INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR 281/14

COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS
(11 January - 5 February 1988)

LECTURES ON THE EKELAND VARIATIONAL PRINCIPLE WITH APPLICATIONS AND DETOURS

Djairo G. De FIGUEIREDO
Departamento de Matematica
Instituto de Ciencias Exactas
Universidade de Brasilia
Brasilia
BRAZIL

LECTURES ON THE EKELAND VARIATIONAL PRINCIPLE WITH APPLICATIONS AND DETOURS

Djairo G. de Figueiredo

Preface

Since its appearance in 1972 the variational principle of Ekeland has found many applications in different fields in Analysis. The best references for those are by Ekeland himself: his survey article [23] and his book with J.-P. Aubin [2]. Not all material presented here appears in those places. Some are scattered around and there lies my motivation in writing these notes. Since they are intended to students I included a lot of related material. Those are the detours. A chapter on Nemytskii mappings may sound strange. However I believe it is useful, since their properties so often used are seldom proved. We always say to the students: go and look in Krasnoselskii or Vainberg! I think some of the proofs presented here are more straightforward. There are two chapters on applications to PDE. However I limited myself to semilinear elliptic. The central chapter is on Brézis proof of the minimax theorems of Ambrosetti and Rabinowitz. To be self contained I had to develop some convex analysis, which was later used to give a complete treatment of the duality mapping so popular in my childhood days! I wrote these notes as a tourist on vacations. Although the main road is smooth, the scenery is so beautiful that one cannot resist to go into the side roads. That is why I discussed some of the geometry of Banach spaces. I would like to thank my colleagues at UNICAMP for their hospitality and Elda Mortari for her patience and cheerful willingness in texing these notes.

Campinas, October 1987

Chapter 1

Minimization of Lower Semicontinuous Functionals.

Let X be a Hausdorff topological space. A functional $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous* if for every $a \in \mathbb{R}$ the set $\{x \in X : \Phi(x) > a\}$ is open. We use the terminology functional to designate a real valued function. A Hausdorff topological space X is *compact* if every covering of X by open sets contains a finite subcovering. The following basic theorem implies most of the results used in the minimization of functionals.

Theorem 1.1. *Let X be a compact topological space and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous functional. Then* (a) *Φ is bounded below, and* (b) *the infimum of Φ is achieved at a point $x_0 \in X$.*

Proof. The open sets $A_n = \{x \in X : \Phi(x) > -n\}$, for $n \in \mathbb{N}$, constitute an open covering of X. By compactness there exists a $n_0 \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{n_0} A_j = X.$$

So $\Phi(x) > -n_0$ for all $x \in X$.

(b) Now let $\ell = \operatorname{Inf}\Phi$, $\ell > -\infty$. Assume by contradiction that ℓ is not achieved. This means that

$$\bigcup_{n=1}^{\infty} \{x \in X : \Phi(x) > \ell + \frac{1}{n}\} = X.$$

By compactness again it follows that there exist a $n_1 \in \mathbb{N}$ such that

$$\bigcup_{n=1}^{n_1} \{x \in X : \Phi(x) > \ell - \frac{1}{n}\} = X.$$

But this implies that $\Phi(x) > \ell + \frac{1}{n_1}$ for all $x \in X$, which contradicts the fact that ℓ is the infimum of Φ. □

In many cases it is simpler to work with a notion of lower semicontinuity given in terms of sequences. A function $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *sequentially lower semicontinuous* if for every sequence (x_n) with $\lim x_n = x_0$, it follows that $\Phi(x_0) \le \liminf \Phi(x_n)$. The relationship between the two notions of lower semicontinuity is expounded in the following proposition

Proposition 1.2. (a) *Every lower semicontinuous function $\Phi : X \to \mathbb{R} \cup \{-\infty\}$ is sequentially lower semicontinuous.* (b) *If X satisfies the First Axiom of Countability, then every sequentially lower semicontinuous function is lower semicontinuous.*

Proof. (a) Let $x_n \to x_0$ in X. Suppose first that $\Phi(x_0) < \infty$. For each $\epsilon > 0$ consider the open set $A = \{x \in X : \Phi(x) > \Phi(x_0) - \epsilon\}$. Since $x_0 \in A$, it follows that there exists $n_0 = n_0(\epsilon)$ such that $x_n \in A$ for all $n \ge n_0$. For such n's, $\Phi(x_n) > \Phi(x_0) - \epsilon$, which implies that $\liminf \Phi(x_n) \ge \Phi(x_0) - \epsilon$. Since $\epsilon > 0$ is arbitrary it follows that $\liminf \Phi(x_n) \ge \Phi(x_0)$. If $\Phi(x_0) = +\infty$ take $A = \{x \in X : \Phi(x) > M\}$ for arbitrary $M > 0$ and proceed in similar way.

(b) Conversely we claim that for each real number a the set $F = \{x \in \Omega : \Phi(x) \le a\}$ is closed. Suppose by contradiction that this is not the case, that is, there exists $x_0 \in \overline{F} \setminus F$, and so $\Phi(x_0) > a$. On the other hand, let O_n be a countable basis of open neighborhoods of x_0. For each $n \in \mathbb{N}$ there exists $x_n \in F \cap O_n$. Thus $x_n \to x_0$. Using the fact that Φ is sequentially lower semicontinuous and $\Phi(x_n) \le a$ we obtain that $\Phi(x_0) \le a$, which is impossible. □

Corollary 1.3. *If X is a metric space, then the notions of lower semicontinuity and sequentially lower semicontinuity coincide.*

Semicontinuity at a Point. The notion of lower semicontinuity can be localized as follows. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a functional and $x_0 \in X$. We say that Φ is *lower semicontinuous at* x_0 if for all $a < \Phi(x_0)$ there exists an open neighborhood V of x_0 such that $a < \Phi(x)$ for all $x \in V$. It is easy

to see that a lower semicontinuous functional is lower semicontinuous at all points $x \in X$. And conversely a functional which is lower semicontinuous at all points is lower semicontinuous. The reader can provide similar definitions and statements for sequential lower semicontinuity.

Some Examples When $X = \mathbb{R}$. Let $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. It is clear that Φ is lower semicontinuous at all points of continuity. If x_0 is a point where there is a jump discontinuity and Φ is lower semicontinuous there, then $\Phi(x_0) = \min\{\Phi(x_0 - 0), \Phi(x_0 + 0)\}$. If $\lim \Phi(x) = +\infty$ as $x \to x_0$ then $\Phi(x_0) = +\infty$ if Φ is to be lower semicontinuous there. If Φ is lower semicontinuous the set $\{x \in \mathbb{R} : \Phi(x) = +\infty\}$ is not necessarily closed. Example: $\Phi(x) = 0$ if $0 \le x \le 1$ and $\Phi(x) = +\infty$ elsewhere.

Functionals Defined in Banach Spaces. In the case when X is a Banach space there are two topologies which are very useful. Namely the norm topology τ (also called the strong topology) which is a metric topology and the weak topology τ^w which is not metric in general. We recall that the weak topology is defined by giving a basis of open sets as follows. For each $\epsilon > 0$ and each finite set of bounded linear functionals $\ell_1, \ldots, \ell_n \in X^*$, X^* is the dual space of X, we define the (weak) open set $\{x \in X : |\ell_1(x)| < \epsilon, \ldots, |\ell_n(x)| < \epsilon\}$. It follows ~~easily that τ~~ is a finer topology than τ^w, i.e. given a weak open set there exists a strong open set contained in it. The converse is not true in general. [We remark that finite dimensionality of X implies that these two topologies are the same]. It follows then that a weakly lower semicontinuous functional $\Phi : X \to \mathbb{R} \cup \{+\infty\}$, X a Banach space, is (strongly) lower semicontinuous. A similar statement holds for the sequential lower semicontinuity, since every strongly convergent sequence is weakly convergent. In general, a (strongly) lower semicontinuous functional is not weakly lower semicontinuous. However the following result holds.

Theorem 1.4. *Let X be a Banach space, and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a convex function. Then the notions of (strong) lower semicontinuity and weak lower semicontinuity coincide.*

Proof. (i) Case of sequential lower semicontinuity. Suppose $x_n \rightharpoonup x_0$ (the half arrow $\rightharpoonup$ denotes weak convergence). We claim that the hypothesis of Φ being (strong) lower semicontinuous implies that

$$\Phi(x_0) \le \liminf \Phi(x_n).$$

Let $\ell = \liminf \Phi(x_n)$, and passing to a subsequence (call it x_n again) we may assume that $\ell = \lim \Phi(x_n)$. If $\ell = +\infty$ there is nothing to prove. If $-\infty < \ell < \infty$, we proceed as follows. Given $\epsilon > 0$ there is $n_0 = n_0(\epsilon)$ such that $\Phi(x_n) \leq \ell + \epsilon$ for all $n \geq n_0(\epsilon)$. Renaming the sequence we may assume that $\Phi(x_n) \leq \ell + \epsilon$ for all n. Since x is the weak limit of (x_n) it follows from Mazur's theorem [which is essentially the fact that the convex hull $co(x_n)$ of the sequence (x_n) has weak closure coinciding with its strong closure] that there exists a sequence

$$y_N = \sum_{j=1}^{k_N} a_j^N x_j, \qquad \sum_{j=1}^{k_N} a_j^N = 1, \qquad a_j^N \geq 0,$$

such that $y_N \to x_0$ as $N \to \infty$. By convexity

$$\Phi(y_N) \leq \sum_{j=1}^{k_N} a_j^N \Phi(x_j) \leq \ell + \epsilon$$

and by the (strong) lower semicontinuity $\Phi(x_0) \leq \ell + \epsilon$. Since $\epsilon > 0$ is arbitrary we get $\Phi(x_0) \leq \ell$. If $\ell = -\infty$, we proceed in a similar way, just replacing the statement $\Phi(x_n) \leq \ell + \epsilon$ by $\Phi(x_n) \leq -M$ for all $n \geq n(M)$, where $M > 0$ is arbitrary.

(ii) Case of lower semicontinuity (nonsequential). Given $a \in \mathbb{R}$ we claim that the set $\{x \in X : \Phi(x) \leq a\}$ is weakly closed. Since such a set is convex, the result follows from the fact that for a convex set being weakly closed is the same as strongly closed. □

Now we discuss the relationship between sequential weak lower semicontinuity and weak lower semicontinuity, in the case of functionals $\Phi : A \to \mathbb{R} \cup \{+\infty\}$ defined in a subset A of a Banach space X. As in the case of a general topological space, every weak lower semicontinuous functional is also sequentially weak lower semicontinuous. The converse has to do with the fact that the weak topology in A ought to satisfy the First Axiom of Countability. For that matter one restricts to the case when A is bounded. The reason is: infinite dimensional Banach spaces X (even separable Hilbert spaces) do not satisfy the First Axiom of Countability under the weak topology. The same statement is true for the weak topology induced in unbounded subsets of X. See the example below

Example (Von Neumann). Let X be the Hilbert space ℓ^2, and let $A \subset \ell^2$ be the set of points x_{mn}, $m, n = 1, 2, \ldots$ whose coordinates are

$$x_{mn}(i) = \begin{cases} 1, & \text{if } i = m \\ m, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}$$

Then 0 belongs to weak closure of A, but there is no sequence of points in A which converge weakly to 0. [Indeed, if there is a sequence $x_{m_j n_j} \to 0$, then $(y, x_{m_j n_j})_{\ell^2} \to 0$, for all $y \in \ell^2$. Take $y = (1, 1/2, 1/3, \ldots)$ and see that this is not possible. On the other hand given any basic (weak) open neighborhood of 0, $\{x \in \ell^2 : (y, x)_{\ell^2} < \epsilon\}$ for arbitrary $y \in \ell^2$ and $\epsilon > 0$, we see that x_{mn} belongs to this neighborhord if we take m such that $|y_m| < \epsilon/2$ and then n such that $|y_n| < \epsilon/2m$].

However, if the dual X^* of X is *separable*, then the induced topology in a *bounded* subset A of X by the weak topology of X is first countable. In particular this is the case if X is *reflexive* and *separable*, since this implies X^* separable. It is noticeable that in the case when X is *reflexive* (with no separability assumption made) the following result holds.

Theorem 1.5. (Browder [19]). *Let X be a reflexive Banach space, A a bounded subset of X, x_0 a point in the weak closure of A. Then there exists an infinite sequence (x_k) in A converging weakly to x_0 in X.*

Proof. It suffices to construct a closed separable subspace X_0 of X such that x_0 lies in the weak closure of C in X_0, where $C = A \cap X_0$. Since X_0 is reflexive and separable, it is first countable and then there exists a sequence (x_k) in C which converges to x_0 in the weak topology of X_0. So (x_k) lies in A and converges to x_0 in the weak topology of X. The construction of X_0 goes as follows. Let B be the unit closed ball in X^*. For each positive integer n, B^n is compact in the product of weak topologies. Now for each fixed integer $m > 0$, each $[\overline{\omega}_1, \ldots, \overline{\omega}_n] \in B^n$ has a (weak) neighborhood V in B^n such that

$$\bigcap_{[\omega_1,\ldots,\omega_n] \in V} \bigcap_{j=1}^{n} \{x \in A : |(\omega_j, x - x_0)| < \frac{1}{m}\} = \emptyset.$$

By compactness we construct a finite set $F_{nm} \subset A$ with the property that given any $[\omega_1, \ldots, \omega_n] \in B^n$ there is $x \in A$ such that $|(\omega_j, x - x_0)| < \frac{1}{m}$ for all $j = 1, \ldots, n$. Now let

$$A_0 = \bigcup_{n,m=1}^{\infty} F_{nm}.$$

Then A_0 is countable and x_0 is in weak closure of A_0. Let X_0 be the closed subspace generated by A_0. So X_0 is separable, and denoting by $C = X_0 \cap A$ it follows that x_0 is in the closure of C in the weak topology of X. Using the Hahn Banach theorem it follows that x_0 is the closure of C in the weak topology of X_0. □

Remark. The Erberlein–Smulian theorem states: "Let X be a Banach space and A a subset of X. Let $\overline{A}$ denote its weak closure. Then $\overline{A}$ is weakly compact if and only if A is weakly sequentially precompact, i.e., any sequence in A contains a subsequence which converges weakly". See Dunford–Schwartz [35; p. 430]. Compare this statement with Theorem 1.5 and appreciate the difference!

Corollary. *In any reflexive Banach space X a weakly lower semicontinuous functional $\Phi : A \to \mathbb{R}$, where A is a bounded subset of X, is sequentially weakly lower semicontinuous, and conversely.*

Chapter 2

Nemytskii Mappings.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geq 1$. A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if (a) for each fixed $s \in \mathbb{R}$ the function $x \mapsto f(x,s)$ is (Lebesgue) measurable in Ω, (b) for fixed $x \in \Omega (a.e.)$ the function $s \mapsto f(x,s)$ is continuous in $\mathbb{R}$. Let $\mathcal{M}$ be the set of all measurable functions $u : \Omega \to \mathbb{R}$.

Theorem 2.1. *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory then the function $x \mapsto f(x, u(x))$ is measurable for all $u \in \mathcal{M}$.*
Proof. Let $u_n(x)$ be a sequence of simple functions converging a.e. to $u(x)$. Each function $f(x, u_n(x))$ is measurable in view of (a) above. Now (b) implies that $f(x, u_n(x))$ converges a.e. to $f(x, u(x))$, which gives its measurability. □

Thus a Carathéodory function f defines a mapping $N_f : \mathcal{M} \to \mathcal{M}$, which is called a *Nemytskii mapping*. The mapping N_f has a certain type of continuity as expresed by the following result.

Theorem 2.2. *Assume that Ω has finite measure. Let (u_n) be a sequence in $\mathcal{M}$ which converges in measure to $u \in \mathcal{M}$. Then $N_f u_n$ converges in measure to $N_f u$.*
Proof. By replacing $f(x,s)$ by $g(x,s) = f(x, s + u(x)) - f(x, u(x))$ we may assume that $f(x,0) = 0$. And moreover our claim becomes to prove that if (u_n) converges in measure to 0 then $f(x, u_n(x))$ also converges in measure to 0. So we want to show that given $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that

$$|\{x \in \Omega : |f(x, u_n(x))| \geq \epsilon\}| < \epsilon \quad \forall n \geq n_0,$$

where $|A|$ denotes the Lebesgue measure of a set A. Let

$$\Omega_k = \{x \in \Omega : |s| < 1/k \Rightarrow |f(x,s)| < \epsilon\}.$$

Clearly $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k (a.e.)$. Thus $|\Omega_k| \to |\Omega|$. So there exists $\bar{k}$ such that $|\Omega| - |\Omega_{\bar{k}}| < \epsilon/2$. Now let

$$A_n = \{x \in \Omega : |u_n(x)| < 1/\bar{k}\}.$$

Since u_n converges in measure to 0, it follows that there exists $n_0 = n_0(\epsilon)$ such that for all $n \geq n_0$ one has $|\Omega| - |A_n| < \epsilon/2$. Now let

$$D_n = \{x \in \Omega : |f(x, u_n(x))| < \epsilon\}.$$

Clearly $A_n \cap \Omega_{\bar{k}} \subset D_n$. So

$$|\Omega| - |D_n| \leq (|\Omega| - |A_n|) + (|\Omega| - |\Omega_{\bar{k}}|) < \epsilon$$

and the claim is proved. □

Remark. The above proof is essentially the one in Ambrosetti-Prodi [2]. The proof in Vainberg [78] is due to Nemytskii and relies heavily in the following result (see references in Vainberg's book; see also Scorza-Dragoni [74] and J.-P. Gossez [47] for still another proof). "Let $f : \Omega \times I \to \mathbb{R}$ be a Carathéodory function, where I is some bounded closed interval in $\mathbb{R}$. Then given $\epsilon > 0$ there exists a closed set $F \subset \Omega$ with $|\Omega \backslash F| < \epsilon$ such that the restriction of f to $F \times I$ is continuous". This is a sort of uniform (with respect to $s \in I$) Lusin's Theorem.

Now we are interested in knowing when N_f maps an L^p space in some other L^p space.

Theorem 2.3. *Suppose that there is a constant $c > 0$, a function $b(x) \in L^q(\Omega)$, $1 \leq q \leq \infty$, and $r > 0$ such that*

$$(2.1) \qquad |f(x,s)| \leq c|s|^r + b(x), \qquad \forall x \in \Omega, \qquad \forall s \in \mathbb{R}.$$

Then (a) N_f maps L^{qr} into L^q, (b) N_f is continuous and bounded (that is, it maps bounded sets into bounded sets).

