INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR 281/12

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
(11 January - 5 February 1988)

---

# SOME RESULTS ON PERIODIC SOLUTIONS OF AUTONOMOUS HAMILTONIAN SYSTEMS

Mario GIRARDI & Michele MATZEU
Department of Mathematics
University of Aquila
Via Roma 33
L'AQUILA
ITALY

---

# SOME RESULTS ON PERIODIC SOLUTIONS OF AUTONOMOUS HAMILTONIAN SYSTEMS

by

Mario GIRARDI and Michele MATZEU
(University of L'AQUILA - Dept. of Mathematics, Via Roma Pal. del Tosto, 67100 L'AQUILA)

INTRODUCTION. The aim of this cycle of lectures is to expose some recent results about periodic solutions of Hamiltonian systems, obtained through the use of variational methods by several authors (see the references).

In the following we shall only consider autonomous Hamiltonian systems which we write in the form

(H) $J\dot{z} = H'(z)$,

where $Jz = (y, -x)$ for every $z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ and $H \in C^1(\mathbb{R}^{2N}; \mathbb{R})$

In this framework, two main problems arise in the study of periodic solutions of (H):

1) PRESCRIBED PERIOD PROBLEM: Fixed any $T > 0$, to find a $T$-periodic solution of (H)

2) FIXED ENERGY PROBLEM: Given a surface $S$ in $\mathbb{R}^{2N}$, to find a trajectory, corresponding with a periodic solution of (H), which stays on $S$; secondly, to state the existence of many such trajectories and count them.

In 1978 Rabinowitz gave, in a celebrated paper [24] a basic result stating that, when $H$ has a superquadratic behaviour, then, for any $T > 0$, there exists a $T$-periodic solution of (H), and, for any $h > 0$, there exists a periodic solution of (H) having energy $h$.

# CHAPTER 1.

## THE PRESCRIBED PERIOD PROBLEM

§ 1 THE DUALITY PRINCIPLE. In his paper, Rabinowitz doesn't prove that the T-periodic solution found for (H) has in fact T as its minimal period. It is a general fact that the variational method enables to exhibit a T-periodic solution as a critical point of a suitable functional, but, if one requires that T is the minimal (the true) period, one has to take into account, in a specific way, the nature of the critical point, that is, whether it is a maximum, a minimum or a mini-max point.

If one considers the functional F directly associated with (H), whose critical points coincide with the T-periodic solutions of (H), that is

$$F(v) = \frac{1}{2}\int_0^T \langle J\dot{v}(t), v(t)\rangle \, dt - \int_0^T H(v(t))\, dt$$

on a suitable space of T-periodic functions, then it is easy to see that F is strongly indefinite, i.e. it is unbounded (both from below and from above) even if perturbed with weakly continuous functionals.

Thus, in many cases, it is more convenient to study another functional, using the duality principle for Hamiltonian systems stated by Clarke and Ekeland in 1980 [9].

Roughly speaking, the principle is based on the idea that an inversion of the operator $\mathcal{L} = J\, d/dt$ yields an integral operator $\mathcal{L}^{-1}$ which has some compactness property. From the other side, if $H'(z) = \mathcal{L}z$, that is $z$ is a solution of (H), then $u = \mathcal{L}z$ solves the equation

$$\mathcal{L}^{-1}u = (H')^{-1}(u),$$

if one suppose that $H'$ is in fact invertible. If $G$ is a $C^1$-function on $\mathbb{R}^{2N}$ such that $G' \equiv (H')^{-1}$, then it is natural to consider the functional

$$F^*(v) = \int_0^T G(v(t))\,dt - \frac{1}{2}\int_0^T \langle \mathcal{L}^{-1}v(t), v(t)\rangle\,dt$$

and to associate the solutions of (H) with the critical points of $F^*$.

Now let us give the precise statement of the duality principle. Let $H$ be a strictly convex $C^1$-function on $\mathbb{R}^{2N}$ and let us consider the Legendre transform $G$, defined on $\mathbb{R}^{2N}$ as

$$G(v) = \sup\{\langle v, z\rangle - H(z) : z \in \mathbb{R}^{2N}\}$$

Then it is well known that $G$ is a $C^1$-convex function such that

(1) $\quad G'(v) = z \quad$ iff $\quad v = H'(z)$

At this point it is easy to state the following

DUALITY PRINCIPLE FOR HAMILTONIAN SYSTEMS. (see [8], [9], [11])

Let $H$ and $G$ as before, and let there exist some $\alpha > 1$ such that

(2) $G(v) \in L^1(0,T;\mathbb{R}^{2N})$ $\quad \forall v \in L^\alpha(0,T;\mathbb{R}^{2N})$.

(3) $G'(v) \in L^\beta(0,T;\mathbb{R}^{2N})$ $\quad \forall v \in L^\alpha(0,T;\mathbb{R}^{2N})$, $\beta = \alpha/_{\alpha-1}$

Putting $\mathcal{L} = J\,d/_{dt}$ with $\operatorname{dom}\mathcal{L} = \{v \in H^{1,\alpha}(0,T;\mathbb{R}^{2N}) : v(0)=v(T), \int_0^T v(t)\,dt = 0\}$, consider the functional $F^*$ defined on $X = \{v \in L^\alpha(0,T;\mathbb{R}^{2N}) : \int_0^T v(t)\,dt = 0\}$ as

$$F^*(v) = \int_0^T G(v(t))\,dt - \frac{1}{2}\int_0^T \langle \mathcal{L}^{-1}v(t), v(t)\rangle\,dt$$

Then $z$ is a $T$-periodic solution of (H) if and only if $u = H'(z)$ (i.e. $z = G'(u)$) is a critical point of $F^*$ (that is $u \in X$, $DF^*(u) = 0$).

§2. DIRECT APPLICATIONS OF THE DUALITY PRINCIPLE. When $H$ satisfies some subquadraticity conditions, it is possible to prove that $F^*$ has a minimum and that any minimum point corresponds, via the duality principle, to a solution of (H) having $T$ as its minimal period.

More precisely, Clarke and Ekeland stated this more general result[(*)]

THEOREM 1. (Clarke and Ekeland [9]). Let $H \in C^1(\mathbb{R}^{2N};\mathbb{R})$ be a convex function such that $H(0)=0$, $H(z)>0$ $\forall z \in \mathbb{R}^{2N}\setminus\{0\}$. Suppose that there are positive numbers $k, K, a, \eta$ with $K > k$ such that

(4) $H(z) \le \frac{1}{2} k |z|^2 + a \quad \forall z \in \mathbb{R}^{2N}$

(5) $H(z) \ge \frac{1}{2} K |z|^2 \quad \forall z \in \mathbb{R}^{2N}, \ |z| < \eta$

(*) The hypothesis of differentiability of $H$ can be omitted: in such a case one has to consider a subdifferential inclusion in place of the differential relation (H).

*Then, for any* $T \in (2\pi/K,\ 2\pi/k)$, *there is a* $T$-*periodic solution of* (H), *having minimal period* $T$.

It is an immediate consequence of Theorem 1 the following

COROLLARY of THEOREM 1. *Let* $H \in C^1(\mathbb{R}^{2N};\mathbb{R})$ *be non-negative and convex, vanishing only at the origin. Suppose that*

(6) $H(z)/|z|^2 \to 0$, *when* $|z| \to +\infty$

(7) $H(z)/|z|^2 \to +\infty$, *when* $|z| \to 0$

*Then, for any* $T > 0$, *there exists a* $T$-*periodic solution of* (H), *having minimal period* $T$.

Here we only give an idea of the proof of Corollary, in the parti-cular case that $H$ is strictly convex and has a "strictly subquadratic" behaviour in the whole $\mathbb{R}^{2N}$, that is $H$ satisfies the conditions

(8) $a_1 |z|^\alpha \leq H(z) \leq a_2 |z|^\alpha \quad \forall z \in \mathbb{R}^{2N}, \quad a_1, a_2 > 0,\ \alpha \in (0,1)$,

which obviously imply (6) and (7). Indeed, when (8) hold, the functional $F^*$ is coercive on the space $X = \{v \in L^\alpha(0,T;\mathbb{R}^{2N}) : \int_0^T v(t)\,dt = 0\}$, as it easily follows from the fact that, in such a case, $F^*$ is the difference of a [illegible] functional having a superquadratic behaviour (in fact (8) imply the "conjugate" relations

(9) $b_1 |v|^\beta \le G(v) \le b_2 |v|^\beta \quad \forall v \in \mathbb{R}^{2N}, \quad b_1, b_2 > 0, \ \beta = \alpha/_{\alpha-1}$ )

and a quadratic term. Moreover, the convexity and continuity of the integral $\int_0^T G(v(t))dt$ on the space $X$ and the compactness of $\mathcal{L}^{-1}$ from $X$ into $L^\infty$ yield the weak lower semicontinuity of $F^*$ on $X$. Therefore $F^*$ has a minimum on $X$. Let $u$ be a minimum point and let $z = G'(u)$ the corresponding solution of (H) via the duality principle. Let us show that $T$ is the minimal period of $z$. In fact, if it was not true, then, for some $m \in N$, $m \geq 2$, $u^*(t) = u\,(t/_m)$ would be $T$-periodic and, since $\mathcal{L}^{-1}u^*(t) = m\,\mathcal{L}^{-1}u\,(t)$, then one would have

(10) $F^*(u^*) = \int_0^T G(u(t))dt - \frac{m}{2}\int_0^T <\mathcal{L}^{-1}u(t), u(t)> = m\,F^*(u) - (m-1)\int_0^T G(u(t))dt \le$

$\le m\,F^*(u)$.

As $F^*(u) < 0$ (it follows from a calculation which shows that $F^*$ is negative on some element of $X$, as a consequence of the second inequality of (9)), then (10) would imply that $F^*(u^*) < F^*(u)$, which is an absurde.

