varepsilon) = \mu\, \nabla\mathcal{K}(v_\varepsilon),$$

i.e.

$$(R(v_\varepsilon + f(v_\varepsilon)) + \lambda^*(v_\varepsilon)\, B(v_\varepsilon + f(v_\varepsilon)),\ h + f'(v_\varepsilon)h) = \mu(\nabla\mathcal{K}(v_\varepsilon), h),$$

for every $h \in \operatorname{Ker} L$. Since $f'(v_\varepsilon)h \in R(Q)$, the definition of $f(v_\varepsilon) = w^*(\lambda^*(v_\varepsilon), v_\varepsilon)$ implies that

$$(A(v_\varepsilon + f(v_\varepsilon)) + \lambda^*(v_\varepsilon) B(v_\varepsilon + f(v_\varepsilon)), f'(v_\varepsilon)h) = 0.$$

Using the fact that $PA = PR$, we have, for $h \in \operatorname{Ker} L$,

$$(R(v_\varepsilon + f(v_\varepsilon)) + \lambda^*(v_\varepsilon) B(v_\varepsilon + f(v_\varepsilon)), h) = \mu(\nabla\chi(v_\varepsilon), h).$$

It follows from the definition of λ^* that

$$0 = (R(v_\varepsilon + f(v_\varepsilon)) + \lambda^*(v_\varepsilon) B(v_\varepsilon + f(v_\varepsilon)), v_\varepsilon) = \mu(\nabla\chi(v_\varepsilon), v_\varepsilon).$$

We obtain from (18) that $\mu = 0$. Finally

$$(R(v_\varepsilon + f(v_\varepsilon)) + \lambda^*(v_\varepsilon) B(v_\varepsilon + f(v_\varepsilon)), h) = 0$$

for every $h \in \operatorname{Ker} L$, i.e. v_ε is a solution of (16). Thus

$$[\lambda(\varepsilon) = \lambda^*(v_\varepsilon), \ u_\varepsilon = v_\varepsilon + f(v_\varepsilon)]$$

is a solution of (13) such that

$$\beta(u_\varepsilon) = \chi(v_\varepsilon) = \varepsilon.$$

By (17), $v_\varepsilon \to 0$ as $\varepsilon \to 0$, so that $\lambda(\varepsilon) = \lambda^*(v_\varepsilon) \to \lambda^*(0) = 0$ as $\varepsilon \to 0$. The other solution is obtained by maximizing g on Z_ε. $\square$

Let now $T(\theta)$ be an isometric representation of S^1 over the Hilbert space X such that

(H4) α and β are S^1-invariant and
$$\operatorname{Ker} L \cap \operatorname{Fix}(S^1) = \{0\}.$$

<u>Remark</u>. By (H4), A, B, L, M, R, S, P and Q are equivariant, $R(L)$ and $\operatorname{Ker} L$ are invariant. Since $\operatorname{Ker} L \cap \operatorname{Fix}(S^1) = \{0\}$, the dimension of $\operatorname{Ker} L$ is even.

Theorem 11. Under the assumptions of theorem 10 and (H_4), for every $\varepsilon>0$ small enough, there exists at least $\frac{1}{2}\dim \operatorname{Ker} L$ S^1-orbits
$$\{[\lambda(\varepsilon), T(\vartheta)u_\varepsilon]: \vartheta\in S^1\}$$
of solutions of (13) such that $\beta(u_\varepsilon)=\varepsilon$. Moreover $\lambda(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Proof. It is easy to verify that the invariance of α and β implies the invariance of g and χ. By (17), for $\varepsilon>0$ small enough,
$$Z_\varepsilon = \{v\in \operatorname{Ker} L : \chi(v)=\varepsilon\}$$
is diffeomorphic to S^{2k-1} where $2k=\dim \operatorname{Ker} L$. Moreover the corresponding diffeomorphism is equivariant. Hence theorem 9 implies the existence of k critical orbits for $g|_{Z_\varepsilon}$.

It is then easy to complete the proof as in theorem 10. $\square$

Let $H\in \mathcal{C}^2(\mathbb{R}^{2N},\mathbb{R})$ be such that $H(0)=0$, $\nabla H(0)=0$. We consider the existence of periodic orbits of the Hamiltonian system

(19) $$J\dot u(t)+\nabla H(u(t))=0$$

on the energy surface $H^{-1}(\varepsilon)$ for small ε.

After the classical results of Liapunov (1900), depending on a non resonance condition, a significant progress was made by Weinstein around 1976. We shall deduce Weinstein theorem from a generalization due to Moser.

Theorem 12. (Moser). Assume that the linearized system

(20) $$J\dot{v}(t) + H''(0)\,v(t) = 0$$

has $2k$ linearly independent T-periodic solutions and that, if $v \neq 0$ is a T-periodic solution of (20), then

$$(H''(0)\,v(t), v(t)) = (H''(0)\,v(0), v(0)) > 0.$$

Then, for every $\varepsilon > 0$ small enough, there exists at least k periodic orbits of (19) on $H^{-1}(\varepsilon)$ whose period are near T.