Proof. It follows from (2.1) using Minkowski inequality

$$(2.2) \qquad ||N_f u||_{L^q} \leq c|||u|^r||_{L^q} + ||b||_{L^q} = c||u||^r_{L^{qr}} + ||b||_{L^q}$$

which gives (a) and the fact that N_f is bounded. Now suppose that $u_n \to u$ in L^{qr}, and we claim $N_f u_n \to N_f u$ in L^q. Given any subsequence of (u_n) there is a further subsequence (call it again u_n) such that $|u_n(x)| \leq h(x)$ for some $h \in L^{qr}(\Omega)$. It follows from (2.1) that

$$|f(x, u_n(x))| \leq c|h(x)|^r + b(x) \in L^q(\Omega).$$

Since $f(x, u_n(x))$ converges a.e. to $f(x, u(x))$, the result follows from the Lebesgue Dominated Convergence Theorem and a standard result on metric spaces. □

It is remarkable that the sufficient condition (2.1) is indeed necessary for a Carathéodory function f defining a Nemytskii map between L^p spaces. Indeed

Theorem 2.4. *Suppose N_f maps $L^p(\Omega)$ into $L^q(\Omega)$ for $1 \leq p < \infty$, $1 \leq q < \infty$. Then there is a constant $c > 0$ and $b(x) \in L^q(\Omega)$ such that*

$$(2.3) \qquad |f(x,s)| \leq c|s|^{p/q} + b(x)$$

Remark. We shall prove the above theorem for the case when Ω is bounded, although the result is true for unbounded domains. It is also true that if N_f maps $L^p(\Omega)$, $1 \leq p < \infty$ into $L^\infty(\Omega)$ then there exists a function $b(x) \in L^\infty(\Omega)$ such that $|f(x,s)| \leq b(x)$. See Vainberg [78].

Before proving Theorem 2.4 we prove the following result.

Theorem 2.5. *Let Ω be a bounded domain. Suppose N_f maps $L^p(\Omega)$ into $L^q(\Omega)$ for $1 \leq p < \infty$, $1 \leq q < \infty$. Then N_f is continuous and bounded.*

Proof. (a) Continuity of N_f. By proceeding as in the proof of Theorem 2.2 we may suppose that $f(x, 0) = 0$, as well as to reduce to the question of continuity at 0. Suppose by contradiction that $u_n \to 0$ in L^p and $N_f u \not\to 0$ in L^q. So by passing to subsequences if necessary we may assume that there is a positive constant a such that

$$(2.4) \qquad \sum_{n=1}^{\infty} ||u_n||^p_{L^p} < \infty \quad \text{and} \quad \int_\Omega |f(x, u_n(x))|^q \geq a, \quad \forall n.$$

Let us denote by

$$B_n = \{x \in \Omega : |f(x, u_n(x))| > (\frac{a}{3|\Omega|})^{1/q}\}$$

In view of Theorem 2.2 it follows that $B_n \to 0$. Now we construct a decreasing sequence of positive numbers ϵ_j, and select a subsequence (u_{n_j}) of (u_n) as follows.

1st step: $\epsilon_1 = |\Omega|$ $u_{n_1} = u_1$.

2nd step: choose $\epsilon_2 < \epsilon_1/2$ and such that

$$\int_D |f(x, u_{n_1}(x))|^q < \frac{a}{3} \quad \forall D \subset \Omega, \quad |D| \leq 2\epsilon_2,$$

then choose n_2 such $|B_{n_2}| < \epsilon_2$.

3rd step: chosse $\epsilon_3 < \epsilon_2/2$ and such that

$$\int_D |f(x, u_{n_2}(x))|^q < \frac{a}{3} \quad \forall D \subset \Omega, \quad |D| \leq 2\epsilon_3.$$

then choose n_3 such that $|B_{n_3}| < \epsilon_3$.

And so on. Let $D_{n_j} = B_{n_j} \setminus \cup_{i=j+1}^{\infty} B_{n_i}$. Observe that the $D_j's$ are pairwise disjoint. Define

$$u(x) = \begin{cases} u_{n_j}(x) & \text{if } x \in D_{n_j}, \quad j = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

The function u is in L^p in view of (2.4). So by the hypothesis of the theorem $f(x, u(x))$ should be in $L^q(\Omega)$. We now show that this is not the case, so arriving to contradiction. Let

$$K_j \equiv \int_{D_{n_j}} |f(x, u(x))|^q = \int_{D_{n_j}} |f(x, u_{n_j}(x))|^q = \int_{B_{n_j}} - \int_{B_{n_j} \setminus D_{n_j}} \equiv I_j - J_j.$$

Next we estimate the integrals in the right side as follows:

$$\begin{aligned} I_j &= \int_{B_{n_j}} |f(x, u_{n_j}(x))|^q = \int_\Omega |f(x, u_{n_j}(x))|^q - \int_{\Omega \setminus B_{n_j}} |f(x, u_{n_j}(x))|^q \\ &\geq a - \frac{a}{3} = \frac{2a}{3} \end{aligned}$$

and to estimate J_j we observe that $B_{n_j} \setminus D_{n_j} \subset \cup_{i=j+1}^{\infty} B_{n_i}$. We see that $|B_{n_j} \setminus D_{n_j}| \leq \sum_{i=j+1}^{\infty} \epsilon_i \leq 2\epsilon_{j+1}$. Consequently $J_j < a/3$. Thus $K_j \geq a/3$. And so

$$\int_\Omega |f(x, u(x))|^q = \sum_{j=1}^{\infty} K_j = \infty.$$

(b) Now we prove that N_f is bounded. As in part (a) we assume that $f(x, 0) = 0$. By the continuity of N_f at 0 we see that there exists $r > 0$ such that for all $u \in L^p$ with $\|u\|_{L^p} \leq r$ one has $\|N_f u\|_{L^q} \leq 1$. Now given any u in L^p let n (integer) be such that $nr^p \leq \|u\|_{L^p}^p \leq (n+1)r^p$. Then Ω can be decomposed into $n + 1$ pairwise disjoint sets Ω_j such that $\int_{\Omega_j} |u|^p \leq r^p$. So

$$\int_\Omega |f(x, u(x))|^q = \sum_{j=1}^{n+1} \int_{\Omega_j} |f(x, u(x))|^q \leq n + 1 \leq \left(\frac{\|u\|_{L^p}}{r}\right)^p + 1 \quad \square$$

Proof of Theorem 2.4. Using the fact that N_f is bounded we get a constant $c > 0$ such that

$$\int_\Omega |f(x, u(x))|^q dx \leq c^q \quad \text{if} \quad \int_\Omega |u(x)|^p \leq 1. \tag{2.5}$$

Now define the function $H : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$H(x, s) = \max\{|f(x, s)| - c|s|^{p/q}; 0\}.$$

Using the inequality $\alpha^q + (1 - \alpha)^q \leq 1$ for $0 \leq \alpha \leq 1$ we get

$$H(x, s)^q \leq |f(x, s)|^q - c^q |s|^p \quad \text{for} \quad H(x, s) > 0. \tag{2.6}$$

Let $u \in L^p$ and $D = \{x \in \Omega : H(x, u(x)) > 0\}$. There exist $n \geq 0$ integer and $0 \leq \epsilon < 1$ such that

$$\int_D |u(x)|^p dx = n + \epsilon.$$

So there are $n + 1$ disjoint sets D_i such that

$$D = \bigcup_{i=1}^{n+1} D_i \quad \text{and} \quad \int_{D_i} |u(x)|^p dx \leq 1.$$

From (2.5) we get

$$\int_D |f(x, u(x))|^q dx = \sum_{i=1}^{n+1} \int_{D_i} |f(x, u(x))|^q dx \leq (n+1)c^q.$$

Then using this estimate in (2.6) we have

$$\int_\Omega H(x, u(x))^q \leq (n+1)c^q - (n+\epsilon)c^q \leq c^q \tag{2.7}$$

which then holds for all $u \in L^p$.

Now using the Lemma below we see that for each positive integer k there exists $u_k \in \mathcal{M}$ with $|u_k(x)| \leq k$ such that

$$b_k(x) \equiv \sup_{|s| \leq k} H(x, s) = H(x, u_k(x)).$$

It follows from (2.7) that $b_k(x) \in L^q(\Omega)$ and $\|b_k\|_{L^q} \leq c$. Now let us define the function $b(x)$ by

$$b(x) \equiv \sup_{-\infty < s < \infty} H(x, s) = \lim_{k \to \infty} b_k(x). \tag{2.8}$$

It follows from Fatou's lemma that $b(x) \in L^q$ and $\|b\|_{L^q} \leq c$. From (2.8) we finally obtain (2.3). □

Lemma. *Let $f : \Omega \times I \to \mathbb{R}$ be a Carathéodory function, where I is some fixed bounded closed interval. Let us define the function*

$$c(x) = \max_{s \in I} f(x, s).$$

Then $c \in \mathcal{M}$ and there exists $\bar{u} \in \mathcal{M}$ such that

$$c(x) = f(x, \bar{u}(x)). \tag{2.9}$$

Proof. (i) For each fixed s the function $x \mapsto f_s(x)$ is measurable. We claim that

$$c(x) = \sup\{f_s(x) : s \in I, \quad s - \text{rational}\}$$

showing then that c is measurable. To prove the claim let $x_0 \in \Omega(a.e.)$ and choose $s_0 \in I$ such that $c(x_0) = f(x_0, s_0)$. Since s_0 is a limit point of rational numbers and $f(x_0, s)$ is a continuous function the claim is proved.

(ii) For each $x \in \Omega(a.e.)$ let $F_x = \{s \in I : f(x, s) = c(x)\}$ which is a closed set. Let us define a function $\bar{u} : \Omega \to \mathbb{R}$ by $\bar{u}(x) = \min_s F_x$. Clearly the function $\bar{u}$ satisfies the relation in (2.9). It remains to show that $\bar{u} \in \mathcal{M}$. To do that it suffices to prove that the sets

$$B_\alpha = \{x \in \Omega : \bar{u}(x) > \alpha\} \quad \forall \alpha \in I$$

are measurable. [Recall that $\bar{u}(x) \in I$ for $x \in \Omega$]. Let β be the lower end of I. Now fixed $\alpha \in I$ we define the function $c_\alpha : \Omega \to \mathbb{R}$ by

$$c_\alpha(x) = \max_{\beta \leq s \leq \alpha} f(x, s)$$

which is measurable by part (i) proved above. The proof is completed by observing that

$$B_\alpha = \{x \in \Omega : c(x) > c_\alpha(x)\}. \qquad □$$

Remark. The Nemytskii mapping N_f defined from L^p into L^q with $1 \leq p < \infty$, $1 \leq q < \infty$ is not compact in general. In fact, the requirement that N_f is compact implies that there exists a $b(x) \in L^q(\Omega)$ such that $f(x, s) = b(x)$ for all $s \in \mathbb{R}$. See Krasnoselskii [53].

The Differentiability of Nemytskii Mappings. Suppose that a Carathéodory function $f(x, s)$ satisfies condition (2.3). Then it defines a mapping from L^p into L^q. It is natural to ask: if $f(x, s)$ has a partial derivative $f_s'(x, s)$ with respect to s, which is also a Carathéodory function, does $f_s'(x, s)$ define a Nemytskii map between some L^p spaces? In view of Theorem 2.4 we see that the answer to this question is no in general. The reason is that (2.3) poses no restriction on the growth of the derivative. Viewing the differentiability of a Nemytskii mapping N_f associated with a Carathéodory function $f(x, s)$ we *start assuming* that $f_s'(x, s)$ is Carathéodory and

$$|f_s'(x, s)| \leq c|s|^m + b(x), \qquad \forall s \in \mathbb{R} \qquad \forall x \in \Omega. \tag{2.10}$$

where $b(x) \in L^n(\Omega)$, $1 \leq n \leq \infty$, $m > 0$. Integrating (2.10) with respect to s we obtain

$$|f(x, s)| \leq \frac{c}{m+1} |s|^{m+1} + b(x)|s| + a(x), \tag{2.11}$$

where $a(x)$ is an arbitrary function. Shortly we impose a condition on $a(x)$ so as to having a Nemytskii map defined between adequate L^p spaces. Using Young's inequality in (2.11) we have

$$|f(x, s)| \leq \frac{c+1}{m+1} |s|^{m+1} + \frac{m}{m+1} b(x)^{(m+1)/m} + a(x).$$

Observe that the function $b(x)^{(m+1)/m}$ is in $L^q(\Omega)$, where $q = m\,n/(m+1)$. So if we pick $a \in L^q$ it follows from Theorem 2.3 that (assuming (2.10)):

$$N_f : L^p \to L^q \qquad p = m\,n \quad \text{and} \quad q = m\,n/(m+1) \tag{2.12}$$

$$N_{f_s'} : L^p \to L^n. \tag{2.13}$$

Now we are ready to study the differentiability of the mapping N_f.

Theorem 2.6. *Assume* (2.10) *and the notation in* (2.12) *and* (2.13). *Then* N_f *is continuously Fréchet differentiable with* $N_f' : L^p \to \mathcal{L}(L^p, L^q)$ *defined by*

$$(2.14)\qquad N_f'(u)[v] = N_{f_s'}(u)\, v \ \ (= f_s'(x, u(x))v(x)), \qquad \forall u,\ v \in L^p.$$

Proof. We first observe that under our hypotheses the function $x \to f_s'(x, u(x))v(x)$ is in $L^q(\Omega)$. Indeed by Hölder's inequality

$$\int_\Omega |f_s'(x, u(x))v(x)|^q \le \Big(\int_\Omega |f_s'(x, v(x)|^{pq/(p-q)}\Big)^{(p-q)/p}\Big(\int_\Omega |v(x)|^p\Big)^{q/p}.$$

Observe that $pq/(p-q) = n$ and use (2.13) above. Now we claim that for fixed $u \in L^p$

$$\omega(v) \equiv N_f(u + v) - N_f(u) - f_s'(x, u)v$$

is $o(v)$ for $v \in L^p$, that is $\|\omega(v)\|_{L^q}/\|v\|_{L^p} \to 0$ as $\|v\|_{L^p} \to 0$. Since

$$\begin{aligned} f(u(x) + v(x)) - f(u(x)) &= \int_0^1 \frac{d}{dt} f(x, u(x) + tv(x))dt \\ &= \int_0^1 f_s'(x, u(x) + tv(x))v(x)dt \end{aligned}$$

we have

$$\int_\Omega |\omega(v)|^q dx = \int_\Omega \Big| \int_0^1 [f_s'(x, u(x) + tv(x)) - f_s'(x, u(x))]v(x)dt\Big|^q dx.$$

Using Hölder's inequality and Fubini we obtain

$$\begin{aligned} &\int_\Omega |\omega(v)|^q dx \le \\ &\le \Big(\int_0^1 \int_\Omega |f_s'(x, u(x) + tv(x)) - f_s'(x, u(x))|^n dx\, dt\Big)^{q/n} \|v\|_{L^p}^q. \end{aligned}$$

Using (2.13) and the fact that $N_{f_s'}$ is a continuous operator we have the claim proved. The continuity of N_f' follows readily (2.14) and (2.13). □

Remark. We observe that in the previous theorem $p > q$, since we have assumed $m > 0$. What happens if $m = 0$, that is

$$|f_s'(x, s)| \le b(x)$$

where $b(x) \in L^n(\Omega)$? First of all we observe that

$$N_{f_s'} : L^p \to L^n \qquad \forall p \ge 1$$

and proceeding as above (supposing $1 \le n < \infty$)

$$N_f : L^p \to L^q \qquad \forall p \ge 1 \qquad \text{and} \qquad q = np/(n - p)$$

and we are precisely in the same situation as in (2.12), (2.13). Now assume $n = +\infty$, i.e., there exists $M > 0$

$$(2.15)\qquad |f_s'(x, s)| \le M \qquad \forall x \in \Omega, \qquad \forall s \in \mathbb{R}.$$

Integrating we obtain

$$(2.16)\qquad |f(x, s)| \le M|s| + b(x)$$

It follows under (2.15) and (2.16) that

$$N_{f_s'} : L^p \to L^\infty \qquad \forall 1 \le p \le \infty$$

$$N_f : L^p \to L^p \qquad (\text{taking} \quad b \in L^p).$$

It is interesting to observe that such an N_f cannot be Fréchet differentiable in general. Indeed:

Theorem 2.7. *Assume* (2.15). *If* $N_f : L^p \to L^p$ *is Fréchet differentiable then there exist functions* ~~$a(x) \in L^\infty$ *and*~~ $b(x) \in L^p$ *such that* $f(x, s) = a(x)s + b(x)$.

Proof. (a) Let us prove that the Gâteaux derivative of N_f at u in the direction v is given by

$$\frac{d}{dv} N_f(u) = f_s'(x, u(x))v(x).$$

First we observe that $f_s'(x, u(x))v(x) \in L^q$. So we have to prove that

$$\omega_t(x) \equiv t^{-1}[f(x, u(x) + tv(x)) - f(x, u(x))] - f_s'(x, u(x))v(x)$$

goes to 0 in L^p as $t \to 0$. As in the proof of Theorem 2.6 we write

$$\omega_t(x) = \int_0^1 [f_s'(x, u(x) + t\tau v(x)) - f_s'(x, u(x))]v(x)d\tau.$$

So

$$\int_\Omega |\omega_t(x)|^p dx \le \int_0^1 \int_\Omega |f_s'(x, u(x) + t\tau v(x)) - f_s'(x, u(x))|^p |v(x)|^p dx d\tau.$$

Now for each $\tau \in [0,1]$ and each $x \in \Omega(a.e.)$ the integrand of the double integral goes to zero. On the other hand this integrand is bounded by $(2M)^p|v(x)|^p$. So the result follows by the Lebesgue Dominated Convergence Theorem.