As for the superquadratic case, a first result was given by Ambrosetti and Mancini, who stated the following

THEOREM 2. (Ambrosetti and Mancini [1]). <u>Let</u> $H \in C^2(\mathbb{R}^{2N};\mathbb{R})$ <u>be strictly convex</u>, <u>let</u> $G$ <u>its Legendre transform and let there exist</u> $a_1, a_2, a_3 > 0$, $\beta > 2$, <u>and</u> $\mu \in (0,1)$ <u>such that</u>:

(11) $a_1 |z|^\beta \le H(z) \le a_2 |z|^\beta \quad \forall z \in \mathbb{R}^{2N}$

(12) $\langle H'(z), z\rangle \ge \beta H(z) \quad \forall z \in \mathbb{R}^{2N}$

(13) $\langle H''(z)\zeta, \zeta\rangle \ge a_3 |z|^{\beta-2} \quad \forall z, \zeta \in \mathbb{R}^{2N}, \ |\zeta| = 1$

(14) $\langle G''(u)u, u\rangle \le \mu \langle G'(u), u\rangle \quad \forall u \in \mathbb{R}^{2N} \setminus \{0\}$

<u>Then, for any</u> $T > 0$, <u>there exists a</u> $T$-<u>periodic solution of</u> (4), <u>having minimal period</u> $T$.

The idea of the proof of Theorem 2 is to consider the problem of minimization of $F^*$ on a suitable manifold $M$ of $L^\alpha$ ($\alpha = \beta/_{\beta-1}$), which contains all the critical points of $F^*$, specifically

$$M = \left\{ v \in L^\alpha(0,T;\mathbb{R}^{2N}) \setminus \{0\} : \int_0^T v(t)\,dt = 0, \ \left(DF^*(v)\right)(v) = 0 \right\}$$

It is possible to show that, as a consequence of (14), $M$ is a regular manifold of $L^\alpha$ and that $F^*$ admits in fact the minimum on $M$. At this point, if one considers a minimum point, $u$, then $z = G'(u)$ is proved to be a solution of (4) having <u>minimal</u> period $T$.

Later on, Ekeland and Hofer were able to eliminate the technical condition (14) and to state this very general result concerning the convex hyperquadratic case

THEOREM 3. (Ekeland and Hofer [10]). Let $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ be strictly convex and such that

(15) $H(z)\,|z|^{-2} \to 0$ when $|z| \to 0$

(16) $\exists\, r > 0,\ \beta > 2 : \langle H'(z), z \rangle \geq \beta H(z)$ for $|z| > r$

Then, for any $T > 0$, there exists a $T$-periodic solution of (H), having minimal period $T$.

The idea of the proof is to show, at first, that, under assumptions (15), (16), the functional $F^*$ has a critical point of Mountain Pass type (see later on in the next section, for a more general case). Secondly, $T$ is proved to be the minimal period of the corresponding solution, as a consequence of a suitable argument based on the Morse index theory and the concept of conjugate points for Hamiltonian systems

§ 3. APPLICATIONS OF THE DUALITY PRINCIPLE TO NON-CONVEX CASES.

In some cases, one can give some suitable modified versions of the duality principle, which enable to deal either with non-convex Hamiltonian functions having a superquadratic or subquadratic behaviour, or with Hamiltonian functions which have the form of the sum of a convex (superquadratic or subquadratic) term plus a quadratic one.

Let $H(z) = \hat{H}(z) + \frac{1}{2}\langle Qz, z\rangle$, where $\hat{H}$ is convex and $Q$ is a $2N \times 2N$ matrix. In this situation, system (H) can be rewritten as

$$(\hat{H}) \quad \hat{\mathcal{L}} z = \left(J \tfrac{d}{dt} - QI\right) z = \hat{H}'(z),$$

then an analogous duality principle can be stated with $\hat{\mathcal{L}}$ in place of $\mathcal{L} = J\,d/dt$ and $\hat{H}$ in place of $H$.

Applying this method and developping some ideas contained in [10], it is possible to state the following theorem, which generalizes the result by Ekeland and Hofer.

THEOREM 4 (Girardi and Matzeu [15]). Let $H(z) = \hat{H}(z) + \frac{1}{2}\langle Qz, z\rangle$ $\forall z \in \mathbb{R}^{2N}$, where $\hat{H}$ satisfies conditions as (15), (16) and $Q = \begin{pmatrix} Q_0 & 0 \\ 0 & Q_0 \end{pmatrix}$, with $Q_0 = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_N \end{pmatrix}$, $0 < |\omega_1| \le \dots \le |\omega_N|$. Then, for any $T < \frac{2\pi}{|\omega_N|}$, there exists a $T$-periodic solution of (H), having minimal period $T$.

As in Theorem 3, the T-periodic solution is found in correspondence with a critical point of Mountain Pass type. Here we'll give some details for the existence of this point, for any $T \neq 2k\pi/\omega_j$ ($k \in \mathbb{Z}$, $j \in \{1, \dots, N\}$), that is for any $T$ such that $\hat{\mathcal{L}}$ is invertible on a space of T-periodic functions. For simplicity, let us suppose that $\hat{H}$ satisfies condition (16) in the whole $\mathbb{R}^{2N}$ and the "strong superquadraticity conditions":

(17) $a_1 |z|^\beta \le \hat{H}(z) \le a_2 |z|^\beta \quad \forall z \in \mathbb{R}^{2N}, \quad a_1, a_2 > 0$

(18) $|\hat{H}'(z)| \le a_3 |z|^{\beta - 1}$

In such a case, for any $T \neq \frac{2k\pi}{\omega_j}$ ($k \in \mathbb{Z}$, $j \in \{1, \dots, N\}$), we can consider a functional $\hat{F}$ defined on $X = L^\alpha(0,T;\mathbb{R}^{2N})$ (with $\alpha = \beta/\beta-1$) as

$$\hat{F}(v) = \int_0^T \hat{G}(v(t))\,dt - \frac{1}{2}\int_0^T \langle \hat{\mathcal{L}}^{-1} v(t), v(t)\rangle\,dt,$$

where $\hat{G}$ is the Legendre transform of $\hat{H}$ and $\hat{\mathcal{L}} = (J\,d/dt - Q\,I)$.

Indeed, $\hat{F}$ satisfies all the assumptions of the Mountain Pass theorem by Ambrosetti and Rabinowitz (see [3]), that is:

(19) $\hat{F} \in C^1(X)$, $\hat{F}(0) = 0$

(20) $\hat{F}(v) > 0$ for $v \neq 0$, $\|v\| < \rho$, $\rho$ sufficiently small

(21) $\exists\ \bar{v} \in X : \hat{F}(\bar{v}) < 0$

(22) $\hat{F}$ satisfies the Palais-Smale condition (PS), that is $\forall \{v_n\} \subset X$, with $D\hat{F}(v_n) \longrightarrow 0$ and $|\hat{F}(v_n)| \leq c$, there exists a subsequence $\{v_{n_j}\}$ strongly converging in $X$.

[(19) is obvious. (20) is deduced from the fact that $\hat{F}$ is the difference of a subquadratic term and a quadratic one. On the other side, choosen $k \in \mathbb{Z}$, $j \in \{1, \dots, N\}$ such that $2k\pi - T\omega_j > 0$, considered any vector $v_0 \in E_{k,j}$, the eigenspace associated with the eigenvalue $\lambda_{kj}$ of $\hat{\mathcal{L}}^{-1}$ given by $\lambda_{kj} = T/(2k\pi - T\omega_j)$, then one has

$$\int_0^T \langle \hat{\mathcal{L}}^{-1} v_0(t), v_0(t)\rangle\,dt > 0,$$

so (21) is verified for any $\bar{v} = \lambda v_0$ with $\lambda > 0$ large enough, as a consequence of the first inequality in (17).

Let us prove (22). Firstly, by virtue of (16), which has been supposed to be verified for all $z \in \mathbb{R}^{2N}$, one has

$$<\hat{G}'(v), v> \leq \alpha \hat{G}(v) \quad \forall v \in \mathbb{R}^{2N},$$

so, putting $\varepsilon_n = \hat{G}'(v_n) - \hat{\mathcal{L}}^{-1} v_n \longrightarrow 0$ in $L^\beta$, one has, by the second inequality in (17),

$$\text{const} \geq \left(1 - \frac{\alpha}{2}\right) \int_0^T \hat{G}(v_n(t))\, dt + \frac{1}{2} \int_0^T <\varepsilon_n(t), v_n(t)> dt \geq \left(1 - \frac{\alpha}{2}\right) \text{const} \|v_n\|^\alpha - \text{const} \|v_n\|$$

which implies the boundedness of $\{v_n\}$ in $X$. Let now $\{v_{n_j}\}$ be a subsequence of $\{v_n\}$ weakly converging in $X$. Then

$$z_{n_j} = \varepsilon_{n_j} + \hat{\mathcal{L}}^{-1} v_{n_j} = \hat{G}'(v_{n_j})$$

strongly converges in $L^\beta$, so $v_{n_j} = \hat{H}'(z_{n_j})$ strongly converges in $X$, as it follows from the continuity of the operator $z \to \hat{H}'(z)$ from $L^\beta$ into $L^\alpha$, deduced from (18).

Therefore, by the theorem of Ambrosetti and Rabinowitz, one can find a critical point $u$ of $\hat{F}$ of Mountain Pass type that is

$$\hat{F}(u) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \hat{F}(\gamma(t))$$

where $\Gamma$ is the set of all continuous paths $\gamma : [0,1] \to X$ with $\gamma(0) = 0$, $\gamma(1) = \bar{v}$.

When $T < 2\pi / |\omega_N|$, it is proved that $z = \hat{\mathcal{L}}^{-1} u$ is in fact a T-periodic solution of (H), having minimal period T.