Remarks. 1. By assumption $H''(0)$ is non singular.

2. The period T is not necessarily the minimal period of the corresponding solutions of (19) and (20).

Proof. We normalize the problem by fixing the period at 1. After the change of variable $s = \tau^{-1} t$, (19) becomes

(21) $$J\dot{z}(s) + \tau\,\nabla H(z(s)) = 0.$$

Any 1-periodic solution of (21) corresponds to a τ-periodic solution of (19). Setting $\tau = T + \lambda$ we obtain the bifurcation problem

$$\nabla\alpha(z) + \lambda\,\nabla\beta(z) = 0$$

where the functionals

$$\alpha(z) = \int_0^1 [(J\dot{z}(s), z(s))/2 + TH(z(s))]\,ds$$

$$\beta(z) = \int_0^1 H(z(s))\,dt,$$

are defined on the Hilbert space $X = W_1^{1,2} = H_1^1$. The functionals α and β are C^2 and invariant representation of $S^1=\mathbb{R}/\mathbb{Z}$ defined on X by the translations in time.

Let $L=\alpha''(0)$. Then $z\in \operatorname{Ker} L$ if and only if

$$J\dot{z}(t)+TH''(0)z(t)=0$$
$$z(0)=z(1)$$

By assumption, $\dim \operatorname{Ker} L=2k$ and $M=\beta''(0)$ is positive definite on $\operatorname{Ker} L$. Since $H''(0)$ is non singular, $\operatorname{Ker} L\cap \operatorname{Fix}(S^1)=\{0\}$. It suffices then to apply theorem 11. □

<u>Corollary</u> (Weinstein). Assume $H\in C^2(\mathbb{R}^{2N},\mathbb{R})$, with $H(0)=0$, $\nabla H(0)=0$ and $H''(0)$ positive definite. For every $\varepsilon>0$ small enough, there exists at least N periodic orbits of (19) on $H^{-1}(\varepsilon)$.

<u>Proof</u>. By assumption (20) has $2N$ linearly independent periodic solutions. The periodic solutions of (20) split into m families with incommensurable periods $T_1,\dots,T_m$ and dimensions $k_1,\dots,k_m$. We note that $k_1+\dots+k_m=2N$. Theorem 12, applied to each of these families, implies the existence of $k_1/2+\dots+k_m/2=N$ periodic orbits of $H^{-1}(\varepsilon)$ for $\varepsilon>0$ small enough. These periodic orbits are distinct within the same family and from one family to another, because they have no common period for $\varepsilon>0$ small enough. □

Examples.

1) Linear case. Assume that the α_i's are positive and incommensurable. Then each energy surface of the quadratic Hamiltonian

$$H(u) = \sum_{i=1}^{N} \alpha_i (u_i^2 + u_{i+N}^2)$$

contains exactly N periodic orbits.

2) Henon - Heiles Hamiltonian. The following Hamiltonian appears in astrophysics:

$$H(u) = (u_1^2 + u_2^2 + u_3^2 + u_4^2)/2 + u_1^2 u_2 - u_2^3/3.$$

For $\varepsilon > 0$ small enough, $H^{-1}(\varepsilon)$ contains at least 2 periodic orbits.

3) Moser example. The only periodic solution corresponding to the following Hamiltonian is the equilibrium:

$$H(u) = (u_1^2 + u_3^2 - u_2^2 - u_4^2) + (u_1^2 + u_2^2 + u_3^2 + u_4^2)(u_3 u_4 - u_1 u_2)$$

(It suffices to differentiate $u_1 u_4 + u_2 u_3$.) But all the orbits of the linearization at 0 are periodic.

Remark. See [2], [4], [5] for other aspects of variational methods in bifurcation theory.

Appendix.

Proposition. Let X be a Banach space and let $\varphi \in \mathcal{C}^1(X, \mathbb{R})$. If φ satisfies (P.S.) and is bounded from below, then φ is coercive, i.e. $\varphi(u) \to +\infty$, $|u| \to \infty$.

Proof. If φ is not coercive, then

$$c = \sup \{ d \in \mathbb{R} : \varphi^d \text{ is bounded} \}$$

is a real number. Let U be an open bounded neighborhood of the compact set K_c and let $\varepsilon \in]0, 1[$ and η be given by the deformation lemma. It follows from the definition of c that $\varphi^{c+\varepsilon} \setminus U$ is unbounded and that $\varphi^{c-\varepsilon}$ is bounded. On the other hand $\eta(1, \varphi^{c+\varepsilon} \setminus U) \subset \varphi^{c-\varepsilon}$ and $\eta(1, \cdot) : X \to X$ maps unbounded sets into unbounded sets. This is a contradiction. $\square$

Remark. The converse is valid in finite dimension only.

I thank Mr. G. Amann for suggesting me this question.