(b) Now suppose N_f is Fréchet differentiable. Then its Fréchet derivative is equal to the Gâteaux derivative, and assuming that $f(x,0) = 0$ we have that

(2.17) $\quad ||u||_{L^p}^{-1}||f(x,u) - f_s'(x,0)u||_{L^{p'}} \to 0 \quad$ as $\quad ||u||_{L^p} \to 0.$

Now for each fixed $\ell \in \mathbb{R}$ and $x_0 \in \Omega$ consider a sequence $u_\ell(x) = \ell\chi_{B_\delta(x_0)}$, i.e., a multiple of the characteristic function of the ball $B_\delta(x_0)$. For such functions the expression in (2.17) raised to the power p can be written as

$$\frac{1}{\ell^p vol B_\delta(x_0)}\int_{B_\delta(x_0)}|f(x,\ell) - f_s'(x,0)\ell|^p dx.$$

So taking the limit as $\delta \to 0$ we obtain

$$\frac{1}{\ell^p}|f(x_0,\ell) - f_s'(x_0,0)\ell| = 0, \qquad x_0 \in \Omega(a.e.)$$

which shows that $f(x_0,\ell) = f_s'(x_0,0)\ell$. Since the previous arguments can be done for all $x_0 \in \Omega(a.e.)$ and all $\ell \in \mathbb{R}$, we obtain that $f(x,s) = a(x)s$ where $a(x) = f_s'(x,0)$ is an L^∞ function. □.

The Potential of a Nemytskii Mapping. Let $f(x,s)$ be a Carathéodory function for which there are constants $0 < m,\ 1 \leq p \leq \infty$ and a function $b(x) \in L^{p/m}(\Omega)$ such that

$$|f(x,s)| \leq c|s|^m + b(x).$$

Denoting by $F(x,s) = \int_0^s f(x,\tau)d\tau$ we obtain that

$$|F(x,s)| \leq c_1|s|^{m+1} + c(x)$$

where $c(x) \in L^{p/(m+1)}(\Omega)$. (See the paragraphs before Theorem 2.7). Then $N_f : L^p \to L^{p/m}$ and $N_F : L^p \to L^{p/(m+1)}$.

In particular, if $p = m+1$, $(\Rightarrow p > 1)$ the inequalities above become

(2.18) $$|f(x,s)| \leq c|s|^{p-1} + b(x), \qquad b(x) \in L^{p'},\ \frac{1}{p} + \frac{1}{p'} = 1$$

$$|F(x,s)| \leq c_1|s|^p + c(x), \qquad c(x) \in L^1$$

and we have that $N_f : L^p \to L^{p'}$ and $N_F : L^p \to L^1$.

Theorem 2.8. *Assume (2.18) Then*

$$\Psi(u) = \int_\Omega F(x,u(x))dx$$

defines a continuous functional $\Psi : L^p(\Omega) \to \mathbb{R}$, which is continuously Fréchet differentiable.

Proof. The continuity of N_F implies that Ψ is continuous. We claim that $\Psi' = N_f$. So all we have to do is proving that

$$\omega(v) \equiv \int_\Omega F(x,u+v) - \int_\Omega F(x,u) - \int_\Omega f(x,u)v = o(v)$$

as $v \to 0$ in L^p. As in the calculations done in the proof of Theorem 2.6 we obtain

$$\omega(v) = \int_\Omega\int_0^1 [f(x,u+tv) - f(x,u)]v\,dt\,dx.$$

Using Fubini's theorem and Hölder's inequality

$$|\omega(v)| \leq \int_0^1 ||N_f(u+tv) - N_f(u)||_{L^{p'}}dt||v||_{L^p}$$

The integral in the above expression goes to zero as $||v||_{L^p} \to 0$ by the Lebesgue Dominated Convergence Theorem with an application of Theorem 2.3. So

$$||v||_{L^p}^{-1}\omega(v) \to 0 \qquad \text{as} \qquad ||v||_{L^p} \to 0. \qquad \square$$

Chapter 3

Semilinear Elliptic Equations I

We consider the Dirichlet problem

$$-\Delta u = f(x,u) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{3.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $\partial\Omega$ denotes its boundary. We assume all along that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. By a *classical solution* of (3.1) we mean a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies the equation at every point $x \in \Omega$ and which vanishes on $\partial\Omega$. By a *generalized solution* of (3.1) we mean a function $u \in H_0^1(\Omega)$ which satisfies (3.1) in the weak sense, i.e.

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f(x,u)v, \qquad \forall\, v \in C_c^\infty(\Omega). \tag{3.2}$$

We see that in order to have things well defined in (3.2), the function $f(x,s)$ has to obey some growth conditions on the real variable s. We will not say which they are, since a stronger assumption will be assumed shortly, when we look for generalized solution as critical points of a functional. Namely let us consider

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) \tag{3.3}$$

where $F(x,s) = \int_0^s f(x,\tau)d\tau$. In order to have $\Phi : H_0^1(\Omega) \to \mathbb{R}$ well defined we should require that $F(x,u) \in L^1(\Omega)$ for $u \in H_0^1(\Omega)$. In view of the Sobolev imbedding theorem $H_0^1 \hookrightarrow L^p$ (continuous imbedding) if $1 \leq p \leq 2N/(N-2)$ if $N \geq 3$ and $1 \leq p < \infty$ if $N = 2$. So using Theorem 2.8 we should require that f satisfies the following condition

$$|f(x,s)| \leq c|s|^{p-1} + b(x) \tag{3.4}$$

where p satisfies the conditions of the Sobolev imbedding and

$$b(x) \in L^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Using Theorem 2.8 we conclude that:

if f satisfies (3.4) the functional Φ defined in (3.3) is continuous Fréchet differentiable, i.e., C^1, *and*

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v - \int_\Omega f(x,u)v, \qquad \forall\, v \in H_0^1 \tag{3.5}$$

where $\langle\ ,\ \rangle$ denotes the inner product in $H_0^1(\Omega)$.

It follows readily that the critical points of Φ are precisely the generalized solutions of (3.1). So the search for solutions of (3.1) is transformed in the investigation of critical points of Φ. ~~In this chapter~~ we study conditions under which Φ has a minimum.

Φ is *bounded below* if the following condition is satisfied:

$$F(x,s) \leq \frac{1}{2}\mu s^2 + a(x) \tag{3.6}$$

where $a(x) \in L^1(\Omega)$ and μ is a constant $0 < \mu \leq \lambda_1$. [Here λ_1 denotes the first eigenvalue of the Laplacian subject to Dirichlet boundary conditions]. Indeed we can estimate

$$\Phi(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\mu\int_\Omega u^2 - \int_\Omega a(x) \geq -\int_\Omega a(x)$$

where we have used the variational characterization of the first eigenvalue.

Φ is *weakly lower semicontinuous in H_0^1* if condition (3.4) is satisfied with $1 \leq p < 2N/(N-2)$ if $N \geq 3$ and $1 \leq p < \infty$ if $N = 2$. Indeed

$$\Phi(u) = \frac{1}{2}\,||u||^2_{H^1} - \Psi(u)$$

where $\Psi(u) = \int_\Omega F(x,u)$ has been studied in Section 1.2, and the claim follows using the fact that the norm is weakly lower semicontinuous and under the hypothesis Ψ is weakly continuous from H_0^1 into $\mathbb{R}$. Let us prove this last statement. Let $u_n \rightharpoonup u$ in H_0^1. Going to a subsequence if necessary we have $u_n \to u$ in L^p with p restricted as above to insure the compact imbedding $H_0^1 \hookrightarrow L^p$. Now use the continuity of the functional Ψ to conclude.

Now we can state the following result

Theorem 3.1. *Assume (3.6) and (3.4) with $1 \le p < 2N/(N-2)$ if $N \ge 3$ and $1 \le p < \infty$ if $N = 2$. Then for each $r > 0$ there exist $\lambda_r \le 0$ and $u_r \in H_0^1$ with $\|u_r\|_{H^1} \le r$ such that $\Phi'(u_r) = \lambda_r u_r$, and Φ restricted to the ball of radius r around 0 assumes its infimum at u_r.*

Proof. The ball $\overline{B}_r(0) = \{u \in H_0^1 : \|u\|_{H^1} \le r\}$ is weakly compact. So applying Theorem 1.1 to the functional Φ restricted to $\overline{B}_r(0)$ we obtain a point $u_r \in \overline{B}_r(0)$ such that

$$\Phi(u_r) = \operatorname{Inf}\{\Phi(u) : u \in \overline{B}_r(0)\}.$$

Now let $v \in \overline{B}_r(0)$ be arbitrary then

$$\Phi(u_r) \le \Phi(tv + (1-t)u_r) = \Phi(u_r) + t\langle \Phi'(u_r), v - u_r\rangle + o(t)$$

which implies

$$(3.7) \qquad \langle \Phi'(u_r), v - u_r\rangle \ge 0.$$

If u_r is an interior point of $\overline{B}_r(0)$ then $v - u_r$ covers a ball about the origin. Consequently $\Phi'(u_r) = 0$. If $u_r \in \partial\overline{B}_r(0)$ we proceed as follows. In the case when $\Phi'(u_r) = 0$ we have the thesis with $\lambda_r = 0$. Otherwise when $\Phi'(u_r) \ne 0$ we assume by contradiction that $\Phi'(u_r)/\|\Phi'(u_r)\| \ne -u_r/\|u_r\|$. Then $v = -r\Phi'(u_r)/\|\Phi'(u_r)\|$ is in $\partial\overline{B}_r(0)$ and $v \ne u_r$. So $\langle v, u_r\rangle < r^2$. On the other hand with such a v in (3.7) we obtain $0 \le \langle -v, v - u_r\rangle \Rightarrow r^2 \le \langle v, u_r\rangle$, contradiction. □

Corollary 3.2. *In addition to the hypothesis of Theorem 3.1 assume that there exists $r > 0$ such that*

$$(3.8) \qquad \Phi(u) \ge a > 0 \quad \text{for} \quad u \in \partial\overline{B}_r(0)$$

where a is some given constant. Then Φ has a critical point.

Proof. Since $\Phi(0) = 0$, we conclude from (3.8) that the infimum of Φ in

$\overline{B}_r(0)$ is achieved at an interior point of that ball. □

Remarks (Sufficient conditions that insure (3.8)).

1) Assume $\mu < \lambda_1$ in condition (3.6). Then

$$(3.9)\ \Phi(u) \ge \frac{1}{2}\int |\nabla u|^2 - \frac{\mu}{2}\int u^2 - C|\Omega| \ge \frac{1}{2}\left(1 - \frac{\mu}{\lambda_1}\right)\int |\nabla u|^2 - C|\Omega|$$

where we have used the variational characterization of the first eigenvalue. It follows from (3.9) that $\Phi(u) \to +\infty$ as $\|u\| \to \infty$, that is, Φ is *coercive*. So (3.8) is satisfied.

2) In particular, if there exists $\overline{\mu} < \lambda_1$ such that

$$\limsup_{|s|\to\infty} \frac{f(x,s)}{s} \le \overline{\mu}$$

then one has (3.6) with a $\mu < \lambda_1$, and Φ is coercive as proved above.

3) (A result of Mawhin-Ward-Willem [60]). *Assume that*

$$(3.10) \qquad \limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2} \le \alpha(x) \le \lambda_1$$

where $\alpha(x) \in L^\infty(\Omega)$ and $\alpha(x) < \lambda_1$ on a set of positive measure. Then under hypotheses (3.4) and (3.10), the Dirichlet problem has a generalized solution $u \in H_0^1(\Omega)$. To prove this statement all it remains to do is to prove that condition (3.8) is satisfied. First we claim that there exists $\epsilon_0 > 0$ such that

$$(3.11) \qquad \Theta(u) \equiv \int_\Omega |\nabla u|^2 - \int_\Omega \alpha(x)u^2 \ge \epsilon_0, \qquad \forall \|u\|_{H_0^1} = 1.$$

Assume by contradiction that there exists a sequence (u_n) in $H_0^1(\Omega)$ with $\|u_n\|_{H^1} = 1$ and $\Theta(u_n) \to 0$. We may assume without loss of generality that $u_n \rightharpoonup u_0$ (weakly) in H_0^1 and $u_n \to u$ in L^2. As a consequence of the fact that $\alpha(x) \le \lambda_1$ in Ω, we have $\Theta(u_n) \ge 0$ and then

$$(3.12) \qquad 0 \le \int |\nabla u_0|^2 - \int \alpha(x)u_0^2 \le 0.$$

On the other hand, $\Theta(u_n) = 1 - \int \alpha(x)u_n^2$ gives $\int \alpha(x)u_0^2 = 1$. From (3.12) we get $\|u_0\|_{H^1} = 1$, which implies that $u_n \to u_0$ (strongly) in H_0^1. This implies that $u_0 \not\equiv 0$. Now observe that $\Theta : H_0^1 \to R$ is weakly lower semicontinuous, that $\Theta(u) \geq 0$ for all $u \in H_0^1$ and $\Theta(u_0) = 0$. So u_0 is a critical point of Θ, which implies that $u_0 \in H_0^1(\Omega)$ is a generalized solution of $-\Delta u_0 = \alpha(x)u_0$. Thus $u_0 \in W^{2,2}(\Omega)$ and it is a strong solution of an elliptic equation. By the Aleksandrov maximum principle (see for instance, Gilbarg-Trudinger [46, p. 246]) we see that $u_0 \neq 0$ a.e. in Ω. Using (3.12) again we have

$$\lambda_1 \int_\Omega u_0^2 \leq \int_\Omega |\nabla u_0|^2 \leq \int \alpha(x)u_0^2 < \lambda_1 \int u_0^2.$$

which is impossible. So (3.11) is proved.

Next it follows from (3.10) that given $\epsilon < \lambda_1\epsilon_0$ (the ϵ_0 of (3.11)) there exists a constant $c_\epsilon > 0$ such that

$$F(x,s) \leq \frac{\alpha(x)+\epsilon}{2} s^2 + c_\epsilon, \qquad \forall x \in \Omega, \qquad \forall s \in \mathbb{R}.$$

Then we estimate Φ as follows

$$\Phi(u) \geq \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2}\int \alpha(x)u^2 - \frac{\epsilon}{2}\int u^2 - c_\epsilon|\Omega|.$$

Using (3.11) we get

$$\Phi(u) \geq \frac{1}{2}\epsilon_0 \int |\nabla u|^2 - \frac{1}{2}\frac{\epsilon}{\lambda_1}\int |\nabla u|^2 - c_\epsilon|\Omega|$$

which implies that Φ is coercive, and in particular (3.8) is satisfied.

Remark. We observe that in all cases considered above we in fact proved that Φ were coercive. We remark that condition (3.8) could be attained without coerciveness. It would be interesting to find some other reasonable condition on F to insure (3.8). On this line see the work of de Figueiredo-Gossez [42].

Final Remark. (*Existence of a minimum without the growth condition* (3.4)). Let us look at the functional Φ assuming the following condition: for some constant $b > 0$ and $a(x) \in L^1(\Omega)$ one has

$$F(x,s) \leq b|s|^p + a(x) \tag{3.13}$$

where $1 \leq p < 2N/(N-2)$ if $N \geq 3$ and $1 \leq p < \infty$ if $N = 2$. For $u \in H_0^1$ we have

$$\int F(x,u(x))dx \leq b\int_\Omega |u(x)|^p dx + \int a(x)dx$$

where the integral on the left side could be $-\infty$. In view of the Sobolev imbedding it is $< +\infty$. So the functional Φ could assume the value $+\infty$. Let us now check its weakly lower semicontinuity at a point $u_0 \in H_0^1(\Omega)$ where $\Phi(u_0) < +\infty$. So $F(x,u_0(x)) \in L^1$. Now take a sequence $u_n \rightharpoonup u_0$ in H_0^1. Passing to subsequence if necessary we may suppose that $u_n \to u_0$ in L^p, $u_n(x) \to u_0(x)$ a.e. in Ω and $|u_n(x)| \leq h(x)$ for some $h \in L^p$.

It follows then from (3.13) that

$$F(x,u_n(x)) \leq b\,h(x)^p + a(x).$$

Since the right side of the above inequality is in L^1 we can apply Fatou's lemma and conclude that

$$\limsup \int_\Omega F(x,u_n(x))dx \leq \int F(x,u_0(x))dx$$

Consequently we have

$$\begin{aligned}\liminf \Phi(u_n) &\geq \liminf \frac{1}{2}\int |\nabla u_n|^2 - \limsup \int F(x,u_n)\\ &\geq \frac{1}{2}\int |\nabla u_0|^2 - \int F(x,u_0).\end{aligned}$$

So $\Phi : H_0^1(\Omega) \to \mathbb{R}\cup\{+\infty\}$ is defined and weakly lower semicontinuous. By Theorem 1.1 Φ has a minimum in any ball $B_r(0)$ contained in H_0^1. If F satisfies condition (3.10) (which by the way implies (3.13)) we see by Remark 3 above that Φ is coercive. Thus Φ has a global minimum in H_0^1. Without further conditions (namely (3.4)) one cannot prove that such a minimum is a critical point of Φ.

Chapter 4

Ekeland Variational Principle

Introduction. We saw in Chapter 1 that a functional bounded below assumes its infimum if it has some type of continuity in a topology that renders (local) compactness to the domain of said functional. However in many situations of interest in applications this is not the case. For example, functionals defined in (infinite dimensional) Hilbert spaces which are continuous in the norm topology but not in the weak topology. Problems with this set up can be handled efficiently by Ekeland Variational Principle. This principle discovered in 1972 has found a multitude of applications in different fields of Analysis. It has also served to provide simple and elegant proofs of known results. And as we see it is a tool that unifies many results where the underlining idea is some sort of approximation. Our motivation to write these notes is to make an attempt to exhibit all these features, which we find mathematically quite interesting.

Theorem 4.1. (Ekeland Principle - weak form). *Let (X, d) be a complete metric space. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and bounded below. Then given any $\epsilon > 0$ there exists $u_\epsilon \in X$ such that*

(4.1) $$\Phi(u_\epsilon) \leq \operatorname{Inf}_X \Phi + \epsilon,$$

and

(4.2) $$\Phi(u_\epsilon) < \Phi(u) + \epsilon\, d(u, u_\epsilon), \quad \forall u \in X \quad \text{with} \quad u \neq u_\epsilon.$$

For future applications one needs a stronger version of Theorem 4.1. Observe that (4.5) below gives information on the whereabouts of the point u_λ. As we shall see in Theorem 4.3 the point u_λ in Theorem 4.2 (or u_ϵ in Theorem 4.1) is a sort of "almost" critical point. Hence its importance.