Another use of the duality principle yields some results for some classes of subquadratic or superquadratic non-convex Hamiltonian functions. As for the subquadratic case, it is possible to give an extension of Theorem 1 to a non-convex case, if a simmetry assumption is made on $H$. One has the following

THEOREM 5 (Girardi and Matzeu [10]). Let $H \in C^2(\mathbb{R}^{2N}; \mathbb{R})$ be such that $H(z) = 0$, $H(z) > 0$ $\forall z \in \mathbb{R}^{2N} \setminus \{0\}$, and let

(23) $H(z) = H(-z)$ $\forall z \in \mathbb{R}^{2N}$

Suppose that there are positive numbers $k, K, a, \eta, L$ with $K > k$, $K > L$, such that

(24) $H(z) \leq k/2 \, |z|^2 + a$ $\forall z \in \mathbb{R}^{2N}$

(25) $H(z) \geq \frac{K}{2} |z|^2$ $\forall z \in \mathbb{R}^{2N}$, $|z| < \eta$

(26) $\langle H'(z), z \rangle > 0$ $\forall z \in \mathbb{R}^{2N}$

(27) $\langle H''(z)\zeta, \zeta \rangle \geq -L$ $\forall z, \zeta \in \mathbb{R}^{2N}$, $|\zeta| = 1$.

Then, for any $T \in \left(2\pi/K, \frac{2\pi}{\min(k, L)}\right)$, there exists a solution $z = z_T$ of (H) having minimal period $T$. The Fourier expansion of $z_T$ in $L^2(0,T; \mathbb{R}^{2N})$ has the form:

(28) $$z_T(t) = \sum_{\substack{k=2h+1 \\ h \in \mathbb{N}}} \left( z_k^{(1)} \cos \frac{2\pi k}{T} t + z_k^{(2)} \sin \frac{2\pi k}{T} t \right), \quad (z_1^{(k)}, z_2^{(k)} \in \mathbb{R}^N)$$

The main idea of the proof is to transform H into the equivalent system

$$\left(J\frac{d}{dt}+LI\right)z = H'(z)+Lz = \widetilde{H}'(z)$$

where $\widetilde{H}(z)=H(z)+\frac{L}{2}|z|^2$ is a convex function to which one can apply the usual duality principle with $\mathcal{I}=J\,d/dt$ replaced by $\widetilde{\mathcal{I}}=J\,d/dt+LI$ and H replaced by $\widetilde{H}$. As it is easily seen the corresponding functional $\widetilde{F}$ is not coercive on the whole space $L^2$, but only on the subspace $L^2_0$ of $L^2$-functions having zero mean. So the idea is to give some conditions to find a subspace of $L^2_0$ which is invariant for the derivative $\widetilde{G}'$ of Legendre transform $\widetilde{G}$ of $\widetilde{H}$. If one assumes (23), then it is possible to verify that the subspace of $L^2_0$ of functions having only odd terms in their trigonometric Fourier expansions has this property, so $\widetilde{F}$ attains its minimum value in this space and every minimum point corresponds, in the usual way, to a solution of (H) which is proved to have minimal period T.

As for the superquadratic case, following the same idea of the modified duality principle as in Theorem 5, and using Theorem 3, together with some results obtained for the proof of Theorem 4, it is possible to show the following:

THEOREM 6 (Girardi and Matzeu, to appear) - <u>Let</u> $H\in C^1(\mathbb{R}^{2N};\mathbb{R})\cap C^2(\mathbb{R}^{2N}\setminus\{0\};\mathbb{R})$ <u>satisfy condition</u> (16) <u>in the whole space, conditions</u> (17), <u>and further</u>.

(29) $|H'(z)|\le a_3\left(|z|^{\beta-1}+1\right)\quad \forall z\in\mathbb{R}^{2N}$

(30) $\langle H''(z)\zeta,\zeta\rangle \ge -c|z|^{\beta-2}\quad \forall z\in\mathbb{R}^{2N}\setminus\{0\},\ \zeta\in\mathbb{R}^{2N},\ |\zeta|=1,\qquad c>0$

<u>Then there exists a positive number</u> $\bar{c}$, <u>depending on</u> $a_1, a_2, \beta$, <u>such that, if</u> $c\in(0,\bar{c})$ <u>then, for any</u> $T>0$, <u>there exists a periodic solution of</u> (H) <u>having minimal period</u> T.

## § 4. THE VARIATIONAL APPROACH VIA THE CLASSICAL FUNCTIONAL.

In this section we give some results obtained through the consideration of the functional F whose critical points give directly the T-periodic solutions of (H), that is

$$F(v) = \frac{1}{2}\int_0^T \langle J\dot{v}(t), v(t)\rangle\,dt - \int_0^T H(v(t))\,dt,$$

without any convexity assumption on H.

Let us suppose that H has a superquadratic behaviour. Then it is possible to state that there exists the minimum positive critical value of the functional F on a suitable space of T-periodic functions. In the general case, this yields the fact that there is no upper bound to minimal periods, as expressed by the following

THEOREM 7 (Girardi and Matzeu [18]). *Let there exist some constants $a_1, a_2 > 0$ and $\beta > 2$ such that*

(31) $H(z) \geq a_1 |z|^\beta \quad \forall z \in \mathbb{R}^{2N}$

(32) $\langle H'(z), z\rangle \geq \beta H(z) \quad \forall z \in \mathbb{R}^{2N}$

(33) $|H'(z)| \leq a_2 |z|^\beta \quad \forall z \in \mathbb{R}^{2N}$

*Then there exists an integer number $\bar{n} \geq 2$, depending on $a_1, a_2, \beta$, such that, for any $T > 0$, there exists a solution $z$ of (H) and an integer number $m \in \{1, \ldots, \bar{n}-1\}$, such that $z$ has minimal period $T/m$.*

Firstly, it is proved that the functional $F$ has a minimum positive critical value on the space $E = \{z \in H^1(0,T;\mathbb{R}^{2N}) ; z(0) = z(T)\}$.

Indeed, the set of critical points of $F$ is not empty, as proved by Rabinowitz [24], and one has, by (32),

$$(34) \quad F(z) \geq \left(\frac{\beta}{2} - 1\right) h_z T$$

for any critical point $z$, where $h_z = H(z(t))$ is a constant independent from $t \in [0,T]$. Then $0$ is an isolated critical value of $F$, since, if it was not true, there would exist a sequence $\{z_n\} \subset E \setminus \{0\}$ such that

$$J \dot{z}_n = H'(z_n), \quad F(z_n) \longrightarrow 0 \quad \text{as } n \to +\infty,$$

so, putting $h_n = H(z_n(t))$ $\forall t \in [0,T]$, one would deduce, by (32) and (34)

$$M_n = \max_{t \in [0,T]} |z_n(t)|^{\beta - 2} \leq \left( h_n / a_1 \right)^{\frac{\beta-2}{\beta}} \leq \left( 2F(z_n) / (\beta-2) a_1 T \right)^{\frac{\beta-2}{\beta}}$$

and $M_n \to 0$ as $n \to +\infty$. On the other side, for any solution $v$ of (H), setting $v = \bar{v} + c$, with $c \in \mathbb{R}^{2N}$ and $\int_0^T \bar{v}(t)\,dt = 0$, and using the Wirtinger inequality for zero-mean $T$-periodic functions, i.e.

$$\|\bar{v}\|_{L^2} \leq \frac{T}{2\pi} \|\dot{v}\|_{L^2}$$

then one has

$$\beta \int_0^T H(v(t))\,dt \leq \frac{T}{2\pi} \|\dot{v}\|^2_{L^2} = \frac{T}{2\pi} \|J\dot{v}\|^2_{L^2} = \frac{T}{2\pi} \|H'(v)\|^2_{L^2} \leq$$

$$\leq \frac{T}{2\pi} a_2^2 \int_0^T |v(t)|^{2\beta - 2}\,dt \leq \frac{T}{2\pi} a_2^2 M \int_0^T |v(t)|^{\beta}\,dt \leq \frac{T}{2\pi} \frac{a_2^2}{a_1} M \int_0^T H(v(t))\,dt,$$

where

$$M = \max\{ |v(t)|^{\beta - 2} : t \in [0,T] \}$$

Therefore, if $v$ solves (H), one has

$$M \geq \frac{2\pi}{T} \frac{a_1}{a_2^2} \beta, \tag{35}$$

so, putting $v = z_n$, one would have

$$M_n \geq \frac{2\pi}{T} \frac{a_1}{a_2^2} \beta,$$

which contradicts the fact that $M_n$ goes to $0$. Therefore the number $c_{min} = \inf \{F(v) : v \in E, F'(v) = 0, F(v) > 0\}$ is positive. It is also a critical value for $F$ (hence the minimum positive critical value of $F$), as it follows from the fact that, if $F(v_n) \longrightarrow c_{min}$, with $F(v_n) > 0$, $F'(v_n) = 0$, then $\{v_n\}$ and $\{\dot v_n\}$ are uniformly bounded (use (34)), so a subsequence of $\{v_n\}$ uniformly converges to a point $\bar v$, which obviously verifies

$$F'(\bar v) = 0 \qquad F(\bar v) = c_{min} .$$

At this point, it is easy to check that every solution $z$ of (H) satisfies, as a consequence of (34) and (35), an estimate of the type

$$C_1 \left(\frac{1}{h}\right)^{\frac{\beta-2}{\beta}} \leq T, \tag{36}$$

where $h$ is the energy level of $z$ and $C_1$ is a suitable constant depending on $a_1, a_2, \beta$. From the other side, if one considers ~~the~~ any solution $z_{min}$ such that $F(z_{min}) = c_{min}$, one can prove that the following estimate from above holds for its period $T$, in dependence of its energy $h$.

$$T \leq C_2 \left(\frac{1}{h}\right)^{\frac{\beta-2}{\beta}}, \quad C_2 = C_2(a_1, a_2, \beta) \tag{37}$$

[(37) can be proved by taking into account that the T-periodic solution $z_R$ exhibited by Rabinowitz in [24] satisfies a suitable estimate from above concerning its critical level $F(z_R)$, and that, obviously, $F(z_{min}) \le F(z_R)$].

At this point the integer $\bar{n}$ of the statement of Theorem 7 is founded to be

$$\bar{n} = \min\{n \in \mathbb{N} : n \ge 2,\ n > C_2/C_1\}$$

In fact, if $F(z) = c_{min}$, then $m \ge \bar{n}$ would imply, by (37),

$$T/m \le T/\bar{n} < T\, C_1/C_2 \le C_2 \left(\frac{1}{h}\right)^{\frac{\beta-2}{\beta}} \frac{C_1}{C_2} = C_1 \left(\frac{1}{h}\right)^{\frac{\beta-2}{\beta}},$$

while, by (36), one has $T/m \ge C_1 \left(\frac{1}{h}\right)^{\frac{\beta-2}{\beta}}$

This technique enables, on the other side, to solve completely the case where H is homogeneous (with degree $\beta$), as, in this case, one can just show that the solutions $z$ with $F(z) = c_{min}$ have minimal period T.

[Let us suppose, by contradiction, that $z$ has a period $T/m$, with $m \ge 2$ and let $z^*(t) = \left(\frac{1}{m}\right)^{\frac{1}{\beta-2}} z(t/m)$. From the homogeneity of H, it follows that $z^*$ is a solution of (H), so $F(z^*) \ge F(z) = c_{min}$. From the other side:

$$F(z^*) = \frac{1}{2}\left(\frac{1}{m}\right)^{\frac{2}{\beta-2}} \frac{1}{m} \int_0^T \langle J\dot{z}(t), z(t)\rangle\, dt - \left(\frac{1}{m}\right)^{\beta/\beta-2} \int_0^T H(z(t/m))\, dt =$$

$$= \left(\frac{1}{m}\right)^{\beta/\beta-2} \left\{ \frac{1}{2} \int_0^T \langle J\dot{z}(t), z(t)\rangle\, dt - \int_0^T H(z(t/m))\, dt \right\},$$

so, by taking into account that $H(z(t)) = H(z(t/m)) = h\ \forall t \in [0,T]$, one concludes that $F(z^*) < F(z)$, which is an absurde].

Some improvements of Theorem 7 have appeared in a following paper [19], where one states the existence of T-periodic solutions having minimal period T, for a suitable T in an interval of the type $[K, \frac{3}{\beta_{32}} K]$, where K is any arbitrary positive number.

By giving a simple condition on the superquadraticity coefficients of H, it is possible to solve completely the problem and also obtain a multiplicity result expressed by the following:

THEOREM 8 (Girardi and Matzeu [20]) Let H satisfy conditions (31), (32), (33) and let the further hypothesis be verified:

(38) $a_2/a_1 < \sqrt{2}\,\beta$.

Then, for any $T>0$, there exist at least N distinct periodic solutions of (H), having minimal period T.

Taking, as it is always possible, $a_1 = 1$, the main idea is to use the mentioned Wirtinger inequality for T-periodic functions of $H^1$ having zero-mean, in order to state that, if

$$F(z) > 0, \quad F'(z) = 0, \quad F(z) < g(\beta)\, 2^{\beta/\beta-2}$$

where

$$g(\beta) = \frac{\beta-2}{2} \left(\frac{2\pi\beta}{a_2^2}\right)^{\beta/\beta-2} \left(\frac{1}{T}\right)^{2/\beta-2},$$

then z has minimal period T.

[In fact if, by contradiction, z had minimal period $T/m$, with $m \geq 2$, putting $z = \bar{z} + c$, where $\bar{z}$ is a zero-mean function of $H^1$ and $c \in \mathbb{R}^{2N}$, by the Wirtinger inequality, one would have:

$$\|\bar{z}\|_{L^2} \leq T/_{4\pi} \|\dot{z}\|_{L^2},$$

so the same calculations made in the proof of Theorem 7 yield here

$$A = \max\{|z(t)|^{\beta-2} : t \in [0,T]\} \geq \frac{4\pi\beta}{T a_2^2},$$

hence one would have, by (31) with $a_1 = 1$,

$$F(z) \geq \frac{\beta-2}{2}\int_0^T H(z(t))dt \geq \frac{\beta-2}{2}\left(\frac{4\pi\beta}{a_2^2}\right)^{\beta/\beta-2}\left(\frac{1}{T}\right)^{2/\beta-2} = g(\beta)\, \ell^{\beta/\beta-2}$$

At this point, one gives an estimate from above for the critical level $c_{min}$. At this purpose we need some definitions and concepts concerning the pseudo-index theory by Benci, developped in [5].

Firstly one considers, in the space $E = H^{1/2}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2N})$ the action $I(w) = \frac{1}{2}\langle Lw, w\rangle - \frac{1}{2}(F'(w))(w)$ $\forall w \in E$, where $\Phi$ is the extension of the functional $\int_0^T \langle J\dot{w}(t), w(t)\rangle dt$ from $C^\infty(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2N})$ into $E$. Let $E = E^+ \oplus E^- \oplus E^0$ be a suitable orthogonal decomposition of $E$ with respect to $I$, where $I$ is respectively positive definite, negative definite and identically zero on $E^+, E^-, E^0$. Putting

$$\mathcal{H}^* = \{h : E \to E : h \text{ is an } S^1\text{-equivariant homeomorphism of the form } h = e^{tL} + \Phi,\ t \in \mathbb{R},\ \Phi \text{ compact}\},$$

and starting from the $S$-index by Benci, defined on the class $\Sigma$ of the closed $S^1$-invariant subsets of $E$ as

$$i(A) = \begin{cases} \text{the minimum integer } k \text{ such that there exists } n \in \mathbb{N} \text{ and a continuous map } \Phi : A \to \mathbb{C}^k \setminus \{0\} \text{ such that } \Phi(u(\cdot + s)) = e^{ins}\Phi(u(\cdot)) \\ \quad \forall u \in A, \forall s \in [0,T] \\ +\infty \text{ if there is no integer } k \text{ having the previous property,} \end{cases}$$

one defines the pseudo-index $i^*$ on $\Sigma$ as

$$i^*(A) = \min \{ i(h(A) \cap S_\rho \cap E^+) ; h \in \mathcal{H}^* \},$$

where $S_\rho$ is the sphere of $E$, $\{v \in E : \|v\| = \rho\}$.

Then, it is possible to show, as a consequence of a very general theorem (Thm. 4.2. of [5]) that the numbers $c_k$ defined, for $k=1,\dots,N$, as

$$c_k = \inf_{A \in \Sigma,\, i^*(A) \geq k} \ \sup_{v \in A} F(v)$$

are in fact critical values of $F$. Moreover they can be estimated, by the consideration of the analogous $\bar{c}_k$'s related to the choice of the other Hamiltonian function $\bar{H}(z) = |z|^\beta$. More precisely one has

$$c_k \leq \bar{c}_k \quad \forall k \in \{1,\dots,N\}$$

and

$$\bar{c}_1 = \dots = \bar{c}_N = g(\beta) \left(\frac{a_2}{\beta}\right)^{2\beta/\beta-2},$$

hence, as $a_2 < \sqrt{2}\,\beta$, one concludes that, for every $z$ such that $F(z) = c_k$, $(k \in \{1,\dots,N\})$ the level $F(z)$ satisfies all conditions in (39), so $z$ has minimal period $T$. If $c_k \neq c_h$ this yields at least $N$ different solutions, when $c_k = c_h$ for some distinct $k$ and $h$ in $\{1,\dots,N\}$, then a further statement of Thm. 4.2. of [5] assures that there infinitely many distinct solutions.

Some ideas of the proof of Theorem 8 have been developed, through a suitable use of the $\mathbb{Z}_p$-pseudo index theory, to obtain some multiplicity results for non-autonomous Hamiltonian systems by Michalek and Tarantello [22].

Some applications of Theorem 8 can also be made to the fixed energy problem, as we shall see in § 3 of Chapter 2.

# CHAPTER 2.

## THE FIXED ENERGY PROBLEM.

§1. POSITION OF THE PROBLEM. First of all, let us recall that, if $z$ is a solution of the Hamiltonian system (H), then $H(z(t)) = \text{const}$ for $t \in [0,T]$, that is $H$ is a first integral of (H). Let $S = \{z \in \mathbb{R}^{2N} : H(z) = \text{const}\}$ be a bounded and regular (i.e. $H'(z) \neq 0 \ \forall z \in S$) surface of $\mathbb{R}^{2N}$.

If $\tilde{H}$ is another $C^1$-function on $\mathbb{R}^{2N}$ such that $S = \{z \in \mathbb{R}^{2N} : \tilde{H}(z) = \text{const}\}$ then $H'(z)$ and $\tilde{H}'(z)$ are different from 0 for any $z \in S$, so there exists a real function $\lambda(z)$ such that

$$\lambda(z) H'(z) = \tilde{H}'(z) \qquad \forall z \in S$$

Furthermore $\lambda(z) \neq 0 \ \forall z \in S$ and $\lambda(z)$ has a constant sign, for example $\lambda(z) \geq c > 0 \ \forall z \in S$.

Therefore, if $z$ is a solution of (H) and $\gamma(t)$ is a primitive of $\lambda(z(t))$ that is $\dot{\gamma}(t) = \lambda(z(t))$, then

$$J \frac{d}{dt} z(\gamma(t)) = J \frac{d}{d\gamma} z(\gamma(t))\, \dot{\gamma}(t) = H'(z(\gamma(t)))\, \dot{\gamma}(t) = \tilde{H}'(z(\gamma(t))),$$

so the Hamiltonian flow doesn't depend on the function $H$, but only on the surface $S$.