Theorem 4.2. (Ekeland Principle - strong form). *Let X be a complete metric space and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function which is bounded below. Let $\epsilon > 0$ and $\bar{u} \in X$ be given such that*

(4.3) $$\Phi(\bar{u}) \leq \operatorname{Inf}_X \Phi + \frac{\epsilon}{2}.$$

Then given $\lambda > 0$ there exists $u_\lambda \in X$ such that

(4.4) $$\Phi(u_\lambda) \leq \Phi(\bar{u})$$

(4.5) $$d(u_\lambda, \bar{u}) \leq \lambda$$

(4.6) $$\Phi(u_\lambda) < \Phi(u) + \frac{\epsilon}{\lambda}\, d(u, u_\lambda) \quad \forall u \neq u_\lambda.$$

Proof. For notational simplification let us put $d_\lambda(x, y) = (1/\lambda) d(x, y)$. Let us define a partial order in X by

$$u \leq v \iff \Phi(u) \leq \Phi(v) - \epsilon\, d_\lambda(u, v).$$

It is straightforward that: (i) (reflexivity) $u \leq u$; (ii) (antisymmetry) $u \leq v$ and $v \leq u$ imply $u = v$; (iii) (transitivity) $u \leq v$ and $v \leq \omega$ imply $u \leq \omega$; all these three properties for all u, v, ω in X. Now we define a sequence (S_n) of subsets of X as follows. Start with $u_1 = \bar{u}$ and define

$$S_1 = \{u \in X : u \leq u_1\}; \quad u_2 \in S_1 \quad s.t. \quad \Phi(u_2) \leq \operatorname{Inf}_{S_1} \Phi + \frac{\epsilon}{2^2}$$

and inductively

$$S_n = \{u \in X : u \leq u_n\}; \quad u_{n+1} \in S_n \quad s.t. \quad \Phi(u_{n+1}) \leq \operatorname{Inf}_{S_n} \Phi + \frac{\epsilon}{2^{n+1}}.$$

Clearly $S_1 \supset S_2 \supset S_3 \supset \cdots$ Each S_n is closed: let $x_j \in S_n$ with $x_j \to x \in X$. We have $\Phi(x_j) \leq \Phi(u_n) - \epsilon\, d_\lambda(x_j, u_n)$. Taking limits using the lower semicontinuity of Φ and the continuity of d we conclude that $x \in S_n$. Now we prove that the diameters of these sets go to zero: $\operatorname{diam} S_n \to 0$. Indeed, take an arbitrary point $x \in S_n$. On one hand, $x \leq u_n$ implies

(4.7) $$\Phi(x) \leq \Phi(u_n) - \epsilon\, d_\lambda(x, u_n).$$

On the other hand, we observe that x belongs also to S_{n-1}. So it is one of the points which entered in the competition when we picked u_n. So

$$\Phi(u_n) \le \Phi(x) + \frac{\epsilon}{2^n}. \tag{4.8}$$

From (4.7) and (4.8) we get

$$d_\lambda(x, u_n) \le 2^{-n} \qquad \forall x \in S_n$$

which gives $\operatorname{diam} S_n \le 2^{-n+1}$. Now we claim that the unique point in the intersection of the S_n's satisfies conditions (4.4) - (4.5) - (4.6). Let then $\cap_{n=1}^{\infty} S_n = \{u_\lambda\}$. Since $u_\lambda \in S_1$, (4.4) is clear. Now let $u \ne u_\lambda$. We cannot have $u \le u_\lambda$, because otherwise u would belong to the intersection of the S_n's. So $u \not\le u_\lambda$, which means that

$$\Phi(u) > \Phi(u_\lambda) - \epsilon\, d_\lambda(u, u_\lambda)$$

thus proving (4.6). Finally to prove (4.5) we write

$$d_\lambda(\overline{u}, u_n) \le \sum_{j=1}^{n-1} d_\lambda(u_j, u_{j+1}) \le \sum_{j=1}^{n-1} 2^{-j}$$

and take limits as $n \to \infty$. □

Remark. The above results and further theorems in this chapter are due to Ekeland. See [37], [38], and his survey paper [39].

Connections With Fixed Point Theory. Now we show that Ekeland's Principle implies a Fixed Point Theorem due to Caristi [22]. See also [23]. As a matter of fact, the two results are equivalent in the sense that Ekeland's Principle can also be proved from Caristi's theorem.

Theorem 4.3. (Caristi Fixed Point Theorem). *Let X be a complete metric space, and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous functional which is bounded below. Let $T : X \to 2^X$ be a multivalued mapping such that*

$$\Phi(y) \le \Phi(x) - d(x,y), \quad \forall x \in X, \quad \forall y \in Tx. \tag{4.9}$$

Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Proof. Using Theorem 4.1 with $\epsilon = 1$ we find $x_0 \in X$ such that

$$\Phi(x_0) < \Phi(x) + d(x, x_0) \qquad \forall x \ne x_0. \tag{4.10}$$

Now we claim that $x_0 \in Tx_0$. Otherwise all $y \in Tx_0$ are such that $y \ne x_0$. So we have from (4.9) and (4.10) that

$$\Phi(y) \le \Phi(x_0) - d(x_0, y) \quad \text{and} \quad \Phi(x_0) < \Phi(y) + d(x_0, y)$$

which cannot hold simultaneously. □

Proof of Theorem 4.1 from Theorem 4.3. Let us use the notation $d_1 = \epsilon\, d$, which is an equivalent distance in X. Suppose by contradiction that there is no u_ϵ satisfying (4.2). So for each $x \in X$ the set $\{y \in X : \Phi(x) \ge \Phi(y) + d_1(x,y);\; y \ne x\}$ is not empty. Let us denote this set by Tx. In this way we have produced a multivalued mapping T in (X, d_1) which satisfies condition (4.9). By Theorem 4.3 it should exist $x_0 \in X$ such that $x_0 \in Tx_0$. But this is impossible: from the very definition of Tx, we have that $x \notin Tx$. □

Remark. If T is a contraction in a complete metric space, that is, if there exists a constant k, $0 \le k < 1$, such that

$$d(Tx, Ty) \le k d(x,y), \qquad \forall x, y \in X,$$

then T satisfies condition (4.9) with $\Phi(x) = \frac{1}{1-k}\, d(x, Tx)$. So that part of the Contraction Mapping Principle which says about the existence of a fixed point can be obtained from Theorem 4.3. Of course the Contraction Mapping Principle is much more than this. Its well known proof uses an iteration procedure (the method of successive approximations) which gives an effective computation of the fixed point, with an error estimate, etc ···

Application of Theorem 4.1 to Functionals Defined in Banach Spaces. Now we put more struture on the space X where the functionals are defined. In fact we suppose that X is a Banach space.. This will allow us to use a Differential Calculus, and then we could appreciate better the meaning of the relation (4.2). Loosely speaking (4.2) has to do with some Newton quocient being small.

Theorem 4.4. *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a lower semicontinuous functional wnich is bounded below. In addition, suppose that Φ*

is Gateaux differentiable at every point $x \in X$. Then for each $\epsilon > 0$ there exists $u_\epsilon \in X$ such that

(4.11) $$\Phi(u_\epsilon) \leq \mathrm{Inf}_X \Phi + \epsilon$$

(4.12) $$\|D\Phi(u_\epsilon)\|_{X^*} \leq \epsilon.$$

Proof. It follows from Theorem 4.1 that there exists $u_\epsilon \in X$ such that (4.11) holds and

(4.13) $$\Phi(u_\epsilon) \leq \Phi(u) + \epsilon\|u - u_\epsilon\| \quad \forall u \in X.$$

Let $v \in X$ and $t > 0$ be arbitrary. Putting $u = u_\epsilon + tv$ in (4.13) we obtain

$$t^{-1}[\Phi(u_\epsilon) - \Phi(u_\epsilon + tv)] \leq \epsilon\|v\|.$$

Passing to the limit as $t \to 0$ we get $-\langle D\Phi(u), v\rangle \leq \epsilon\|v\|$ for each given $v \in X$. Since this inequality is true for v and $-v$ we obtain $|\langle D\Phi(u), v\rangle| \leq \epsilon\|v\|$, for all $v \in X$. But then

$$\|D\Phi(u)\|_{X^*} = \sup_{v\in V\ v\neq 0} \frac{\langle D\Phi(u), v\rangle}{\|v\|} \leq \epsilon. \quad \square$$

Remark 1. The fact that Φ is Gateaux differentiable does not imply that Φ is lower semicontinuous. One has simple examples, even for $X = \mathbb{R}^2$.

Remark 2. In terms of functional equations Theorem 4.4 means the following. Suppose that $T : X \to X^*$ is an operator which is a *gradient*, i.e., there exists a functional $\Phi : X \to \mathbb{R}$ such that $T = D\Phi$. The functional Φ is called the *potential* of T. If Φ satisfies the conditions of Theorem 4.4, then that theorem says that the equation $Tx = x^*$ has a solution x for *some* x^* in a ball of radius ϵ around 0 in X^*. And this for all $\epsilon > 0$. As a matter of fact one could say more if additional conditions are set on Φ. Namely

Theorem 4.5. *In addition to the hypotheses of Theorem 4.4 assume that there are constants $k > 0$ and C such that*

$$\Phi(u) \geq k\|u\| - C.$$

Let B^* denote the unit ball about the origin in X^*. Then $D\Phi(X)$ is dense in kB^*.

Proof. We should prove that given $\epsilon > 0$ and $u^* \in kB^*$ there exists $u_\epsilon \in X$ such that $\|D\Phi(u_\epsilon) - u^*\|_{X^*} \leq \epsilon$. So consider the functional

$\Psi(u) = \Phi(u) - \langle u^*, u\rangle$. It is easy to see that Ψ is lower semicontinuous and Gateaux differentiable. Boundedness below follows from (4.14). So by Theorem 4.4 we obtain u_ϵ such that $\|D\Psi(u_\epsilon)\|_{X^*} \leq \epsilon$. Since $D\Psi(u) = D\Phi(u) - u^*$, the result follows. $\square$

Corollary 4.6. *In addition to the hypotheses of Theorem 4.4 assume that there exists a continuous function $\varphi : [0,\infty) \to \mathbb{R}$ such that $\varphi(t)/t \to +\infty$ as $t \to +\infty$ and $\Phi(u) \geq \varphi(\|u\|)$ for all $u \in X$. Then $D\Phi(X)$ is dense in X^*.*

Proof. Let $k > 0$. Choose $t_0 > 0$ such that $\varphi(t)/t \geq k$ for $t > t_0$. So $\Phi(u) \geq k\|u\|$ if $\|u\| > t_0$. If $\|u\| \leq t_0$, $\Phi(u) \geq C$ where $C = \min\{\varphi(t) : 0 \leq t \leq t_0\}$. Applying Theorem 4.5 we see that $D\Phi(X)$ is dense in kB^*. Since k is arbitrary the result follows. $\square$

For the next result one needs a very useful concept, a sort of compactness condition for a functional Φ. We say that a C^1 functional satisfies the *Palais-Smale condition* [or (PS) condition, for short] if every sequence (u_n) in X which satisfies

$$|\Phi(u_n)| \leq \text{const. and} \quad \Phi'(u_n) \to 0 \quad \text{in} \quad X^*$$

possesses a convergent (in the norm) subsequence.

Theorem 4.7. *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional which satisfies the (PS) condition. Suppose in addition that Φ is bounded below. Then the infimum of Φ is achieved at a point $u_0 \in X$ and u_0 is a critical point of Φ, i.e., $\Phi'(u_0) = 0$.*

Proof. Using Theorem 4.4 we see that for each positive integer n there is $u_n \in X$ such that

(4.15) $$\Phi(u_n) \leq \mathrm{Inf}_X \Phi + \frac{1}{n} \qquad \|\Phi'(u_n)\| \leq \frac{1}{n}.$$

Using (PS) we have a subsequence (u_{n_j}) and an element $u_0 \in X$ such that $u_{n_j} \to u_0$. Finally from the continuity of both Φ and Φ' we get (4.15).

(4.16) $$\Phi(u_0) = \mathrm{Inf}_X \Phi \qquad \Phi'(u_0) = 0 \qquad \square$$

Remarks. 1) As a matter of fact the result is true without the continuity of Φ'. The mere existence of the Fréchet differential at each point suffices. Indeed, we have only to show that the first statement in (4.16) implies the second. This is a standard fact in the Calculus of Variations. Here it goes its simple proof: take $v \in X$, $||v|| = 1$, arbitrary and $t > 0$. So

$$\Phi(u_0) \leq \Phi(u_0 + tv) = \Phi(u_0) + t\langle \Phi'(u_0), v\rangle + o(t)$$

from which follows that $||\Phi'(u_0)||_{X^*} = \sup_{||v||=1} \langle \Phi'(u_0), v\rangle \leq \frac{o(t)}{t}$ for all $t > 0$. Making $t \to 0$ we get the result.

2) The boundedness below of Φ it could be obtained by a condition like the one in Corollary 4.6. Observe that a condition like $\Phi(u) \to +\infty$ as $||u|| \to \infty$ (usually called coerciveness) [or even the stronger one $\Phi(u)/||u|| \to +\infty$ as $||u|| \to +\infty$] does not guarantee that Φ is bounded below. See Chapter 1.

3) Theorem 4.7 appears in Chang [24] with a different proof and restricted to Hilbert spaces. Possibly that proof could be extended to the case of general Banach using a the flow given by subgradient, like in [68], instead of the gradient flow.

Chapter 5

Variational Theorems of Min–Max Type

Introduction. In this chapter we use Ekeland Variational Principle to obtain a general variational principle of the min-max type. We follow closely Brezis [13], see also Aubin–Ekeland [6]. From this result we show how to derive the Mountain Pass Theorem of Ambrosetti and Rabinowitz [4], as well as the Saddle Point and the Generalized Mountain Pass Theorems of Rabinowitz, [66] and [67] respectively.

Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional. Let K be a compact metric space and $K_0 \subset K$ a closed subset. Let $f_0 : K_0 \to X$ be a given (fixed) continuous mapping. We introduce the family

$$\Gamma = \{f \in C(K, X) : \quad f = f_0 \quad \text{on} \quad K_0\} \tag{5.1}$$

where $C(K, X)$ denotes the set of all continuous mappings from K into X. Now we define

$$c = \operatorname{Inf}_{f\in\Gamma} \operatorname{Max}_{t\in K} \Phi(f(t)), \tag{5.2}$$

where we observe that without further hypotheses c could be $-\infty$.

Theorem 5.1. *Besides the foregoing notations assume that*

$$\operatorname{Max}_{t\in K} \Phi(f(t)) > \operatorname{Max}_{t\in K_0} \Phi(f(t)), \quad \forall f \in \Gamma. \tag{5.3}$$

Then given $\epsilon > 0$ there exists $u_\epsilon \in X$ such that

$$\begin{aligned} c \ &\leq \ \Phi(u_\epsilon) \leq c + \epsilon \\ &||\Phi'(u_\epsilon)||_{X^*} \leq \epsilon \end{aligned}$$

Remark. Observe that the functional Φ is not supposed to satisfy the (PS) condition, see Chapter 1. The theorem above says that under the hypotheses there exists a *Palais—Smale sequence*, that is, (u_n) in X such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$. Consequently, if one assumes in addition that Φ satisfies the (PS) condition, then there exists a critical point at level $c : \Phi'(u_0) = 0$ and $\Phi'(u_0) = c$.

The proof uses some facts from Convex Analysis which we now expound in a generality slight greater than actually needed here.

The Subdifferential of a Convex Function. Let X be a Banach space and $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ a convex lower semicontinuous functional, with $\Phi(x) \not\equiv +\infty$. Let us denote the domain of Φ by $\text{dom}\Phi = \{x \in X : \Phi(x) < \infty\}$. We define the *subdifferential* of Φ, $\partial\Phi : X \to 2^{X^*}$, by

(5.4) $$\partial\Phi(x) = \{\mu \in X^* : \Phi(y) \geqslant \Phi(x) + \langle \mu, y - x\rangle, \quad \forall y \in X\}.$$

We observe that $\partial\Phi(x)$ could be the empty set for some $x \in X$. Clearly this is the case if $x \notin \text{dom}\Phi$. However the following property has a straightforward proof.

(5.5) $$\partial\Phi(x) \quad \text{is a convex } w^*\text{–closed set.}$$

In general $\partial\Phi(x)$ is not bounded. [To get a good understanding with pictures (!) consider simple examples. (i) $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(x) = 0$ if $|x| \leq 1$ and $\Phi(x) = +\infty$ otherwise. (ii) $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(x) = 1/x$ if $0 < x \leq 1$ and $\Phi(x) = +\infty$ otherwise. (iii) $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by $\Phi(x) = |\tan x|$ if $|x| < \pi/2$ and $\Phi(x) = +\infty$ otherwise].
However the following result is true.

Proposition 5.2. *Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous functional, with $\Phi(x) \not\equiv +\infty$. Let $x_0 \in \text{dom}\Phi$ and suppose that Φ is continuous at x_0. Then $\partial\Phi(x_0)$ is bounded and non-empty.*
Proof. Given $\epsilon > 0$ there exists $\delta > 0$ such that

(5.6) $$|\Phi(x) - \Phi(x_0)| < \epsilon \quad \text{if} \quad |x - x_0| < \delta.$$

Let $v \in X$ with $\|v\| = 1$ be arbitrary and take a (fixed) t_0, with $0 < t_0 < \delta$. For each $\mu \in \partial\Phi(x)$ taking in (5.4) $y = x_0 + t_0 v$ we obtain

$$\Phi(x_0 + t_0 v) \geq \Phi(x_0) + \langle \mu, t_0 v\rangle,$$

where $\langle\ ,\ \rangle$ denotes the duality pairing between X^* and X. And using (5.6) we get $\langle \mu, v\rangle \leq \epsilon/t_0$, which implies $\|\mu\|_{X^*} \leq \epsilon/t_0$. It remains to prove that $\partial\Phi(x_0) \neq \emptyset$. This will be accomplished by using the Hahn-Banach theorem applied to sets in the cartesian product $X \times \mathbb{R}$. Let

$$A = \{(x, a) \in X \times \mathbb{R} : x \in B_\delta(x_0), \quad a > \Phi(x)\}$$

where $B_\delta(x_0)$ denotes the open ball of radius δ around x_0. It is easy to check that A is open and convex. Also, the point $(x_0, \Phi(x_0)) \notin A$. So there exists a non-zero functional $(\nu, r) \in X^* \times \mathbb{R}$ such that

(5.7) $$\langle \nu, x_0\rangle + r\Phi(x_0) \leq \langle \nu, x\rangle + r\,a, \quad \forall (x, a) \in A.$$

By taking $x = x_0$ in (5.7) we conclude conclude that $r > 0$. So calling $\mu = \nu/r$ we obtain

$$\langle \mu, x_0\rangle + \Phi(x_0) \leq \langle \mu, x\rangle + a \quad \forall (x, a) \in A.$$

By the continuity of Φ we can replace a in the above inequality by $\Phi(x)$, and so we get

(5.8) $$\Phi(x) \geq \Phi(x_0) + \langle \mu, x - x_0\rangle \quad \forall x \in B_\delta(x_0).$$

To extend inequality (5.8) to all $x \in X$ and so finishing the proof we proceed as follows. Given $y \notin B_\delta(x_0)$, there exist $x \in B_\delta(x_0)$ and $0 < t < 1$ such that $x = ty + (1 - t)x_0$. By convexity $\Phi(x) \leq t\Phi(y) + (1 - t)\Phi(x_0)$. This together with (5.8) completes the proof. □

Remark. If follows from Proposition 5.2 and (5.5) that $\partial\Phi(x)$ is a non-empty convex w^*–compact set at the points x of continuity of Φ where $\Phi(x) < +\infty$.