Then two general questions arise in a natural way

Pb. 1 EXISTENCE: given a smooth, bounded surface $S$ in $\mathbb{R}^{2N}$, does it carry some periodic Hamiltonian trajectory?

Pb. 2 MULTIPLICITY : If the answer to the previous question is positive, how many Hamiltonian trajectories lie on $S$ ?

The solution to these problems obviously depends on some geometrical properties of $S$, in view of the previous remark.

A first type of results concerns the small oscillations, that is the periodic solutions near an equilibrium of the system. We don't speak of this type of results in our lectures, but we only recall a classical theorem:

THEOREM 1. (Weinstein [28], Moser [23]) Let $H$ be $C^2$ in a neighbourhood of $z=0$ $(H(0)=0, H'(0)=0)$ and let the Hessian $H''(0)$ be positive definite. Then, for every $\varepsilon > 0$ small enough, the surface $H(z)=\varepsilon$ carries at least $N$ Hamiltonian trajectories.

This result generalises the well known Liapunov center theorem, which states the same conclusion under a suitable "non resonance" condition on eigenvalues of the Hessian.

Let us now expose some results about Pb 1 and Pb 2.

§2. SOME EXISTENCE RESULTS. A first result was obtained by Weinstein in 1978, expressed by the following

THEOREM 2 (Weinstein [29], Seifert [25]). Let $S=\partial\Omega$, where $\Omega$ is a bounded convex domain in $\mathbb{R}^{2N}$. Then $S$ carries at least one periodic Hamiltonian trajectory

A generalization of Theorem 2 is the following

THEOREM 3 (Rabinowitz [24]) Let $S$ be strictly starshaped (i.e. radially diffeomorphic to a sphere). Then $S$ carries at least one periodic Hamiltonian trajectory

Before giving some idea of the proof of Theorem 3, let us give some notice of further developments about Pb. 1.

Recently, in 1986, C. Viterbo proved in [27] that any manifold of "contact type" in $\mathbb{R}^{2N}$ carries at least one periodic trajectory, later on this result has been extended by Hofer and Zehnder [21].

Let us give now an idea of the proof of Theorem 3.

Firstly, define, for any $z \in \mathbb{R}^{2N}$, $\alpha(z)$ as the (unique) positive number such that

$$z = \alpha(z)\, w$$

where $w$ is the (unique) point of $S$ lying on the ray of $z$. Then

$$S = \{ z \in \mathbb{R}^{2N} : \alpha(z) = 1 \}$$

Let $r = \min\{|z| : z \in S\}$, $R = \max\{|z| : z \in S\}$ and let $d = \min\{ \langle n(z), z \rangle : z \in S \}$, where $n(z)$ is the unitary exterior normal vector to $S$ at $z$. Recall that $\langle n(z), z \rangle$ represents the distance between the tangent hyperplane of $S$ at $z$ and the origin of the coordinates. So $d$ represents the ray of the maximal sphere contained in the envelope of the tangent hyperplanes of $S$.

One can easily check the following properties for $\alpha(z)$

$$\frac{1}{R}|z| \le \alpha(z) \le \frac{1}{r}|z|$$

$$\langle z, \alpha'(z)\rangle = \alpha(z)$$

$$\max\{\, |\alpha'(z)| : z \in S \} = 1/d$$

Then, if one takes any number $\beta \geq 1$, one can define

$$H(z) = \alpha(z)^{\beta},$$

so that $H$ is a function belonging to $C^2(\mathbb{R}^{2n}\setminus\{0\};\mathbb{R}) \cap C^1(\mathbb{R}^{2n};\mathbb{R})$ such that

$(H_1)$ $\frac{1}{R^{\beta}} \le H(z) \le \frac{1}{r^{\beta}}|z|^{\beta}$

$(H_2)$ $\beta H(z) = \langle H'(z), z\rangle$

$(H_3)$ $|H'(z)| \le \frac{\beta}{d\, r^{\beta-1}}|z|^{\beta-1}$

Moreover

$$S = \{ z \in \mathbb{R}^{2n} : H(z) = 1\}$$

Let us finally remark that, if $S$ is the boundary of a convex set, then $d \geq r$ and $H$ is a strictly convex function.

Now the problem of finding a periodic Hamiltonian trajectory on $S$ can be formulated in various ways. The most natural way is to consider the action integral

$$I(z) = \frac{1}{2}\int_0^{2\pi} \langle J\dot{z}(t), z(t)\rangle\, dt$$

in a suitable given space $E$ of $2\pi$-periodic functions, for example

$$E = \{ z \in H^1(0,2\pi; \mathbb{R}^{2N}) : z(0) = z(2\pi) \}$$

and consider the manifold

$$S = \{ z \in E : \frac{1}{2\pi} \int_0^{2\pi} H(z) = 1 \}$$

Then a critical point of $I$ on $S$ verifies

$$J\dot{z} = \lambda H'(z),$$

so, putting $\lambda = \frac{T}{2\pi}$, the function $z(\tau) = z(t \frac{2\pi}{T})$ verifies

$$\begin{cases} J\dot{z} = H'(z) \\ z(0) = z(T) \\ z(t) \in S \end{cases}$$

But, due to the fact that the function $H$ is homogeneous of degree $\beta$, one can also observe that, given a $T$-periodic solution $z$ of (H), with $H(z(t)) = h$, then

$$w(t) = h^{1/\beta} z(h^{\frac{2-\beta}{\beta}} t)$$

is a periodic solution (with period $2\pi h^{\frac{2-\beta}{\beta}}$) such that $H(w(t)) = 1$.

Then, in order to find a periodic solution on $S$, one can arbitrarily fix $T, \beta$ and show that the functional

$$F(z) = \frac{1}{2} \int_0^T \langle J\dot{z}, z \rangle - \int_0^T H(z)$$

has a critical point in a suitable space.

At this point it is clear how to obtain the proof of Theorem 3. For example, one can fix some $\beta > 2$ and consider the "superquadratic Hamiltonian function $H(z)=(\alpha(z))^{\beta}$ and to find, for any $T>0$, a $T$-periodic solution of the corresponding Hamiltonian system, as a consequence of the mentioned theorem by Rabinowitz [24].

§ 3. SOME MULTIPLICITY RESULTS. The first multiplicity result "in large", (that is not local as the theorem by Weinstein and Moser) was obtained by Ekeland and Lasry in 1978. They stated the following result:

THEOREM 4 (Ekeland and Lasry [7], see also Ambrosetti and Mancini [8]). Let $S$ be a $C^2$-boundary of a compact, strictly convex set of $\mathbb{R}^{2N}$. Then, if $R^2<2r^2$, $S$ carries at least $N$ periodic Hamiltonian trajectories.

Let us remark that, in fact, by some simple linear examples, it is easy to show that there exist some surfaces in $\mathbb{R}^{2N}$ which carry "only" $N$ Hamiltonian trajectories. Let $Q$ the $2N\times 2N$ matrix
$Q=\begin{pmatrix}\omega_1 & & & & 0\\ & \ddots & & & \\ & & \omega_N & & \\ & & & \omega_1 & \\ & & & & \ddots \\ 0 & & & & \omega_N\end{pmatrix}$, let $H(z)=\frac{1}{2}\langle Qz,z\rangle$ and consider the linear system

$$J\dot z = Qz\ ,\quad \text{so}\quad \dot z=-JQz$$

~~Hence~~ any solution has the form

$$z(t)=z(0)\,e^{-JQt}$$

Therefore, it is clear that any surface ("ellipsoid") $E_h=\{z\in\mathbb{R}^{2N}: \frac{1}{2}\langle Qz,z\rangle=h\}$ carries only $N$ periodic trajectories, if $\omega_i/\omega_j\notin\mathbb{Q}$ for $i\neq j$ ($N$ normal modes of period $2\pi/\omega_i$)

Later on, in case that $N\geq 2$, Ekeland and Lasry have been able to eliminate the condition on the rays $R$ and $r$ and to state the following:

THEOREM 5. (Ekeland and Lasry [13]). Let $S$ be a $C^2$-boundary of a compact strictly convex set of $\mathbb{R}^{2N}$, with $N \geq 2$. Then $S$ carries at least two periodic Hamiltonian trajectories.

Theorem 4 was extended in 1985 to a star-shaped surface by Berestycki, Lasry, Mancini and Ruf. Before stating this theorem, let us introduce some notations.

Let $Q$ be a $2N \times 2N$ diagonal matrix as before. Then if $S$ is a $C^2$ ~~bounded~~ strictly star-shaped surface in $\mathbb{R}^{2N}$, then $S$ verifies, for some suitable positive numbers $\alpha, \beta$, the following inclusion:

$$S \subset \{z \in \mathbb{R}^{2N} : \alpha \leq \frac{1}{2} \langle Qz, z \rangle \leq \beta\}.$$

Putting $d = \min\{\langle n(z), z \rangle : z \in S\}$, one can state the following

THEOREM 6 (Berestycki, Lasry, Mancini and Ruf [7]). Let $S$ as before. There exists a constant $\delta = \delta(d, \alpha, \omega_1, \ldots, \omega_N) > 0$, such that, if $\beta^2 < (1+\delta)\alpha^2$, then $S$ carries at least $N$ periodic Hamiltonian trajectories

An explicit expression of the function $\delta$ is also given in [7].

For simplicity, here we consider only a particular case of the previous theorem, that is when $\omega_1 = \cdots = \omega_N = 1$. In this case the theorem can be expressed in the following form:

COROLLARY of THEOREM 6. Let $S$ be a $C^2$ bounded strictly star-shaped surface in $\mathbb{R}^{2N}$ and let $R = \max\{|z| : z \in S\}$. If $R^2 < 2d^2$, then $S$ carries at least $N$ periodic Hamiltonian trajectories

Before giving an idea of the proof of this result, let us remark that, in this last form, it is clear that, if $S$ is the boundary of a convex set, then $d \geq 2$ and this result becomes the theorem by Ekeland and Lasry. However, if you look at the general statement of Theorem 6, it is clear that even the local theorem, that is the Weinstein theorem mentioned above, can be seen as a particular case.