Corollary 5.3. *If $\Phi : X \to \mathbb{R}$ is convex and continuous with $\text{dom}\Phi = X$, then $\partial\Phi(x)$ is a non-empty convex w^*-compact subset of X^* for all $x \in X$.*

The One-Sided Directional Derivative. Let $\Phi : X/\mathbb{R}$ be convex continuous function. It follows from convexity that the function: $t \in (0, \infty) \mapsto$

$t^{-1}[\Phi(x+ty)-\Phi(x)]$ is increasing as t increases for every $x,\ y \in X$ fixed. Now let $\mu \in \partial\Phi(x)$. One has

$$\frac{\Phi(x+ty)-\Phi(x)}{t} \geq \langle \mu, y\rangle \quad \forall t > 0 \tag{5.9}$$

It follows then that the limit as $t \to 0$ of the left side of (5.9) exists and it is $\geq \mathrm{Max}\{\langle\mu, y\rangle : \mu \in \partial\Phi(x)\}$. Actually one has equality, as proved next.

Proposition 5.4. *Let $\Phi : X \to \mathbb{R}$ be convex and continuous. Then for each $x,\ y \in X$ one has*

$$\lim_{t\downarrow 0}\frac{\Phi(x+ty)-\Phi(x)}{t} = \mathrm{Max}_{\mu\in\partial\Phi(x)}\langle\mu, y\rangle. \tag{5.10}$$

Proof. Let x and y in X be fixed and let us denote the left side of (5.10) by $\Phi'_+(x;y)$. In view of the discussion preceeding the statement of the present proposition, it suffices to exhibit a $\mu \in \partial\Phi(x)$ such that $\Phi'_+(x;y) \leq \langle\mu, y\rangle$. To do that we consider the following two subsets of $X \times \mathbb{R}$:

$$A = \{(z,a) \in X \times \mathbb{R} : a > \Phi(z)\}$$

$$B = \{(x+ty, \Phi(x)+t\Phi'_+(x;y)) : t \geq 0\},$$

which are the interior of the epigraph of Φ and a half-line respectively. It is easy to see that they are convex and A is open.

So by the Hahn-Banach theorem they can be separated: there exists a non-zero functional $(\nu, r) \in X^* \times \mathbb{R}$ such that

$$\langle\nu, z\rangle + ra \geq \langle\nu, x+ty\rangle + r\{\Phi(x)+t\Phi'_+(x;y)\} \tag{5.11}$$

for all $(z,a) \in A$ and all $t \geq 0$. Making $z = x$ and $t = 0$ in (5.11) we conclude that $r > 0$. So calling $\mu = -\nu/r$ and replacing a by $\Phi(z)$ [here use the continuity of Φ] we obtain

$$-\langle\mu, z\rangle + \Phi(z) \geq -\langle\mu, x+ty\rangle + \Phi(x) + t\Phi'_+(x;y) \tag{5.12}$$

which holds for all $z \in X$ and all $t \geq 0$. Making $t = 0$ in (5.12) we conclude that $\mu \in \partial\Phi(x)$. Next taking $z = x$ we get $\Phi'_+(x;y) \leq \langle\mu, y\rangle$, completing the proof. □

The Subdifferential of a Special Functional. Let K be a compact metric space and $C(K, \mathbb{R})$ be the Banach space of all real valued continuous

functions $x : K \to \mathbb{R}$, endowed with the norm $\|x\| = \max\{|x(t)| : t \in K\}$. To simplify our notation let us denote $E = C(K, \mathbb{R})$. By the Riesz representation theorem, see Dunford-Schwartz, [35; p. 234] the dual E^* of E is isometric isomorphic to the Banach space $\mathcal{M}(K, \mathbb{R})$ of all regular countably additive real-valued set functions μ (for short: Radon measures) defined in the σ-field of all Borel sets in K, endowed with the norm given by the total variation:

$$\|\mu\| = \sup\{\sum_{i=1}^{k}|\mu(E_i)| : \bigcup_{i=1}^{k} E_i \subset E,\ E_i \cap E_j = \emptyset; \quad \forall k = 1,2,\ldots\}$$

Next we recall some definitions. We say that a *Radon measure* μ is *positive*, and denote $\mu \geq 0$ if $\langle\mu, x\rangle \geq 0$ for all $x \in E$ such that $x(t) \geq 0$ for all $t \in K$. We say that a *Radon measure* μ has *mass one* if $\langle\mu, \mathbb{1}\rangle = 1$, where $\mathbb{1} \in E$ is the function defined by $\mathbb{1}(t) = 1$ for all $t \in K$. We say that a *Radon measure* μ *vanishes* in an open set $U \subset K$ if $\langle\mu, x\rangle = 0$ for all $x \in E$ such that the support of x is a compact set K_0 contained in U. Using partition of unit, one can prove that if μ vanishes in a collection of open sets U_α, then μ also vanishes in the union $\cup U_\alpha$. So there exists a largest open set $\tilde{U}$ where μ vanishes. The *support of the measure* μ, denoted by suppμ, is defined by supp$\mu = K\backslash\tilde{U}$. For these notions in the more general set-up of distributions, see Schwartz [72]. We shall need the following simple result.

Lemma 5.5. *Let $x \in E$ be a function such that $x(t) = 0$ for all $t \in$ suppμ. Then $\langle\mu, x\rangle = 0$.*

Proof. For each subset $A \subset K$ let us denote by $A_\epsilon = \{t \in K : \mathrm{dist}(t, A) < \epsilon\}$, where $\epsilon > 0$. By Urysohn's Theorem there exists for each $n = 1, 2, \ldots,$ a function $\varphi_n \in E$ such that $\varphi_n(t) = 0$ for $t \in (\mathrm{supp}\mu)_{1/n}$ and $\varphi_n(t) = 1$ for $t \notin (\mathrm{supp}\mu)_{2/n}$. Then the sequence $\varphi_n x$ converges to x in E and $\langle\mu, \varphi_n x\rangle \to \langle\mu, x\rangle$. Since the support of each $\varphi_n x$ is a compact set contained in $\tilde{U} = K\backslash$suppμ, we have $\langle\mu, \varphi_n x\rangle = 0$, and then the result follows. □

Proposition 5.6. *Using the above notation consider the functional $\Theta : E \to \mathbb{R}$ defined by*

$$\Theta(x) = \mathrm{Max}\{x(t) : t \in K\}.$$

Then Θ is continuous and convex. Moreover, for each $x \in E$,

$$\mu \in \partial\Theta(x) \Leftrightarrow \mu \geq 0,\ \langle\mu, \mathbb{1}\rangle = 1,\ \mathrm{supp}\mu \subset \{t \in K : x(t) = \Theta(x)\}. \tag{5.13}$$

Proof. The convexity of Θ is straightforward. To prove the continuity, let $x, y \in E$. Then

$$\Theta(x) - \Theta(y) = x(\bar{t}) - \mathrm{Max}_K y \leq x(\bar{t}) - y(\bar{t}),$$

where $\bar{t} \in K$ is a point where the maximum of x is achieved. From the above inequality one obtains

$$|\Theta(x) - \Theta(y)| \leq ||x - y||$$

ii) Let us prove (5.13) $\Leftarrow$. We claim that

$$\Theta(y) \geq \Theta(x) + \langle \mu, y - x \rangle \quad \forall y \in E. \tag{5.14}$$

The function $z = x - \Theta(x)\mathbf{1}$ is in E and $z(t) = 0$ for $t \in \mathrm{supp}\mu$. Using Lemma 5.5 we have that $\langle \mu, z \rangle = 0$ which implies $\langle \mu, x \rangle = \Theta(x)$. So (5.14) becomes $\Theta(y) \geq \langle \mu, y \rangle$. But this follows readily from the fact that $\mu \geq 0$ and the function $u = \Theta(y)\mathbf{1} - y$ is ≥ 0.

iii) Let us prove (5.13) $\Rightarrow$. Now we have that (5.14) holds by hypothesis. Let $z \in E$, $z \geq 0$, be arbitrary and put $y = x - z$ in (5.14); we obtain

$$\mathrm{Max}_K(x - z) - \mathrm{Max}_K(x) \geq -\langle \mu, z \rangle.$$

Since the left side of the above inequality is ≥ 0 we get $\langle \mu, z \rangle \geq 0$. Next let $C \in \mathbb{R}$ be arbitrary and put $y = x + C\mathbf{1}$ in (5.14); we obtain

$$\mathrm{Max}_K(x + C\mathbf{1}) - \mathrm{Max}_K(x) \geq \langle \mu, C\mathbf{1} \rangle = C\langle \mu, \mathbf{1} \rangle.$$

Since the left side of the above inequality is C we obtain $C\langle \mu, \mathbf{1} \rangle \leq C$, which implies $\langle \mu, \mathbf{1} \rangle = 1$. Finally, in order to prove that the support of μ is contained in the closed set $S = \{t \in K : x(t) = \Theta(x)\}$ it suffices to show that μ vanishes in any open set $U \subset K \backslash S$. Let $z \in E$ be a function with compact support K_0 contained in U. Let

$$\ell = \Theta(x) - \mathrm{Max}_{K_0}(x) > 0,$$

and choose $\epsilon > 0$ such that $\pm \epsilon z(t) < \ell$, for all $t \in K$. Thus $x(t) \pm \epsilon z(t) < \Theta(x)$, and $\Theta(x \pm \epsilon z) = \Theta(x)$. So using (5.14) with $y = x \pm \epsilon z$ we obtain $\pm\epsilon\langle \mu, z \rangle \leq 0$ which shows that $\langle \mu, z \rangle = 0$. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. (i) Viewing the use of Ekeland Variational Principle we observe that Γ is a complete metric space with the distance defined by

$$d(f, g) = \mathrm{Max}\{||f(t) - g(t)|| : t \in K\}, \quad \forall f, g \in \Gamma.$$

Next define the functional $\Psi : \Gamma \to \mathbb{R}$ by

$$\Psi(f) = \mathrm{Max}_{t \in K}\Phi(f(t)).$$

It follows from (5.3) that Ψ is bounded below. Indeed $\Psi(f) \geq b$ for all $f \in \Gamma$, where

$$b = \mathrm{Max}_{t \in K_0}\Phi(f_0(t)).$$

Next we check the continuity of Ψ at $f_1 \in \Gamma$. Given $\epsilon > 0$, choose $\delta > 0$ such that $|\Phi(x) - \Phi(y)| \leq \epsilon$ for all $y \in f_1(K)$ and all $x \in X$ such that $||x - y|| \leq \delta$. Now for each $f \in \Gamma$ such that $d(f, f_1) \leq \delta$ we have

$$\Psi(f) - \Psi(f_1) = \Phi(f(\bar{t})) - \mathrm{Max}_{t \in K}\Phi(f_1(t)) \leq \Phi(f(\bar{t})) - \Phi(f_1(\bar{t}))$$

where $\bar{t} \in K$ is the point where the maximum of $\Phi(f(t))$ is achieved. Since $||f(\bar{t}) - f_1(\bar{t})|| \leq d(f, f_1) < \delta$ we conclude that $\Psi(f) - \Psi(f_1) \leq \epsilon$. And reverting the roles of f and f_1 we obtain that $|\Psi(f) - \Psi(f_1)| \leq \epsilon$, showing that Ψ is continuous. Thus by Ekeland Variational Principle, given $\epsilon > 0$ there exists a $f_\epsilon \in \Gamma$ such that

$$c \leq \Psi(f_\epsilon) \leq c + \epsilon \tag{5.15}$$

$$\Psi(f_\epsilon) \leq \Psi(f) + \epsilon d(f, f_\epsilon), \quad \forall f \in \Gamma. \tag{5.16}$$

(ii) Now we denote $\Gamma_0 = \{k \in C(K, X) : k(t) = 0, \quad \forall t \in K_0\}$. For any $k \in \Gamma_0$ and any $r > 0$, we have

$$\Phi(f_\epsilon(t) + rk(t)) = \Phi(f_\epsilon(t)) + r\langle \Phi'(f_\epsilon(t)), k(t) \rangle + o(rk(t)).$$

So

$$\mathrm{Max}_{t \in K}\Phi(f_\epsilon(t) + rk(t)) \leq \mathrm{Max}_{t \in K}\{\Phi(f_\epsilon(t)) + r\langle \Phi'(f_\epsilon(t)), k(t) \rangle\} + o(r||k||)$$

where $||k|| = \max_{t \in K} ||k(t)||$. Using this in (5.16) with $f = f_\epsilon + rk$ we obtain

$$\mathrm{Max}_{t \in K}\Phi(f_\epsilon(t)) \leq \mathrm{Max}\{\Phi(f_\epsilon(t)) + r\langle \Phi'(f_\epsilon(t)), k(t) \rangle\} + \epsilon r||k|| \tag{5.17}$$

$t \epsilon$

Now we are in the framework of Proposition 5.6: the functions $x(t) = \Phi(f_\epsilon(t))$ and $y(t) = \langle \Phi'(f_\epsilon(t)), k(t)\rangle$ are in $E = C(K, \mathbb{R})$. So (5.17) can be rewritten as

$$\frac{\Theta(x+ry)-\Theta(x)}{r} \geq -\epsilon\|k\|$$

Taking limits as $r \downarrow 0$ and using Proposition 5.4 we get

$$\text{Max}_{\mu\in\partial\Theta(x)}\langle \mu, y\rangle \geq -\epsilon\|k\| \tag{5.18}$$

Observe that in (5.18) y depends on k. Replacing k by $-k$ we obtain from (5.18):

$$\text{Min}_{\mu\in\partial\Theta(x)}\langle \mu, y\rangle \leq \epsilon\|k\| \tag{5.19}$$

where $\partial\Theta(x)$ is the set of all Radon measures μ in K such that $\mu \geq 0$, $\langle \mu, \mathbb{1}\rangle = 1$ and $\text{supp}\mu \subset K_1$ where $K_1 = \{t \in K : \Phi(f_\epsilon(t)) = \text{Max}_{t\in K}\Phi(f_\epsilon(t))\}$. Dividing (5.19) through by $\|k\|$ and taking Sup we get

$$\text{Sup}_{\substack{k\in\Gamma_0\\ \|k\|\leq 1}} \text{Min}_{\mu\in\partial\Theta(x)}\langle \mu, \langle \Phi'(f_\epsilon(\cdot)), k(\cdot)\rangle\rangle \leq \epsilon \tag{5.20}$$

Using Von Neumann min-max Theorem [7] we can interchange the Sup and Min in the above expression. Now we claim

$$\text{Sup}_{\substack{k\in\Gamma_0\\ \|k\|\leq 1}} \langle \mu, \langle \Phi'(f_\epsilon(\cdot)), k(\cdot)\rangle\rangle = \text{Sup}_{\substack{k\in C(K,X)\\ \|k\|\leq 1}} \langle \mu, \langle \Phi'(f_\epsilon(\cdot)), k(\cdot)\rangle\rangle \tag{5.21}$$

Indeed, since K_0 and K_1 are disjoint compact subsets of K one can find a continuous function $\varphi : K \to \mathbb{R}$ such that $\varphi(t) = 1$ for $t \in K_1$, $\varphi(t) = 0$ for $t \in K_0$ and $0 \leq \varphi(t) \leq 1$ for all $t \in K$. Given any $k \in C(K, X)$ with $\|k\| \leq 1$ we see that $k_1(\cdot) = \varphi(\cdot)k(\cdot) \in \Gamma_0$, $\|k_1\| \leq 1$ and

$$\langle \mu, \langle \Phi'(f_\epsilon(\cdot)), k(\cdot)\rangle\rangle = \langle \mu, \langle \Phi'(f_\epsilon(\cdot)), k_1(\cdot)\rangle\rangle$$

because $\text{supp}\mu \subset K_1$. So (5.21) is proved. Since $\mu \geq 0$ the right side of (5.21) is less or equal to

$$\langle \mu, \text{Sup}_{\substack{k\in C(K,X)\\ \|k\|\leq 1}} \langle \Phi'(f_\epsilon(\cdot)), k(\cdot)\rangle\rangle$$

But the Sup in the above expression is equal to $\|\Phi'(f_\epsilon(\cdot))\|$. So coming back to (5.20) interchanged we get

$$\text{Min}_{\mu\in\partial\Theta(x)}\langle \mu, \|\Phi'(f_\epsilon(\cdot))\|\rangle \leq \epsilon.$$

Let $\bar\mu \in \partial\Theta(x)$ the measure that realizes the above minimum:

$$\langle \bar\mu, \|\Phi'(f_\epsilon(t))\|\rangle \leq \epsilon$$

Since $\bar\mu$ has mass one and it is supported in K_1, it follows that, there exists $\bar t \in K_1$ such that $\|\Phi'(f_\epsilon(\bar t))\| \leq \epsilon$. Let $u_\epsilon = f_\epsilon(\bar t)$. Since $\bar t \in K_1$ it follows that

$$\Phi(u_\epsilon) = \text{Max}_{t\in K}\Phi(f_\epsilon(t)) \equiv \Psi(f_\epsilon).$$

So from (5.15) we have $c \leq \Phi(u_\epsilon) \leq c+\sqrt{\epsilon}$, completing the proof. □

Remark 1. In the above proof, Von Neumann min-max theorem was applied to the function $G : \mathcal{M}(K, \mathbb{R}) \times C(K, X) \to \mathbb{R}$, ($\mathcal{M}(K, \mathbb{R})$ endowed with the w^*-topology) defined by

$$G(\mu, k) = \langle \mu, \langle \Phi'(f_\epsilon(\cdot)), k(\cdot)\rangle\rangle.$$

Observe that G is continuous and linear in each variable separately and that the sets $\Theta(x)$ and $\{k \in C(K, X) : \|k\| \leq 1\}$ are convex, the former one being w^*-compact.