The proof can be broken in some steps.

Step 1. Variational formulation. Let $E = H^{1/2}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^{2N})$, let $L$ denote the bounded linear self-adjoint operator that extends to $H^{1/2}$ the quadratic form $f(z) = \frac{1}{2}\int_0^{2\pi} \langle J\dot{z}, z\rangle$ and consider the action integral

$$I(z) = \frac{1}{2} \langle Lz, z\rangle \quad \forall z \in E$$

Consider the manifold

$$S^* = \{z \in E : \frac{1}{2\pi}\int_0^{2\pi} H(z) = 1\},$$

where $H(z)$ is defined on the surface $S$ by the "gauge" function $\alpha(z)$, putting $H(z) = (\alpha(z))^2$. Then $S^*$ is a $C^1$-manifold in $E$ which is radially diffeomorphic to the unit sphere of $E$.

By a simple regularity argument (see [6]), one can see that, if $z \in S^*$ is a critical point of $I$ on $S^*$, then there exists some $\lambda \in \mathbb{R}$ (a Lagrange multiplier) such that

$$J\dot{z} = \lambda H'(z)$$

and $z$ belongs to $C^1(\mathbb{R}; \mathbb{R}^{2N})$

The problem now is to find N critical points of $I$ on $S^*$. Let us remark that, in order to obtain <u>different</u> periodic trajectories on $S$, we have to take into account the following problems:

(i) if $z(t)$ is a solution on $S$, then also $z(t+\vartheta)$ is a solution on $S$ $\forall \vartheta \in [0,2\pi]$

(ii) if $z(t)$ is a solution on $S$ and its minimal period is not $2\pi$ but $2\pi/m$ for some integer $m \geq 2$, then also $z^1(t) = z(t/m)$ is a solution and describes the same trajectory.

The first question can be solved by a suitable $S^1$-pseudo index theory related to the manifold $S^*$, which one has to use also to prove the existence of critical points; the second one can be solved by proving that the critical points founded have $2\pi$ as their minimal period.

Therefore the next step in the proof will be the definition of a suitable pseudo-index theory on $S^*$ and the related minimax principle which allows to find N critical points of $I(z)$ on $S^*$.

Step 2. <u>Pseudo index theory on $S^*$ and minimax principle</u>

The main result is the following

PROPOSITION 1. <u>There exist</u> $N$ <u>critical</u> <u>values of</u> $I$ <u>on</u> $S^*$, <u>say</u> $c_1 \leq \dots \leq c_N$ <u>such that</u>

$$\pi r^2 \leq c_1 \leq \dots \leq c_N \leq \pi R^2,$$

and, if $c_i = c_j$ for some $i \neq j$, then there exist infinitely many distinct critical orbits at level $I(z) = c_i$.

Here we only give some definitions and properties which yield Proposition 1, recalling that it is a natural extension of "linking" theorems (see [4], [5], [6], [7]). Let us define now the pseudo-index on $S^*$.

Let us consider the $S^1$-index mentioned in § 4. Following the same other notations of § 4, let us put:

$\mathcal{U} \equiv$ the family of all self-adjoint linear equivariant isomorphism $U: E \to E$ such that $U(E^+) \subset E^+$

$T_{S^*} = \{ h : S^* \to S^* : h$ is an equivariant homeomorphism and $\exists\, g : S^* \to \mathbb{R}^+$ continuous and $\exists\, U \in \mathcal{U}$ such that $h - gU$ is compact $\}$

Then, for any subset $A$ of $S^*$, with $A \in \Sigma$, the pseudo-index of $A$ on $S^*$ is defined as

$$i^*_{S^*}(A) = \min \{ i(h(A) \cap E^+) : h \in T_{S^*} \}$$

Let us recall the following results (for details see [4], [5], [7])

PROPOSITION 2. Let $S_1^*$ and $S_2^*$ be two $C^1$-manifolds radially diffeomorphic to a unit sphere of $E$ and invariant for the $S^1$-action and let $p : S_1^* \to S_2^*$ be the radial projection, then

$$i^*_{S_1^*}(A) = i^*_{S_2^*}(p(A)) \quad \forall A \in S_1^* \cap E$$

PROPOSITION 3. Let $H_h$ be a $2h$-dimensional invariant subspace of $E^+$ and let $E = H_h \oplus W$. Then, for $A \in \Sigma \cap S^*$,

$$i^*_{S^*}(A) \geq h+1 \Rightarrow A \cap W \neq \emptyset$$

Moreover

$$i^*_{S^*}\left[(H_h \oplus E^- \oplus E^0) \cap S^*\right] = h$$

By standard arguments based on the Ljusternik-Schnirelman theory, provided that the action integral on $S^*$ verifies the Palais-Smale condition (see [7]), one has the following

MINIMAX PRINCIPLE. Let $Z_a = \{z \in S^* : I'|_{S^*}(z) = 0,\ I(z) = a\}$ and define

$$a_k = \inf_{A \in \mathcal{Q}_k} \sup_{z \in A} I(z) \qquad \mathcal{Q}_k = \{A \in \Sigma \cap S^* : i^*_{S^*}(A) \geq k\}.$$

Then $a_k$ is a critical value for $I|_{S^*}$; moreover if $c_k = \dots = c_{k+p} = c$ then $i^*_{S^*}(Z_c) \geq p+1$. Furthermore, if $i^*_{S^*}(Z_c) \geq 2$, then $Z_c$ contains infinitely many distinct critical orbits (see [7]).

By using the minimax principle, it is clear that the functional $I(z)$ has infinitely many critical values on any $C^1$-manifold $S^*$ which is radially homeomorphic to a unit sphere.

At this point, in order to state Proposition 1, we need two lemmas

LEMMA 1. Let $Z = \{z \in S^* ; I'|_{S^*}(z) = 0, I(z) > 0\}$. Then, if $z \in Z$, $I(z) \geq \pi d^2$

Proof - Put $M = \max\{|H'(z)| : z \in H^{-1}(1)\}$ and

Then
$$d(z) = \langle n(z), z \rangle = \langle H'(z), z \rangle \frac{1}{|H'(z)|} = \frac{2}{|H'(z)|}$$

$$d = \min\{d(z) : z \in S\} = 2/M$$

Let $u$ be a critical point of $I/_{S^*}$. Then one has

$$I(u) = \frac{1}{2}\int_0^{2\pi} \langle J\dot u, u \rangle = \frac{1}{2}\lambda \int_0^{2\pi} \langle H'(u), u \rangle = \frac{\lambda}{2}\, 2 \cdot 2\pi = \lambda 2\pi = T.$$

On the other hand, by using the Wirtinger inequality, one has

$$2\pi\lambda = \frac{1}{2}\int_0^{2\pi} u J\dot u = \frac{1}{2}\int_0^{2\pi} \tilde u J \dot{\tilde u} \leq \frac{1}{2}\|\dot u\|_2^2,$$

where $u = \tilde u + c$, $c \in \mathbb{R}^{2N}$ and $\int_0^{2\pi} \tilde u = 0$. Then

$$2\pi\lambda \leq \frac{1}{2}\int_0^{2\pi} |\dot u|^2 = \frac{1}{2}\int_0^{2\pi} |\lambda H'(u)|^2 \leq \frac{\lambda^2}{2}\int_0^{2\pi} |H'(u)|^2 \leq \frac{\lambda^2}{2} M^2 2\pi = \pi\lambda^2 M^2,$$

so

$$\lambda \geq 2/M^2,$$

that is

$$I(u) = 2\pi\lambda \geq \frac{4\pi}{M^2} = \pi d^2 \qquad \blacksquare$$

Now we apply the minimax principle and ... Lemma 1 to a sphere $S_R$ of radius $R$. Let $S_R = \{z \in \mathbb{R}^{2N} : |z|^2 = R\}$ and let $S_R^* = \{u \in H^{1/2} : 1/2\pi \int_0^{2\pi} |u|^2 = R\}$.

By the previous lemma, if $u$ is a critical point of $I_{/S_R}$ with $I(u)>0$, then $I(u) \geq \pi R^2$. Now we want to prove that, putting $b_1 \leq \dots \leq b_N$, the first $N$ critical values of $I$ on $S_R$, one has

LEMMA 2. <u>One has</u>

$$b_1 = b_2 = \dots = b_N = \pi R^2$$

<u>Proof</u>. – By the Palais-Smale condition, there exists the minimum positive critical value of $I$ on $\dot{S}_R^a$, $b_{min}$ say. If $u = I^{-1}(b_{min})$, then $u$ is a solution of a linear Hamiltonian system, then

$$u = (\xi \cos \gamma t + \eta \sin \gamma t ; \xi \sin \gamma t - \eta \cos \gamma t)$$

for some $\xi, \eta \in \mathbb{R}^N$. Provided that $u \in I^{-1}(b_{min})$, $u$ has minimal period $2\pi$ (if not $u^*(t) = u(t \frac{m}{2\pi})$ is such that $I(u^*) = \frac{1}{m} I(u)$]

Therefore $\gamma = 1$, then

$$\{u \in S_R^* : I'_{/S_R^*}(u) = 0, I(u) = b_{min}\} =$$

$$= \{\xi \cos t + \eta \sin t ; \xi \sin t - \eta \cos t : |\xi|^2 + |\eta|^2 = R^2\}$$

let now denote

$$H_N = \{(\xi \cos t + \eta \sin t ; \xi \sin t - \eta \cos t) : \xi, \eta \in \mathbb{R}^N\}$$

$H_N$ is a $2N$-dimensional linear subspace of $E^+$, so, putting

$$A_N = (H_N \oplus E^- \oplus E^0) \cap S_R,$$

one has, by Proposition 3, that

$$i^*(A_N^\phi) = N$$

but an easy computation shows that

$$\sup_{u \in A_N} I(u) = \pi R^2,$$

then

$$\pi R^2 \le b_{min} \le b_1 \le \cdots \le b_N \le \sup_{u \in A_N} I(u) = \pi R^2$$

and Lemma 2 is proved

Now let $c_1 \le \cdots \le c_N$ be the critical values of mini-max type obtained via the minimax principle. One can take $A = A_N$ in Proposition 2, so $i^*_{S^*}(A_N) = i^*_{S^*}(p(A_N)) = N$. Then, taking, for any $u \in A_N$, $\lambda$ as the (unique) element in $(0,1]$ such that $\lambda u \in S^*$, one has

$$c_N \le \sup_{w \in p(A_N)} I(w) = \sup_{u \in A_N} I(\lambda(w)u) \le \sup_{\substack{u \in A_N \\ \lambda \in (0,1]}} \lambda^2 I(u) \le \sup_{u \in A_N} I(u),$$

which implies

$$c_1 \le \cdots \le c_N \le \pi R^2$$

To complete the proof of Proposition 1, it is easy to use Proposition 2, with p given by the projection of the sphere $S^*_r$ into $S^*$, in order to obtain

$$\pi r^2 \le c_1$$

STEP 3. Now we can easily prove the Corollary of Theorem 6.
Firstly we can always reduce to the case that

$$c_1 < c_2 \dots < c_N,$$

as, if not, there exist infinitely many distinct orbits and then infinitely many periodic trajectories on $S$ (provided that there exists a lower bound for the positive critical values, that is for the periods).

Then we have at least $N$ solutions $z_1(t), \dots, z_N(t)$, with distinct orbits that is $z_i(t+\vartheta_1) \neq z_j(t+\vartheta_2)$ $\forall \vartheta_1, \vartheta_2 \in [0, 2\pi]$

At this point, we have only to solve the problem of the minimality of the period $2\pi$

If $z(t) \in I^{-1}(c_i)$ and has not minimal period $2\pi$, let $2\pi/m$, $m \geq 2$ be the minimal period and let $z^*(t) = z(t/m)$, say the "primitive" of $z$. Then

$$I(z^*) = \frac{1}{m} I(z) \leq \frac{1}{2} I(z) \leq \frac{\pi R^2}{2} < \pi d^2$$

and the proof is complete.

§ 3. SOME OTHER RESULTS AND EXTENSIONS. In this section we want to illustrate some different results which can be obtained for Pb 2 for the star-shaped surfaces.

Let us state the following simple lemma

LEMMA 1. Let $\bar{n}$ be the minimum integer number such that

$$R^2 < \sqrt{\bar{n}}\, r d$$

and let $z_i$ be a solution, that is $I'_{l,s}(z_i)=0$, at level $c_i$ ($i \in \{1,\dots,N\}$). Then the minimal period of $z_i$ is $\frac{2\pi}{m}$ with $1 \le m \le \bar{n}-1$.

Proof - Let $2\pi/m$ be the minimal period of $z_i$. Then, using the Wirtinger inequality and putting $z_i = \tilde{z}_i + c$ with $c \in \mathbb{R}^{2N}$ and $\int_0^{2\pi} \tilde{z}_i = 0$, one has

$$\pi r^2 \le I(z_i) = \frac{1}{2}\int_0^{2\pi} \langle J\dot{z}_i, z_i\rangle = \frac{1}{2}\int_0^{2\pi} \langle J\dot{\tilde{z}}_i, \tilde{z}_i\rangle \le \frac{1}{2m}\|\dot{z}_i\|^2_{L^2} \le \frac{1}{2m}\int_0^{2\pi} |\lambda H'(z_i)|^2 \le$$

$$\le \frac{1}{2m}\lambda^2 M^2 2\pi = \frac{\pi\lambda^2 M^2}{m},$$

hence

$$\lambda \ge \frac{\sqrt{m}}{2}\, r d$$

If $m \ge \bar{n}$, then one would have

$$\lambda \ge \frac{\sqrt{\bar{n}}}{2}\, r d,$$

so

$$I(z_i) = 2\pi\lambda \ge \pi\sqrt{\bar{n}}\, r d > \pi R^2,$$

which is an absurde ∎

Then one can give the following

THEOREM 7. (Girardi and Matzeu [20]). Let $S$ be as in Theorem 6 and let

$$R^2 < \sqrt{3}\, r d$$

Then $S$ carries at least $N$ distinct periodic Hamiltonian trajectories

Proof. By Lemma 1, we know that the minimal period of a solution $z_i \in I^{-1}(c_i)$, for $i \in \{1,\dots,N\}$, can be only $2\pi$ or $\pi$. Let $z_i^*$ the the corresponding function of minimal period $2\pi$.

Given any two solutions of minimax type, $z_i$ and $z_j$, we want to prove that $I(z_i^*) \neq I(z_j^*)$ for $i \neq j$. On the other side, if $I(z_i) \neq I(z_j)$ and if the two solutions $z_i$ and $z_j$ have the same minimal period (either $2\pi$ or $\pi$), then obviously $I(z_i^*) \neq I(z_j^*)$. Therefore the only case to deal is when one of them, say $z_i$, has minimal period $2\pi$, and the other, say $z_j$, has minimal period $\pi$. In this case, if $I(z_i^*)$ was equal to $I(z_j^*)$, one would have

$$I(z_j) = 2\,I(z_j^*) = 2\,I(z_i^*) = 2\,I(z_i) \geq 2\pi r^2 \geq 2\pi r d \geq \pi\sqrt{3}\, r d > \pi R^2,$$

which is a contradiction ∎

Theorem 7 can be generalized to the following

THEOREM 8 (Girardi and Matzeu [20]). Let $S$ as before and let $\bar{n}$ be an integer number such that $r < \sqrt{\bar{n}}\, d$. If

$$R^2 < \min\left\{\sqrt{\bar{n}}\, r d,\ \frac{\bar{n}-1}{\bar{n}-2}\, r^2\right\},$$

then $S$ carries at least $N$ distinct periodic Hamiltonian trajectories

Proof. The condition $R^2 < \sqrt{\bar{n}}\, r d$ guarantees that the minimal period of every $z_i$ at level $c_i$ is $2\pi/m$ with $m \leq \bar{n}-1$ and the condition $R^2 < \frac{\bar{n}-1}{\bar{n}-2} r^2$ guarantees that the functional $I$ assumes different values on the primitives $z_i^*$ ∎

Theorem 8 means that one can also obtain the multiplicity result for "strongly" star-shaped surfaces (that is with $d$ very small), but this is "payed" by the fact that, in this case, one has to take $R$ more near to $\varepsilon$.

Let us conclude now, showing how all the previous results can be improved by making some more assumptions on the surface $S$. In particular we'll expose the case where the surface $S$ is symmetric with respect to the origin.

Let $S$ verify all the assumptions of Theorem 6 and also suppose that $S$ is symmetric with respect to the origin, that is, if $z \in S$, then also $-z \in S$. In this case, it is easy to verify that the "gauge" function of $S$ becomes an even function: in fact, putting $H(z) = (\rho(z))^2$ and $S = \{z \in \mathbb{R}^{2N} : H(z) = 1\}$, one has that $H(z) = H(-z)$ $\forall z \in \mathbb{R}^{2N}$.

Now we state a suitable variational principle on a suitable subspace of $E = H^{1/2}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^{2N})$ in order to obtain "symmetric" periodic trajectories on the surface $S$ (see [14]). For any $u \in E$, let

$$u = \sum_{k=-\infty}^{+\infty} a_k e^{ikt}$$

be a Fourier expansion and define the subspace $E_1$ of $E$ as

$$E_1 = \{u \in E : u = \sum_{k=-\infty}^{+\infty} a_{2k+1} e^{i(2k+1)t}\}$$

Clearly, all functions in $E_1$ have zero mean, and verify the property

$$u(t+\pi) = -u(t) \quad \forall t \in [0, 2\pi]$$

Let us consider now the usual variational formulation restricted in $E_1$

that is, given the functional $I(u) = \frac{1}{2} \langle Lu, u\rangle$ and the manifold $S_1^* = \{u \in E_1 : \frac{1}{2\pi}\int_0^{2\pi} H(u) = 1\}$, one can consider the critical point of $I$ on the surface $S_1^*$ and state the following

LEMMA. If $u \in S_1^*$ is a critical point of $I$ restricted to $S_1^*$, then there exists some $\lambda > 0$ such that

$$J\dot{u} = \lambda H'(u)$$

Proof. Denote $\mathcal{H} \in C^1(E;\mathbb{R})$ the weakly continuous functional

$$\mathcal{H}(u) = \frac{1}{2\pi}\int_0^{2\pi} H(u)$$

If $u$ is a critical point of $I$ on $S_1^*$, then one has

$$\langle Lu, v\rangle = \lambda \mathcal{H}'(u)(v) \qquad \forall v \in E_1$$

We want to prove now that the previous identity holds in fact for any $v \in E$, so one has

$$\langle Lu, v\rangle = \lambda \mathcal{H}'(u)(v) \qquad \forall v \in E$$

and one can conclude as in the unconstrained case.