Remark 2. Let (g_α) be an arbitrary family of functions in $C(K, \mathbb{R})$, which are uniformly bounded. Then $g \equiv \sup_\alpha g_\alpha \in C(K, \mathbb{R})$. Since $g_\alpha \le g$ we have that for $\mu \in \mathcal{M}(K, \mathbb{R})$, $\mu \ge 0$, one has $\langle \mu, g_\alpha \rangle \le \langle \mu, g \rangle$. This given

$$\mathrm{Sup}_\alpha \langle \mu, g_\alpha \rangle \le \langle \mu, \mathrm{Sup} g_\alpha \rangle.$$

Now we turn to showing that Theorem 5.1 contains as special cases all three min-max theorems cited in the Introduction to this Chapter.

Theorem 5.7. (Mountain Pass Theorem [4]). *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional which satisfies the (PS) condition. Let S be a closed subset of X which disconnets X. Let x_0 and x_1 be points of X which are in distinct connected components of $X \setminus S$. Suppose that Φ is bounded below in S, and in fact the following condition is verified*

$$\mathrm{Inf}_S \Phi \ge b \quad \text{and} \quad \mathrm{Max}\{\Phi(x_0), \Phi(x_1)\} < b. \tag{5.22}$$

Let

$$\Gamma = \{f \in C([0,1]; X]) : f(0) = x_0, f(1) = x_1\}.$$

Then

$$c = \mathrm{Inf}_{f \in \Gamma} \mathrm{Max}_{t \in [0,1]} \Phi(f(t))$$

is $> -\infty$ and it is a critical value. That is there exists $x_0 \in X$ such that $\Phi(x_0) = c$ and $\Phi'(x_0) = 0$.

Remark. The connectedness referred above is arcwise connectedness. So $X \setminus S$ is a union of open arcwise connected components, see Dugundji [34, p. 116]. Thus x_0 and x_1, being in distinct components implies that any arc in X connecting x_0 and x_1 intersept S. For instance S could be a hyperplane in X or the boundary of an open set, [in particular, the boundary of a ball].

Proof of Theorem 5.7 It is an immediate consequence of Theorem 5.1. In view of the above remark, (5.22) implies (5.3). □

Theorem 5.8. (Saddle Point Theorem [66]). *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional which satisfies the (PS) condition. Let $V \subset X$*

be a finite dimensional subspace and W a complement of V : $V \oplus W$. Suppose that there are real numbers $r > 0$ and $a < b$ such that

$$\mathrm{Inf}_W \Phi \ge b \qquad \mathrm{Max}_{\partial D} \Phi \le a \tag{5.23}$$

where $D = V \cap B_r(0)$, $B_r(0) = \{x \in X : \|x\| < r\}$ and $\partial D = \{x \in V : \|x\| = r\}$. Let

$$\Gamma = \{f \in C(\overline{D}, X) : f(x) = x \ \forall x \in \partial D\},$$

and

$$c = \mathrm{Inf}_{f \in \Gamma} \mathrm{Sup}_{x \in \overline{D}} \Phi(f(x))$$

Then $c > -\infty$ and it is a critical value.

Proof. It suffices to show that (5.23) implies (5.3) and the result follows from Theorem 5.1. The sets K and K_0 of said theorem are $\overline{D}$ and ∂D respectively. Let $f \in K$. Since the right side of (5.3) in view of (5.23) is $\le a$, it suffices to prove that there is $x \in \overline{D}$ such that $f(x) \in W$ and then use (5.23) again. Let $P : X \to X$ be the linear projection over V along W. So $f(x) \in W$ is equivalent to $Pf(x) = 0$. Thus the question reduces in showing that the continuous mapping

$$Pf : \overline{D} \to V$$

has a zero. Since V is finite dimensional and Pf = identity on ∂D the result follows readily from Brouwer fixed point theorem. □

Remark. The last step in the previous proof is standard. It can be proved in few lines using the Brouwer theory of topological degree. Consider the homotopy $H(t, \bullet) \equiv tPf + (1-t)id : \overline{D} \to V$. Since $Pf(x) = x$ for $x \in \partial D$ it follows that the homotopy is admissible and $deg(H(t, \bullet), D, 0)$ =const. Thus $deg(Pf, D, 0) = deg(id, D, 0) = 1$ and consequently Pf has a zero. Another proof using Brouwer fixed point instead uses the mapping $R \circ Pf : \overline{D} \to D$ where R is the radial retraction over D : $R(v) = v$ if $\|v\| \le r$ and $R(v) = rv/\|v\|$ elsewhere.

Theorem 5.9. (Generalized Mountain Pass Theorem [67]). *Let X be a Banach space and $\Phi : X \to \mathbb{R}$ a C^1 functional which satisfies the (PS) condition. As in the previous theorem let $X = V \oplus W$, V finite dimensional. Let $w_0 \in W$ be fixed and let $\rho < R$ be given positive real numbers. Let $Q = \{v + rw_0 : v \in V, \|v\| \le R, 0 \le r \le R\}$. Suppose that*

$$\mathrm{Inf}_{W \cap \partial B_\rho} \Phi \ge b \ , \ \mathrm{Max}_{\partial Q} \Phi \le a \ , \ a < b, \tag{5.24}$$

where ∂B_ρ is the boundary of the ball $B_\rho(0)$. Let

$$\Gamma = \{f \in C(Q, X) : f(x) = x, x \in \partial Q\},$$

and

$$c = \text{Inf}_{f\in\Gamma}\text{Sup}_{x\in Q}\Phi(f(x))$$

Then $c > -\infty$ and it is a critical value.

Proof. We apply Theorem 5.1 with $K = Q$ and $K_0 = \partial Q$. It suffices then to show that (5.24) implies (5.3). First we see that the right side of (5.3) is $\leq a$ in view of (5.24). So by (5.24) again it is enough to show that for each given $f \in \Gamma$ there exists $x \in Q$ such that

$$f(x) \in W \cap \partial B_\rho. \tag{5.25}$$

To prove that we use degree theory again. Let us define a mapping $g : Q \to V \oplus \mathbb{R}\omega_0$, as follows

$$g(v + r\omega_0) = (Pf(v + r\omega_0), ||(I - P)f(v + r\omega_0)||)$$

Clearly g is continuous and $g(v+r\omega_0) = v + r\omega_0$ if $v + r\omega_0 \in \partial Q$. The point $(0, \rho)$ is in the interior of Q relative to $V \oplus \mathbb{R}\omega_0$. So there exists $\bar{v} + \bar{r}\omega_0 \in Q$ such that $g(\bar{v} + \bar{r}\omega_0) = (0, \rho)$. This proves (5.25). □

A useful and popular form of the Mountain Pass Theorem. The following result follows from Theorem 5.7 and Theorem 5.10 below. *Φ is C^1, satisfies (PS) and it is unbounded below. Suppose that u_0 is a strict local minimum of Φ. Then Φ possesses a critical point $u_1 \neq u_0$.* The definition of *strict minimum* is: there exists $\epsilon > 0$ such that

$$\Phi(u_0) < \Phi(u). \quad \forall 0 < ||u - u_0|| < \epsilon.$$

Theorem 5.10. (On the nature of local minima). *Let $\Phi \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition. Suppose that $u_0 \in X$ is a local minimum,* i.e. *there exists $\epsilon > 0$ such that*

$$\Phi(u_0) \leq \Phi(u) \quad \text{for} \quad ||u - u_0|| \leq \epsilon.$$

Then given any $0 < \epsilon_0 \leq \epsilon$ the following alternative holds: either (i) *there exists $0 < \alpha < \epsilon_0$ such that*

$$\text{Inf}\{\Phi(u) : ||u - u_0|| = \alpha\} > \Phi(u_0)$$

or (ii) *for each α, with $0 < \alpha < \epsilon_0$, Φ has a local minimum at a point u_α with $||u_\alpha - u_0|| = \alpha$ and $\Phi(u_\alpha) = \Phi(u_0)$.*

Remark. The above result shows that at a strict local minimum, alternative (i) holds. The proof next is part of the proof of Theorem 5.11 below given in de Figueiredo-Solimini [43].

Proof. Let ϵ_0 with $0 < \epsilon_0 \leq \epsilon$ be given, and suppose that (i) does not hold. So for any given fixed α, with $0 < \alpha < \epsilon_0$, one has

$$\text{Inf}\{\Phi(u) : ||u - u_0|| = \alpha\} = \Phi(u_0) \tag{5.26}$$

Let $\delta > 0$ be such that $0 < \alpha - \delta < \alpha + \delta < \epsilon_0$. Consider Φ restricted to the ring $\mathcal{R} = \{u \in X : \alpha - \delta \leq ||u - u_0|| \leq \alpha + \delta\}$. We start with u_n such that

$$||u_n - u_0|| = \alpha \quad \text{and} \quad \Phi(u_n) \leq \Phi(u_0) + \frac{1}{n},$$

where the existence of such u_n is given by (5.26). Now we apply the Ekeland variational principle and obtain $v_n \in \mathcal{R}$ such that

$$\Phi(v_n) \leq \Phi(u_n), \quad ||u_n - v_n|| \leq \frac{1}{n} \quad \text{and} \tag{5.27}$$

$$\Phi(v_n) \leq \Phi(u) + \frac{1}{n}||u - v_n|| \quad \forall u \in \mathcal{R}. \tag{5.28}$$

From the second assertion in (5.27) it follows that v_n is in the interior of $\mathcal{R}$ for large n. We then take in (5.28) $u = v_n + tw$, where $\omega \in X$ with norm 1 is arbitrary and $t > 0$ is sufficiently small. Then using Taylor's formula and letting $t \to 0$ we get $||\Phi'(v_n)|| \leq \frac{1}{n}$. This together with the first assertion in (5.27) and (PS) gives the existence of a subsequence of v_n (call it v_n again) such that $v_n \to v_\alpha$. So $\Phi(v_\alpha) = \Phi(u_0)$, $\Phi'(v_\alpha) = 0$ and $||v_\alpha - u_0|| = \alpha$. □

A weaker form of the Mountain Pass Theorem. The following weaker form of the result presented in the last section appears inRabinowitz [68]. He uses a sort of dual version of the Mountain Pass Theorem. The proof presented here is due to de Figueiredo-Solimini [43].

Proposition 5.11. *Let $\Phi \in C^1(X, \mathbb{R})$ satisfy (PS) condition. Suppose that*

$$\text{Inf}\{\Phi(u) : ||u|| = r\} \geq \text{Max}\{\Phi(0), \Phi(e)\} \tag{5.29}$$

where $0 < r < ||e||$. Then Φ has a critical point $u_0 \neq 0$.

Proof. The case when there is strict inequality in (5.29) is contained in

Theorem 5.7. Therefore let us assume equality in (5.29). If e is local minimum we are through. So we may assume that there exists a point e' near e where $\Phi(e') < \Phi(e)$. Therefore replacing e by e' two things may occur: either (i) we gain inequality in (5.29) and again Theorem 5.7 applies and we finish, or (ii) equality persists and we have

$$(5.30) \qquad \operatorname{Inf}\{\Phi(u) : \|u\| = r\} = \Phi(0) > \Phi(e).$$

So we assume that (5.30) holds. Also we may assume that

$$(5.31) \qquad \operatorname{Inf}\{\Phi(u) : \|u\| \leq r\} = \Phi(0)$$

because otherwise Theorem 5.7 would apply again and we would finish. But (5.31) says that 0 is a local minimum. So we can apply Theorem 5.10 and conclude. □

Corollary 5.12 *Let $\Phi \in C^1(X, \mathbb{R})$ satisfy (PS) condition. Suppose that Φ has two local minima. Then Φ has at least one more critical point.*

Proof. Use Theorems 5.10 and 5.7.

Chapter 6

Semilinear Elliptic Equations II

Introduction. In this chapter we continue the study of the Dirichlet problem:

$$(6.1) \qquad -\Delta u = f(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and $\partial\Omega$ denotes its boundary. In order to minimize technicalities, we assume all along this chapter the following minimal assumption on the nonlinearity:

$$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \qquad \text{is a continuous function of both variables.}$$

As in Chapter 3 we search the critical points of the functional

$$(6.2) \qquad \Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(x,u).$$

If not stated on the contrary all integrals are taken over the whole of Ω. We assume the following additional condition of f:

$$(6.3) \qquad |f(x,s)| \leq c|s|^{p-1} + b(x)$$

where $c > 0$ is a constant, $b(x) \in L^{p'}(\Omega)$ with $(1/p) + (1/p') = 1$, and $1 \leq p < \infty$ if $N = 2$, or $1 \leq p \leq 2N/(N-2)$ if $N \geq 3$. As we proved in Chapter 3, under this hypothesis the functional $\Phi : H_0^1 \to \mathbb{R}$ defined in (6.3) is continuously Fréchet differentiable. In Chapter 3 a further condition was required on F (see (3.6) there) which was sufficient to guarantee that Φ is bounded below. Here we are interested in the cases when Φ is not bounded below any longer. We assume the following condition

$$(6.4) \qquad \liminf_{s\to+\infty} \frac{f(x,s)}{s} > \lambda_1 \quad \text{uniformly in } \overline{\Omega}$$

where λ_1 is the first eigenvalue of $(-\Delta, H_0^1)$.

Lemma 6.1. *Under* (6.4) *the functional Φ is unbounded below.*

Proof. It follows from (6.4) that there are constants $\mu > \lambda_1$ and c such that $f(x,s) \geq \mu s - c$, for all $s > 0$. Therefore we can find constants μ' and c' with $\mu > \mu' > \lambda_1$ such that $F(x,s) \geq \frac{1}{2}\mu' s^2 - c'$, for all $s > 0$. Thus for $t > 0$ we have

$$\Phi(t\varphi_1) \leq \frac{1}{2}\lambda_1 t^2 \int \varphi_1^2 - \frac{1}{2}\mu' t^2 \int \varphi_1^2 + c'|\Omega|. \quad \square$$

Viewing the future applications of the variational theorems of Chapter 5 we now state conditions which insure the (PS) condition for Φ.

Lemma 6.2. *Assume condition* (6.3) *with* $1 \leq p < 2N/(N-2)$ *if* $N \geq 3$ *and* $1 \leq p < \infty$ *if* $N = 2$. *The Φ satisfies the (PS) condition if every sequence* (u_n) *in* H_0^1, *such that*

$$(6.5) \qquad |\Phi(u_n)| \leq \text{const}, \quad \Phi'(u_n) \to 0$$

is bounded.

Proof. All we have to prove is that (u_n) contains a subsequence which converges in the norm of H_0^1. Since (u_n) is bounded, there is a subsequence (u_{n_j}) converging weakly in H_0^1 to some u_0 and strongly in any L^p to the same u_0, with $1 \leq p < 2N/(N-2)$ if $N \geq 3$ and $1 \leq p < \infty$ if $N = 2$. On the other hand the second assertion in (6.5) means that

$$(6.6) \qquad \left|\left| \int \nabla u_n \nabla v - \int f(x,u_n)v \right|\right| \leq \epsilon_n ||v||_{H^1}, \quad \forall v \in H_0^1$$

where $\epsilon_n \to 0$. Put $v = u_n - u_0$, and taking limits over the subsequence we obtain that

$$\int \nabla u_{n_j} \nabla(u_{n_j} - u_0) \to 0.$$

[Here we have used the continuity properties of the Nemytskii mappings, see Chapter 2]. So $||u_{n_j}||_{H^1} \to ||u_0||_{H^1}$. This together with the fact that $u_{n_j} \to u_0$ (weakly) in H_0^1, gives that $u_{n_j} \to u_0$ (strongly) in H_0^1. $\square$

Palais–Smale Condition for Asymptotically Linear Problems. Assume that the limits below exist as L^∞ functions

$$(6.7) \qquad \lim_{s\to-\infty} \frac{f(x,s)}{s} = \alpha(x) \quad \text{and} \quad \lim_{s\to+\infty} \frac{f(x,s)}{s} = \beta(x).$$

It follows then that there are positive constants c_1 and c_2 such that

$$(6.8) \qquad |f(x,s)| \leq c_1|s| + c_2, \quad \forall s \in \mathbb{R}, \quad \forall x \in \Omega.$$

Lemma 6.3. *Assume* (6.7) *above. In addition suppose that the problem below has only the solution* $v \equiv 0$:

$$(6.9) \qquad -\Delta v = \beta(x)v^+ - \alpha(x)v^- \quad \text{in } \Omega \quad \text{and} \quad v = 0 \quad \text{on } \partial\Omega.$$

Then the functional Φ satisfies (PS) condition. Here $v^+ = \max(v,0)$ *and* $v^- = v^+ - v$.

Remark 1. It suffices to consider (6.9) in the H_0^1 sense. That is

$$\int \nabla v_0 \nabla v = \int [\beta(x)v_0^+ - \alpha(x)v_0^-]v, \quad \forall v \in H_0^1$$

and $v_0 \in H_0^1$. We will use without further mentionning a result of Stampacchia [1]: if $v \in H_0^1$ then v_0^+, v_0^- and in general $G(v)$, where $G : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, are all H_0^1 functions.

Remark 2. If α and β are constants, then the pairs (α,β) such that problem (6.9) has non-trivial solutions constitute the so-called singular set $\sum$. In the case of $N = 1$ this set has been completely characterized by Fučik [44]. It is not known a similar result for $N \geq 2$. However some information about $\sum$ has been obtained, see Dancer [29], Gallouet-Kavian [45] and Magalhaes [59].