Indeed let $w \in E$ and let $w = w_1 + w_2 + w_0$, where $w_1 \in E_1$, $w_2 \in E_2 = \{u \in E : u = \sum_{k=-\infty}^{+\infty} a_k e^{ikt}\}$ and $w_0 \in E_0 \simeq \mathbb{R}^{2N}$. Then

$$\langle Lu, w\rangle = \langle Lu, w_1\rangle + \langle Lu, w_2\rangle$$

Let $u_n \to u$ in $C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{2N}) \cap E_1$. Then, by definition of $L$, one has that

$$\langle Lu, w_2\rangle = \lim_{n\to\infty} \int_0^{2\pi} \langle J\dot{u}_n, w_2\rangle,$$

but $u_n \in E_1$, then $\int_0^{2\pi} \langle J\dot{u}_n, w_2 \rangle = 0$ (Note that, if $u \in E_1$, $v \in E_2$ then $\int_0^{2\pi} \langle u, v \rangle = 0$ and, if $u \in H^1$, then $J\dot{u} \in E_1$). Therefore

$$\langle Lu, w \rangle = \langle Lu, w_1 \rangle$$

Recall now that $H'$ is a continuous map from $L^2(\mathbb{R}/2\pi; \mathbb{R}^{2N})$ into itself (as $|H'(z)| \le 2/\alpha \, |z|$). Then $u_n \to u$ in $C^\infty$ implies that $H'(u_n) \to H'(u)$ in $L^2$. As $u_n \in E_1$ and $H'(-z) = H'(z)$, one has that $H'(u_n(t+\pi)) = H'(-u_n(t))$ and then $H'(u_n) \in E_1$, so $H'(u) \in E_1$.

It follows that

$$\int_0^{2\pi} \langle H'(u_n), w_0 \rangle = \int_0^{2\pi} \langle H'(u_n), w_2 \rangle = 0 \qquad \forall n$$

hence

$$\mathcal{H}'(u)(w_0) = \mathcal{H}'(u)(w_2) = 0$$

and the lemma is proved. ∎

Let us state now the multiplicity result for symmetric surfaces

THEOREM 3 (Girardi [14]) <u>Let $S$ as before and let $S$ be symmetric with respect to the origin. Then $S$ carries at least one symmetric periodic Hamiltonian trajectory on $S$. Moreover, if</u>

$$R^2 < 3r\alpha,$$

<u>then $S$ carries at least $N$ symmetric periodic Hamiltonian trajectories</u>

<u>Proof</u>. Using the same pseudo-index theory introduced in § 4 of Ch. 1 for the general case, with $E$ replaced by $E_1$, then one obtains the existence of $N$ critical points $c_1 \le \ldots \le c_N$, verifying the estimates:

$$\pi r^2 \le c_i \le \pi R^2 \qquad \forall i \in \{1, \dots N\}$$

(Recall that the space $H_N$ is contained in $E_1$). At this point, let us remark that there are two different and better results in $E_1$. The first is that the estimate of a lower bound for critical values can be improved. In fact one has the following

LEMMA 1. *If $u$ is a critical point of $I$ on $S_1^*$, then*

$$I(u) \ge \pi r d$$

*Proof*. Recall that, if $u \in E_1$, then $u$ has zero mean, so

$$2\pi r^2 \le \int_0^{2\pi} |u|^2 \le \int_0^{2\pi} |\dot u|^2 = \frac{T^2}{4\pi^2} \int_0^{2\pi} |H'(u)|^2 \le \frac{T^2}{4\pi^2} M^2 2\pi,$$

then $T = I(u) \ge \pi r \frac{2}{M} = \pi r d$ ∎

The second fact is that a function $u \in E_1 \setminus \{0\}$, whose minimal period is not $2\pi$, cannot have minimal period $\pi$, since, if $u(t+\pi) = u(t)$, then, as $u \in E_1$ one has $u(t+\pi) = -u(t)$, which implies $u(t) = 0 \ \forall t \in [0, 2\pi]$.

This allows to state immediately the following

LEMMA 2. *If $u \in I^{-1}(c_i)$ and $R^2 < 3rd$, then $u$ has minimal period $2\pi$.*

*Proof* - If $u$ had not minimal period $2\pi$, then its minimal period would be $2\pi/m$ with $m \ge 3$. It follows that

$$I(u^*) = I(u(t/m)) = \frac{1}{m} I(u) \le \frac{1}{3} I(u) \le \frac{\pi R^2}{3} < \frac{1}{3}\pi\, 3rd = \pi r d$$

and one would have a contradiction with Lemma 1 ∎

Let us remark that Theorem 9 can also be generalized (as in the case of Theorem 6) to a symmetric surface which is near to an ellipsoid.

Let us also observe that, for the symmetric surfaces which are boundaries of convex sets, the condition becomes

$$R^2 < 3r^2$$

which yields an improvement of the theorem by Ekeland and Lasry.

Other cases where the Hamiltonian surfaces have "more" symmetric properties have been also studied. For example, Van Groesen in [26] considered the case that S is the boundary of a convex set and the "gauge" function $\alpha(z)$ verifies

$$\forall z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \quad \alpha(x,y) = \alpha(-x,y) = \alpha(x,-y) = \alpha(-x,-y).$$

Such a type of surfaces can also be treated in the star-shaped case, by a suitable variational principle.

REFERENCES


[1] A. AMBROSETTI, G. MANCINI: "Solutions of minimal period for a class of convex Hamiltonian systems", Math. Ann., 255 (1981), pp. 405-421

[2] A. AMBROSETTI, G. MANCINI: "On a theorem by Ekeland and Lasry concerning the number of periodic Hamiltonian trajectories", J. Diff. Eq., 43 (1981), pp 1-6

[3] A. AMBROSETTI, P. RABINOWITZ: "Dual variational methods in critical point theory and applications", J. Funct. Analysis, t. 14 (1973), pp. 349-381

[4] V. BENCI: "On the critical point theory for indefinite functionals in presence of symmetries", Trans. Am. Math. Soc. 274 (1982), pp. 533-572

[5] V. BENCI: "A geometrical index for a group $S^1$ and some applications to the study of periodic solutions of O.D.E." Comm. Pure Appl. Math., 34 (1981), pp. 393-432

[6] V. BENCI, P. RABINOWITZ: "Critical point theory for indefinite functionals", Inv. Math., 52 (1979), pp. 336-352

[7] H. BERESTYCKI, J.M. LASRY, G. MANCINI, B. RUF: "Existence of multiple periodic orbits on star-shaped Hamiltonian surfaces", Comm. Pure Appl. Math, 38 (1985), pp. 253-289

[8] F. CLARKE: "Periodic solutions of Hamiltonian inclusions", J. Diff. Eq., 40 (1981), pp 1-6

[9] F. CLARKE, I. EKELAND: "Hamiltonian trajectories having prescribed minimal period", Comm. Pure Appl. Math., 33 (1980), pp. 103-116

[10] I. EKELAND, H. HOFER: "Periodic solutions with prescribed period for convex autonomous Hamiltonian systems", Inv. Math. 81 (1985), pp. 155-188

[11] I. EKELAND, J.M. LASRY: "Problèmes variationnels non convexes en dualité", C.R. Acad. Sc Paris, t. 291, série A, (1980), pp. 493-496

[12] I. EKELAND, J.M. LASRY: "On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface", Ann. Math., 112 (1980), pp. 283-319

[13] I. EKELAND, L. LASSOUED: "Multiplicité des trajectoires fermées de systèmes Hamiltonien convexes", preprint 1986

[14] M. GIRARDI: "Multiple orbits for Hamiltonian systems on starshaped surfaces with symmetries", Ann. Inst. H. Poincaré, vol. I, n° 4 (1984) pp 285-294

[15] M. GIRARDI, M. MATZEU: "Periodic solutions of convex autonomous Hamiltonian systems with a quadratic growth and superquadratic at infinity", Ann. Mat. Pura e Appl. (IV), Vol. CXLVII (1987), pp. 21-72

[16] M. GIRARDI, M. MATZEU: "Some results on solutions of minimal period to Hamiltonian systems", in Nonlinear Oscillations for Conservative Systems, Proceedings, Venice 9-12/1/1985

[17] M. GIRARDI, M. MATZEU. "Periodic solutions of prescribed minimal period for Hamiltonian systems: an extension of a theorem by Ekeland and Hofer to the non convex case", preprint, November 1987.

[18] M. GIRARDI, M. MATZEU: "Some results on solutions of minimal period to superquadratic Hamiltonian systems", Nonlinear Analysis 7, n.5 (1983), pp 475-482

[19] M. GIRARDI, M. MATZEU: "Solutions of prescribed minimal period to convex and non-convex Hamiltonian systems", Boll. UMI, (6) 4-B (1985) pp 951-967

[20] M. GIRARDI, M. MATZEU: "Solutions of minimal period for a class of non-convex Hamiltonian systems and applications to the fixed energy problem", Nonlinear Analysis, 10, no 4 (1986), pp 371-382

[21] H. HOFER, E. ZEHNDER : "Periodic solutions on hypersurfaces and a result of C. Viterbo", Inv. Math., 90 (1987), pp. 1-9

[22] R. MICHALEK, G. TARANTELLO : "Subharmonic solutions with prescribed minimal period for nonautonomous Hamiltonian systems", to appear on Comm. Pure Appl. Math.

[23] J. MOSER : "Periodic orbits near an equilibrium point and a theorem by A. Weinstein", Comm. Pure Appl. Math., t. 29 (1976), pp. 724-747

[24] P. RABINOWITZ : "Periodic solutions of Hamiltonian systems", Comm. Pure Appl. Math., t. 31 (1978), pp. 157-184

[25] H. SEIFERT : "Periodische bewegungen mechanischer systeme", Math. Z. 51 (1948), pp. 197-216

[26] E.W.C. VAN GROESEN : "Existence of multiple normal mode trajectories of even classical Hamiltonian systems", J. Diff. Eq. 57 (1985), pp. 70-89

[27] C. VITERBO : "Proof of the Weinstein conjecture in $\mathbb{R}^{2N}$", preprint September 1986

[28] A. WEINSTEIN : "Normal modes for nonlinear Hamiltonian systems", Inv. Math., t. 20 (1973), pp. 47-57

[29] A. WEINSTEIN : "Periodic orbits for convex Hamiltonian systems", Ann. Math., 108 (1978), pp. 507-518