Proof fo Lemma 6.3. We use Lemma 6.2. Suppose by contradiction that there is a sequence (u_n) in H_0^1 such that

$$(6.10) \qquad |\Phi(u_n)| = \left|\frac{1}{2}\int |\nabla u_n|^2 - \int F(x,u_n)\right| \leq \text{Const}$$

$$(6.11) \quad |\langle\Phi'(u_n), v\rangle| = \left|\int \nabla u_n \nabla v - \int f(x,u_n)v\right| \leq \epsilon_n ||v||_{H^1}, \quad \forall v \in H_0^1$$

(6.12) $$\|u_n\|_{H^1} \to \infty, \qquad \epsilon_n \to 0.$$

Let $v_n = u_n/\|u_n\|_{H^1}$, and (passing to a subsequence if necessary) assume that a $v_0 \in H_0^1$ can be found such that $v_n \rightharpoonup v_0$ (weakly) in H_0^1, $v_n \to v_0$ (strongly) in L^2, $v_n \to v_0$ a.e. and $|v_n| \le h$ for some L^2-function h. Now we claim that

(6.13) $$\|u_n\|_{H^1}^{-1} f(x, u_n(x)) \to \beta(x)v_0^+ - \alpha(x)v_0^- \quad \text{in} \quad L^2.$$

We prove that using the argument in Costa–de Figueiredo–Gonçalves [27]. Let us denote $\alpha_n \equiv \|u_n\|$ and $\ell(x) \equiv \beta(x)v_0^+ - \alpha(x)v_0^-$. It suffices to show that every subsequence of $f_n(x) \equiv \alpha_n^{-1} f(x, \alpha_n v_n(x))$ possesses a further subsequence which converges to $\ell(x)$ in L^2. Using (6.8)

(6.14) $$|f_n(x)| \le \alpha_n^{-1}|c_1\alpha_n|v_n(x)| + c_2| \le c_1 h(x) + \alpha_n^{-1}c_2.$$

In the set $A = \{x : v_0(x) \ne 0\}$, $f_n(x) \to \ell(x)$ a.e.. So by the Lebesgue Dominated Convergence theorem $f_n\chi_A \to \ell$ in L^2. In the set $B = \{x : v_0(x) = 0\}$ it follows from (6.14) that $f_n(x) \to 0$ a.e. So similarly $f_n\chi_B \to 0$ in L^2. So the claim in (6.13) is proved. Now dividing (11) by $\|u_n\|_{H^1}$ and passing to the limit we obtain

$$\int \nabla v_0 \nabla v - \int [\beta(x)v_0^+ - \alpha(x)v_0^-]v = 0 \quad \forall v \in H_0^1.$$

In view of (6.9) it follows that $v_0 = 0$. Next use (6.11) again with $v = v_n$, divide it through by $\|u_n\|_{H^1}$ to obtain

$$\left|\int |\nabla v_n|^2 - \int \frac{f(x, u_n)}{\|u_n\|_{H^1}} v_n\right| \le \frac{\epsilon_n}{\|u_n\|_{H^1}} \|v_n\|_{H^1}.$$

In the above inequality the first term is equal to 1 and the other two converge to zero, impossible!

Palais–Smale Condition for Superlinear Problems. Assume that

(6.15) $$\liminf_{|s|\to\infty} \frac{f(x, s)}{s} = +\infty,$$

that is, the problem is superlinear at both $+\infty$ and $-\infty$. In this case the following lemma provides sufficient conditions for (PS).

Lemma 6.4. *Φ satisfies (PS) condition if one assumes:* (i) *condition* (6.3) *with $1 \le p < 2N/(N-2)$ in the case $N \ge 3$ and $1 \le p < \infty$ in the case $N = 2$, and* (ii) *the following condition introduced by Ambrosetti and Rabinowitz [7]: there is a $\theta > 2$ and $s_0 > 0$ such that*

(6.16) $$0 < \theta F(x, s) \le s f(x, s) \quad \forall x \in \overline{\Omega} \quad \forall |s| \ge s_0.$$

Proof. Let (u_n) be a sequence in H_0^1 satisfying conditions (6.10) and (6.11) above. Replace v by u_n in (6.11). Multiply (6.10) by θ and subtract (6.11) from the expression obtained:

$$(\frac{\theta}{2} - 1)\int |\nabla u_n|^2 \le \int [\theta F(x, u_n) - u_n f(x, u_n)] + \epsilon_n\|u_n\|_{H^1} + C$$

Using (6.16) we obtain that $\|u_n\|_{H^1} \le C$. The proof is completed using Lemma 6.2. □

Remark. Condition (6.16) implies that F is superquadratic. Indeed, from (6.16): $\theta/s \le f(x, s)/F(x, s)$. Integrating from s_0 to s : $\theta[\ell n|s| - \ell n s_0] \le \ell n F(x, s) - \ell n F(x, s_0)$, which implies $F(x, s) \ge F(x, s_0)s_0^{-\theta}|s|^\theta$, for $|s| \ge s_0$. Using (6.16) again we obtain $f(x, s) \ge \theta F(x, s_0)s_0^{-\theta}|s|^{\theta-1}$. Observe that this inequality is stronger than the requirement put in (6.15). So there is some room between (6.15) and (6.16). Thus, how about Palais–Smale in the case when f satisfies (6.15) but not (6.16)? There is a partial answer to this question in [42].

Palais–Smale Condition for Problems of the Ambrosetti–Prodi Type. Now we assume

(6.17) $$\limsup_{s\to-\infty} \frac{f(x, s)}{s} < \lambda_1 \quad \text{and} \quad \liminf_{s\to+\infty} \frac{f(x, s)}{s} > \lambda_1$$

where the conditions above are to hold uniformly for $x \in \overline{\Omega}$. The first limit could be $-\infty$ and the second could be $+\infty$.

Lemma 6.5. *Assume* (6.3) *and the first assertion in* (6.17). *Let (u_n) be a sequence in $H_0^1(\Omega)$ such that*

(6.18) $$\left|\int \nabla u_n \nabla v - \int f(x, u_n)v\right| \le \epsilon_n\|v\|_{H^1} \quad \forall v \in H_0^1.$$

where $\epsilon_n \to 0$. [We may visualize the u_n's as "almost" critical points of Φ, or as "approximate" solutions of (6.1), vaguely speaking!] *Then there exists a constant* $M > 0$ *such that* $\|u_n^-\|_{H^1} \le M$.

Proof. If follows from the assumption that there exists $0 < \mu < \lambda_1$ and a constant c such that

$$f(x,s) > \mu s - c \quad \text{for} \quad s \le 0. \tag{6.19}$$

Replacing v by u_n^- in (6.18) we can estimate

$$\int |\nabla u_n^-|^2 \le -\int f(x,u_n)u_n^- + \epsilon_n \|u_n^-\|_{H^1}.$$

Using (6.19) we obtain

$$\int |\nabla u_n^-|^2 \le \mu \int (u_n^-)^2 + c\int u_n^- + \epsilon_n \|u_n^-\|_{H^1}.$$

Finally using Poincaré and Schwarz inequalities we complete the proof. □

Lemma 6.6. *Assume* (6.17) *and that* f *has linear growth,* i.e.

$$|f(x,s)| \le c_1|s| + c_2 \quad \forall x \in \overline{\Omega}, \quad \forall s \in \mathbb{R} \tag{6.20}$$

where c_1, c_2 *are given positive constants. Then the functional* Φ *satisfies (PS) condition.*

Proof. Assume by contradiction that there exists a sequence (u_n) in H_0^1 satisfying conditions (6.10), (6.11) and (6.12) above. As in Lemma 6.3, let $v_n = u_n/\|u_n\|_{H^1}$ and assume that $v_n \rightharpoonup v_0$ in H_0^1, $v_n \to v_0$ in L^2 and a.e. (Ω), and that there is an L^2-function h such that $|v_n(x)| \le h(x)$. It follows from Lemma 6.5 that $v_n^- \to 0$ in H_0^1, and we may assume that $v_n^- \to 0$ a.e. (Ω). So $v_0 \ge 0$ in Ω. First we claim that the sequence

$$g_n \equiv \chi_n \frac{f(x,u_n)}{\|u_n\|} \to 0 \quad \text{in} \quad L^2$$

where χ_n is the characteristic function of the set $\{x : u_n(x) \le 0\}$. Indeed this follows easily from the Lebesgue Dominated Convergence Theorem, observing that (6.20) implies

$$|g_n| \le c_1 \frac{u_n^-}{\|u_n\|_{H^1}} + \frac{c_2}{\|u_n\|_{H^1}} \to 0 \quad \text{a.e.} \tag{6.21}$$

and $|g_n| \le c_1 h + c_2/\|u_n\|_{H^1}$. On the other hand, the sequence (or passing to a subsequence of it):

$$\gamma_n \equiv (1-\chi_n)\frac{f(x,u_n)}{\|u_n\|_{H^1}} \rightharpoonup \gamma \quad \text{in} \quad L^2 \tag{6.22}$$

where γ is some L^2 function and $\gamma \ge 0$. Indeed using (6.20) we have that

$$|\gamma_n| \le c_1 h + c_2/\|u_n\|_{H^1} \le c_1 h + 1 \in L^2.$$

The positiveness of γ comes from the following consideration. From the second assertion in (6.17) there exists $r > 0$ such that $f(x,s) \ge 0$ for $s \ge r$. Let ξ_n be characteristic function of the set $\{x \in \Omega : u_n(x) \ge r\}$. Clearly $\xi_n\gamma_n \rightharpoonup \gamma$ in L^2. And the assertion that $\gamma \ge 0$ follows from the fact that $\xi_n\gamma_n$ is in the cone of non-negative functions of L^2, which is closed and convex. Now go back to (6.11), divide it through by $\|u_n\|_{H^1}$ and pass to the limit using (6.21) and (6.22). We obtain

$$\int \nabla v_0 \nabla v - \int \gamma v = 0 \quad \forall v \in H_0^1. \tag{6.23}$$

It follows from the second assertion in (6.17) that there are constants $\mu > \lambda_1$ and $c > 0$ such that $f(x,s) \ge \mu s - c$ for $x \in \Omega$ and $s \ge 0$. So $\gamma_n \ge \mu v_n^+ - c\|u_n\|_{H^1}^{-1}$. Passing to the limit we obtain $\gamma \ge \mu v_0$. Using (6.23) with $v = \varphi_1$, where $\varphi_1 > 0$ in Ω is a first eigenfunction of $(-\Delta, H_0^1(\Omega))$, we obtain

$$\lambda_1 \int v_0\varphi_1 = \int \nabla v_0 \nabla \varphi_1 = \int \gamma\varphi_1 \ge \mu \int v_0\varphi_1.$$

Since $\mu > \lambda_1$ we conclude that $v_0 \equiv 0$. So from (6.23) $\gamma \equiv 0$. Finally use (6.11) again, divided through by $\|u_n\|_{H^1}$ and with $v = v_n$:

$$\left|\int |\nabla v_n|^2 - \int \frac{f(x,u_n)}{\|u_n\|} v_n\right| \le \epsilon_n \|u_n\|_{H^1} = \epsilon_n.$$

The first term is equal to 1 and the other two go to zero, impossible! □

Lemma 6.7. *Assume* (6.3) *and* (6.17) *and suppose that there are constants* $\theta > 2$ *and* $s_0 > 0$ *such that*

$$0 < \theta F(x,s) \le s f(x,s), \quad \forall x \in \Omega, \quad \forall s \ge s_0. \tag{6.24}$$

Then Φ *satisfies (PS) condition.*

Proof. Let (u_n) be a sequence in H_0^1 for which (6.10) and (6.11) holds. By

Lemma 6.2 we should show that $||u_n||_{H^1} \leq$ const. We know from Lemma 6.5 that $||u_n^-||_{H^1} \leq$const. Using the first assertion in (6.17) we see that there are constants $0 < \mu < \lambda_1$ and $c > 0$ such that $F(x,s) \leq \frac{\mu}{2} s^2 - cs$, for $x \in \Omega$ and $s \leq 0$.

$$\int F(x, -u_n^-) \leq \frac{\mu}{2} \int (u_n^-)^2 - c \int u_n^- \leq \text{const.}$$

So from (6.10) we obtain

$$\frac{1}{2} \int |\nabla u_n^+|^2 - \int F(x, u_n^+) \leq \text{const.} \tag{6.25}$$

Using (6.11) with $v = u_n^+$ we obtain

$$\left| \int |\nabla u_n^+|^2 - \int f(x, u_n^+) u_n^+ \right| \leq \epsilon_n ||u_n^+||_{H^1}. \tag{6.26}$$

Multiplying (6.25) by θ and subtracting (6.26) from it we get

$$(\frac{\theta}{2} - 1) \int |\nabla u_n^+|^2 \leq \int [\theta F(x, u_{n+}) - f(x, u_n^+) u_n^+] + \epsilon_n ||u_n^+||_{H^1} + \text{const.}$$

Using (6.24) we conclude that $||u_n^+||_{H^1} \leq$ const. □

Remark. If $f(x,s) \geq 0$ for $x \in \Omega$ and $s \leq 0$, then the eventual solutions of (6.1) are ≥ 0. Indeed, let $u \in H_0^1$ be a solution of (6.1), that is:

$$\int_\Omega \nabla u \nabla v = \int_\Omega f(x,u) v \quad \forall v \in H_0^1.$$

Let $v = u^-$. Then

$$-\int |\nabla u^-|^2 = \int f(x,u) u^- \geq 0 \Rightarrow \int |\nabla u^-|^2 = 0.$$

So $u^- = 0$, proving the claim. Observe that under this hypothesis on f, the first assertion in (6.17) holds. So Lemmas 6.6 and 6.7 provide sufficient conditions for (PS) on a class of semilinear elliptic equations with positive solutions. For example: (i) $f = |u|^p$ for any $1 \leq p < \infty$ if $N = 2$ or $1 \leq p < (N+2)/(N-2)$ if $N \geq 3$. (ii) $f = (u^+)^p$ with the same restrictions on p.

Existence results for (1). To illustrate the use of the theorems proved in Chapter 5 we now consider some examples.

f 9 b

Example 1. Consider the following Dirichlet problem

$$-\Delta u = f(u) + h(x) \quad \text{in} \quad \Omega \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{6.27}$$

where $h \in C(\overline{\Omega})$ [a weaker condition would suffice] and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{s \to -\infty} \frac{f(s)}{s} = \alpha \quad \text{and} \quad \lim_{s \to +\infty} \frac{f(s)}{s} = \beta. \tag{6.28}$$

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $(-\Delta, H_0^1(\Omega))$.

Theorem 6.8. (Dolph [33]). *If $\lambda_k < \alpha, \beta < \lambda_{k+1}$, then problem (6.27) has a solution for every h.*

Proof. We look for critical points of the functional $\Phi : H_0^1 \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2} \int |\nabla u|^2 - \int F(u) - \int hu$$

where $F(s) = \int_0^s f$. It is easy to see that (α, β) does not belong to the singular set $\sum$. So by Lemma 6.3, Φ satisfies (PS). Let V be the finite dimensional subspace generated by the first k eigenfunctions of $(-\Delta, H_0^1)$, and $W = V^\perp$. Let μ and $\overline{\mu}$ be such that $\lambda_k < \mu < \alpha$, $\beta < \overline{\mu} < \lambda_{k+1}$. It follows from (6.28) that there exists $s_0 > 0$ such that $\mu < s^{-1} f(s) < \overline{\mu}$ for $|s| \geq s_0$. A straightforward computation shows that there exist constants C and $\overline{C}$ such that

$$\frac{1}{2} \mu s^2 - C \leq F(s) \leq \frac{1}{2} \overline{\mu} s^2 + \overline{C}, \quad \forall s. \tag{6.29}$$

Now if $v \in V$ we estimate

$$\Phi(v) \leq \frac{1}{2} \int |\nabla v|^2 - \frac{\mu}{2} \int v^2 + C|\Omega| + ||h||_{L^2} ||v||_{L^2}$$

and using the inequality $\int |\nabla v|^2 \leq \lambda_k \int v^2$, for $v \in V$, and the Poincaré inequality we obtain

$$\Phi(v) \leq \frac{1}{2}(1 - \frac{\mu}{\lambda_k}) \int |\nabla v|^2 + C|\Omega| + ||h||_{L^2} \lambda_1^{-1} ||\nabla v||_{L^2}.$$

So $\Phi(v) \to -\infty$ as $||v|| \to \infty$ with $v \in V$. On the other hand if $w \in W$ we estimate

$$\Phi(w) \geq \frac{1}{2} \int |\nabla w|^2 - \frac{\overline{\mu}}{2} \int w^2 - \overline{C}|\Omega| - ||h||_{L^2} ||w||_{L^2}$$

and using the inequality $\int |\nabla w|^2 \geq \lambda_{k+1} \int w^2$ for $\omega \in W$, we obtain

$$\Phi(w) \geq \frac{1}{2}(1 - \frac{\overline{\mu}}{\lambda_{k+1}}) \int |\nabla w|^2 - \overline{C}|\Omega| - ||h||_{L^2}\lambda_1^{-1}||\nabla w||_{L^2}.$$

So $\Phi(w) \to +\infty$ as $||w|| \to \infty$ with $w \in W$. Therefore the result follows from the Saddle Point Theorem, Theorem 5.8. □

Remark. Condition (6.29) suffices to having the functional with the "shape" of the Saddle Point Theorem. However we do not know if it will give (PS). Observe that (PS) was obtained above with hypothesis on the derivative of F with respect to s, namely f. Of course we are willing to assume growth conditions on f to assure the differentiability of Φ, see Chapter 2. But even so, it is not known if (PS) holds.

As a second example we consider problem (1) again and prove

Theorem 6.9. (Ambrosetti-Rabinowitz [4]). *Suppose that f satisfies the conditions of Lemma 6.4. In addition, assume*

$$\lim_{s\to 0} s^{-1} f(x,s) < \lambda_1. \tag{6.30}$$

Then problem (1) *has a nontrivial solution.*

Proof. It follows from (6.30) that $f(x,0) = 0$, and therefore $u \equiv 0$ is a solution of (6.1). Observe that condition (6.16) implies (see Remark after the proof of Lemma 6.4) that (6.15) holds. So Lemma 6.1 implies that Φ is not bounded below. Also (PS) holds, by Lemma 6.4. We plan to apply the Mountain Pass Theorem, Theorem 5.7. For that matter we study the functional Φ near $u = 0$. Given $0 < \mu < \lambda_1$, it follows from (6.30) that there exists $\delta > 0$ such that $|f(x,s)| \leq \mu|s|$ for $|s| \leq \delta$. Therefore

$$|F(x,s)| \leq \frac{\mu}{2}|s|^2 \quad \forall |s| \leq \delta.$$

On the other hand using (6.3) we can find a constant $k > 0$ such that

$$|F(x,s)| \leq k|s|^p \quad \forall |s| > \delta.$$

Here without loss of generality we may suppose $p > 2$. Therefore adding the two previous inequalities:

$$|F(x,s)| \leq \frac{\mu}{2}|s|^2 + k|s|^p \quad \forall s \in \mathbb{R}.$$

Thus Φ can be estimated as follows

$$\Phi(u) \geq \frac{1}{2}\int |\nabla u|^2 - \frac{\mu}{2}\int u^2 - k\int |u|^p$$

and using Poincare inequality and Sobolev imbedding we obtain

$$\Phi(u) \geq \frac{1}{2}(1 - \frac{\mu}{\lambda_1}) \int |\nabla u|^2 - k(\int |\nabla u|^2)^{p/2}.$$

Since $p > 2$ we see that there exists $r > 0$ such that if $||u||_{H^1} = r$ then $\Phi(u) \geq a$ for some constant $a > 0$. So the result follows immediately from Theorem 5.7. □

A Problem of the Ambrosetti-Prodi Type. As a third example we consider the Dirichlet problem

$$-\Delta u = f(x,u) + t\varphi_1 + h \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{6.31}$$

where t is a real parameter, $\varphi_1 > 0$ is a first eigenfunction of $(-\Delta, H_0^1)$ and $h \in C^\alpha(\overline{\Omega})$, with $\int h\varphi_1 = 0$, is fixed. Assume that f is locally lipschitzian in $\overline{\Omega} \times \mathbb{R}$. Then we prove.

Theorem 6.10. *Assume* (6.17) *and* (6.20) [*or* (6.3), (6.17), (6.24)]. *Then it follows that there exists $t_0 \in \mathbb{R}$ such that for all $t \leq t_0$, problem* (31) *has at least two solutions in $C^{2,\alpha}(\overline{\Omega})$.*

Remark. There is an extensive literature on problems of the Ambrosetti-Prodi type, starting with work of Ambrosetti-Prodi [3]. We mention Kazdan-Warner [52], Amann-Hess [1], Berger-Podolak [11], H. Berestycki [9], McKenna [57], Ruf [71] Solimini [76], ⋯ There has been several recent papers by Lazer-McKenna which we don't survey them here. The result above and the proof next are due to de Figueiredo-Solimini [43]. See a similar result by K. C. Chang [25].

Proof. It follows from either Lemma 6.6 [or Lemma 6.7] that the functional below satisfies (PS):

$$\Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(x,u) - \int (t\varphi_1 + h)u. \tag{6.32}$$

Also the second assertion in (6.17) implies by Lemma 6.1 that Φ is not bounded below. In the previous example we also proved the existence of two solutions; however there we had the first solution ($u \equiv 0$) to start. Here we have first to obtain a solution of (6.31) which is a local minimum, and then apply the Mountain Pass Theorem. We break some steps of the proof in a series of lemmas.

Remark. For the next lemma we recall some definitions. We present them in the particular framework of the problem considered here. We refer to Gilbarg-Trudinger [46] for more general definitions.

(i) A function $\omega \in H_0^1$ is said to be a *weak subsolution* of the Dirichlet problem

(*) $-\Delta u + Mu = g(x)$ in Ω $u = 0$ on $\partial\Omega$, where $g \in L^2$ and M is a real constant, if

$$\int \nabla\omega\nabla\psi + M\int \omega\psi \le \int g\psi, \quad \forall\psi \in H_0^1, \quad \psi \ge 0.$$

(ii) A function $\omega \in C^{2,\alpha}(\overline{\Omega})$ is a *classical subsolution* of (*) if

$$-\Delta\omega + M\omega \le g \quad \text{in} \quad \Omega \quad \omega = 0 \quad \text{on} \quad \partial\Omega.$$

(iii) Weak supersolution and classical supersolution are defined likewise by reverting the inequalities in the equations above.

(iv) Every classical subsolution [supersolution] is also weak subsolution [supersolution].

Lemma 6.11. *Assume* (6.17). *Then there exist constants* $0 < \underline{\mu} < \lambda_1 < \overline{\mu}$ *and* $C > 0$ *such that*

$$f(x,s) > \underline{\mu}s - C \quad \text{and} \quad f(x,s) > \overline{\mu}s - C \tag{6.33}$$

for all $x \in \Omega$ *and all* $s \in \mathbb{R}$.

Proof. The first assertion in (6.17) gives $0 < \underline{\mu} < \lambda_1$ and $C_1 > 0$ such that $f(x,s) > \underline{\mu}s - c_1$ for all $x \in \Omega$ and $s \le 0$. The second assertion in (6.17) gives $\lambda_1 < \overline{\mu}$ and $C_2 > 0$ such that $f(x,s) > \overline{\mu}s - C_2$ for all $x \in \Omega$ and $s \ge 0$. Let $C = \max\{C_1, C_2\}$ and (6.33) follows. □

Lemma 6.12. *Assume* (6.17). *Then for each* $t \in \mathbb{R}$ *problem* (6.31) *has a classical subsolution* ω_t. *Moreover given any classical supersolution* W_t *of* (6.31) [*or in particular any solution of* (6.31)] *one has* $\omega_t(x) < W_t(x)$, $\forall x \in \Omega$, *and*

$$\frac{\partial\omega_t}{\partial\nu}(x) > \frac{\partial W_t}{\partial\nu}(x), \quad \forall x \in \partial\Omega.$$

Proof. Let $M_t = \mathrm{Sup}_\Omega|t\varphi_1(x) + h(x)|$. The Dirichlet problem

$$-\Delta u = \underline{\mu}u - C - M_t \quad \text{in} \quad \Omega \qquad u = 0 \quad \text{on} \quad \partial\Omega$$

with $\underline{\mu}$ and C as in (6.33) above, has a unique solution $\omega_t \in C^{2,\alpha}(\overline{\Omega})$, which is < 0 in Ω and $\frac{\partial\omega_t}{\partial\nu} < 0$ in $\partial\Omega$, see Lemma 6.13 below. It follows from the inequality in (6.33) that ω_t is a classical subsolution. To prove the second statement, use (6.33) again

$$-\Delta(W_t - \omega_t) \ge f(x, W_t) + t\varphi_1 + h - \underline{\mu}\omega_t - C - M_t > \underline{\mu}(W_t - \omega_t)$$

and apply Lemma 6.13. □

Lemma 6.13. *Let* $\alpha(x)$ *be an* L^∞ *function such that* $\mathrm{Sup}_\Omega\alpha(x) < \lambda_1$. *Let* $u \in H_0^1$ *be such that*

$$-\Delta u \ge \alpha(x)u \quad \text{in} \quad H_0^1 - \text{sense}. \tag{6.34}$$

Then $u \ge 0$. *Moreover, if* $u \in C^{2,\alpha}(\overline{\Omega})$ *then* $u > 0$ *in* Ω *and* $\frac{\partial u}{\partial\nu} < 0$ *on* $\partial\Omega$.

Proof. Expression (6.34) means

$$\int \nabla u\nabla\psi \ge \int \alpha(x)u\psi, \quad \forall\psi \in H_0^1, \quad \psi \ge 0.$$

Take $\psi = u^-$ and use Poincaré's inequality

$$-\lambda_1\int(u^-)^2 \ge -\int|\nabla u^-|^2 \ge -\int\alpha(x)(u^-)^2 > -\lambda_1\int(u^-)^2$$

which implies $u^- = 0$. So $u = u^+$. If $u \in C^{2,\alpha}(\overline{\Omega})$ then we use the classical maximum principle to

$$-\Delta u + \alpha^- u \ge \alpha^+ u,$$

where the right side is ≥ 0 by the first part of this lemma. □

D4

Lemma 6.14. *Assume* (6.17). *Then there exists a* $t_0 \in \mathbb{R}$ *such that for all* $t \le t_0$ (6.31) *has a classical supersolution* W_t.

Proof. Let $k = \int f(x,0)\varphi_1$ and $f_1(x) = f(x,0) - k\varphi_1$. So the equation in (6.31) may be rewritten as

$$(6.35) \qquad -\Delta u = f(x,u) - f(x,0) + (k+t)\varphi_1 + h + f_1.$$

Now let W_t be the solution of the Dirichlet problem

$$-\Delta u = h + f_1 \quad \text{in} \quad \Omega \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

We see that W_t is a supersolution of (6.31) provided t is such that

$$f(x, W_t) - f(x,0) + (k+t)\varphi_1 \le 0$$

or

$$t \le -k - \mathrm{Sup}_\Omega \frac{f(x,W_t) - f(x,0)}{\varphi_1}.$$

So it remains to prove that the above Sup is $< +\infty$. It suffices then to show that the function $g(x) \equiv [f(x, W(x)) - f(x,0)]/\varphi_1(x)$ is bounded in a neighborhood of $\partial\Omega$. Boundedness in any compact subset of Ω follows from $\varphi_1(x) > 0$ there. If x_0 is a point on $\partial\Omega$, we use the Lipschitz condition on f to estimate

$$|g(x)| \le K \left|\frac{W_t(x)}{\varphi_1(x)}\right|$$

where K is a local lipschitz constant in a neighborhood N of x_0. The function $W_t(x)/\varphi_1(x)$ is bounded in N: at the points $x \in N \cap \partial\Omega$, $W_t(x)/\varphi_t(x) = |\nabla W_t(x)|/|\nabla\varphi_1(x)|$ by L'Hôspital rule. □

Remark. The above proof is taken from Kannan and Ortega [51]. Another proof of existence of a supersolution for this class of problems can be seen in Kazdan and Warner [52].

Proof of Theorem 6.10 Continued. From now on we fix $t \le t_0$ determined by Lemma 6.14. So by Lemma 6.12, $w_t \le W_t$, and in fact $w_t < W_t$ in Ω. Let

$$(6.36) \qquad C = \{u \in H_0^1 : w_t \le u \le W_t\}.$$

which is closed convex subset of H_0^1. *Plan of action:* (i) restrict Φ to C and show that it has a minimum u_0 in C which is a critical point of Φ: (ii) show that u_0 is indeed a local minimum of Φ in H_0^1; (iii) obtain a 2^{nd} solution of (6.31) using the Mountain Pass Theorem, Proposition 5.11. To accomplish the first step we need the following result.

Proposition 6.15. *Let* $\Phi : X \to \mathbb{R}$ *be a* C^1 *functional defined in a Hilbert space* X. *Let* C *be a closed convex subset of* X. *Suppose that* (i) $K \equiv I - \Phi'$ *maps* C *into* C, (ii) Φ *is bounded below in* C *and* (iii) Φ *satisfies (PS) in* C. *Then there exists* $u_0 \in C$ *such that* $\Phi'(u_0) = 0$ *and* $\mathrm{Inf}_C \Phi = \Phi(u_0)$.

Proof. Apply the Ekeland variational principle to $\Phi : C \to \mathbb{R}$. So given $\epsilon > 0$ there is $u_\epsilon \in C$ such that $\Phi(u_\epsilon) \le Inf_C\Phi + \epsilon$ and

$$(6.37) \qquad \Phi(u_\epsilon) \le \Phi(u) + \epsilon||u - u_\epsilon|| \quad \forall u \in C$$

Put in (6.37) $u = (1-t)u_\epsilon + tKu_\epsilon$ with $0 < t < 1$ and use Taylor's formula to expand $\Phi(u_\epsilon + t(Ku_\epsilon - u_\epsilon))$ about u_ϵ. We obtain

$$t||\Phi'(u_\epsilon)||^2 \le \epsilon t||\Phi'(u_\epsilon|| + o(t)$$

which implies $||\Phi'(u_\epsilon|| < \epsilon$. We then use (PS) to conclude. □

Back to the Proof of Theorem 6.10. The idea now is to apply Proposition 6.15 to the functional Φ defined in (6.31) and C defined in (6.36). However a difficulty appears in the verification of condition (i). The way we see to solve this question is to change the norm in H_0^1 as follows. We choose $M > 0$ such that the function

$$(6.38) \qquad s \mapsto g(x,s) \equiv f(x,s) + Ms + t\varphi_1(x) + h(x)$$

is increasing in $s \in [a,b]$, for each $x \in \overline{\Omega}$ fixed, where $a = \min w_t$ and $b = \max W_t$. The norms in H_0^1 given by

$$||u||^2_{H^1} = \int |\nabla u|^2 \quad \text{and} \quad ||u||^2 = \int |\nabla u|^2 + M\int u^2$$

are equivalent. Let us denote by $\langle\, ,\, \rangle$ the inner product in H_0^1 corresponding to the second norm. Next we rewrite (6.31)

$$(6.39) \qquad -\Delta u + Mu = g(x,u) \quad \text{in} \quad \Omega \quad u = 0 \quad \text{on} \quad \partial\Omega$$

The functional associated to (6.39) is

$$\Psi(u) = \frac{1}{2}\langle u,u\rangle - \int G(x,u), \quad G(x,s) = \int_0^s g(x,\xi)d\xi,$$

which is also C^1, it satisfies (PS) and it has the same critical points as the original functional Φ. Now we show that for such a functional, condition (i) of Proposition 6.15 holds. Indeed, let $u \in C$ and let $v = (I - \Psi')u$. This means

$$\langle v,\psi\rangle = \langle u,\psi\rangle - \langle u,\psi\rangle + \int g(x,u)\psi$$

for all $\psi \in H_0^1$. Then

$$\langle v - w_t,\psi\rangle \geq \int [g(x,u) - g(x,w_t)]\psi, \quad \forall\psi \in H_0^1, \quad \psi \geq 0$$

and we obtain $v \geq w_t$ using the weak maximimum principle. Similarly $v \leq W_t$. Now we apply Proposition 6.15 and get a critical point u_0 of Φ. However the proposition insures only that u_0 is a minimum of Φ restricted to C. To apply Proposition 5.11, in order to obtain a second solution, we shold now prove that u_0 is indeed a local minimum. [This is not trivial since C has empty interior in H_0^1]. Observe that $u_0 \in H_0^1$ is a solution (6.31). It follows then from the L^p regularity theory of elliptic equations that $u_0 \in C^{2,\alpha}(\overline{\Omega})$. This is proved by a standard bootstrap argument. Suppose now that u_0 is *not* a local minimum of Φ. This means that for every $\epsilon > 0$ there exists $u_\epsilon \in B_\epsilon \equiv \overline{B_\epsilon(u_0)}$ such that $\Psi(u_\epsilon) < \Psi(u_0)$. Now consider the functional Φ restricted to B_ϵ and use Theorem 3.1: there exists $v_\epsilon \in B_\epsilon$ and $\lambda_\epsilon \leq 0$ such that

$$(6.40) \qquad \Psi(v_\epsilon) = \operatorname{Inf}_{B_\epsilon}\Psi \leq \Psi(u_\epsilon) < \Psi(u_0)$$

$$(6.41) \qquad \Psi'(v_\epsilon) = \lambda_\epsilon(v_\epsilon - u_0).$$

Using again a bootstrap argument as above we conclude that $v_\epsilon \in C^{2,\alpha}(\overline{\Omega})$. (6.41) means

$$(6.42) \qquad \langle v_\epsilon,\psi\rangle - \int g(x,v_\epsilon)\psi = \lambda_\epsilon\langle v_\epsilon - u_0,\psi\rangle, \quad \forall\psi \in H_0^1.$$

Clearly $v_\epsilon \to u_0$ in H_0^1 as $\epsilon \to 0$. We have seen that u_0 and the v_ϵ's are $C^{2,\alpha}$ functions. Now we show that $v_\epsilon \to u_0$ in the norm of $C^{1,\alpha}(\overline{\Omega})$. From (6.42) we obtain

$$(6.43) \qquad (1 - \lambda_\epsilon)\langle v_\epsilon - u_0,\psi\rangle = \int [g(x,v_\epsilon) - g(x,u_0)]\psi \qquad \forall\psi \in H_0^1.$$

Again a bootstrap in the equation (6.43) gives the claimed convergence. On the other hand, it follows from Lemma 6.12 that $w_t < u_0$ in Ω and $\frac{\partial w_t}{\partial\nu} > \frac{\partial u_0}{\partial\nu}$ in $\partial\Omega$. Therefore by the above convergence we have similar inequalities for v_ϵ in place of u_0. A similar argument with W_t. Thus $v_\epsilon \in C$ and we have a contradiction!

Final Remark. As said before, problems of the Ambrosetti-Prodi have been extensively studied in the literature. A direction not touched in these notes is the question of obtain more than two solutions. A remarkable progress has been made by H. Hofer and S. Solimini through a delicate analysis of the nature of the critical points.

Chapter 7

Support Points And Support Functionals

Introduction. Let X be a Banach space and C a closed convex subset of X. We *always assume* that $C \neq X$ and $C \neq \emptyset$. A point $x_0 \in C$ is said to be a *support point* if there exists a bounded linear functional $f \in X^*$ such that $f(x_0) = \mathrm{Sup}_C f$. A given functional $f \in X^*$ is said to be a *support functional* if there exists $x_0 \in C$ such that $f(x_0) = Sup_C f$. We *always assume* that $f \neq 0$. The terminology "support" comes from the geometric fact that the hyperplane H determined by f, where $H = \{x \in X : f(x) = f(x_0)\}$, touches C at x_0 and leaves C in one of half spaces determined by H. Two basic questions will be studied in this chapter.

Problem 1. Given C a closed convex subset of X. Are all points in the boundary ∂C of x support points? If not, how large is the set of support points?

Problem 2. Given C a closed convex subset of X. Are all functionals $f \in X^*$ support functionals? If not, how large is the set of support functionals of a given C?

Six Remarks and Examples. 1) Of course the above questions make sense if f is bounded on C, i.e., there exists $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in C$. This will be achieved in particular if C is bounded. In many cases studied here we assume that C is a closed bounded convex subset of X.

2) Given f and C, it is not true in general that f supports C at some

point. For example:

$$C = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0 \ \text{ and } \ xy \geq 1\} \ \text{ and } \ f(x, y) = -y.$$

However if C is a closed convex bounded subset of $\mathbb{R}^N$ (or more generally of a reflexive Banach space X) then any continuous linear functional f supports C. This follows readily from Theorem 1.1: C is weakly compact and $-f$ is weakly continuous.

3) The previous result is false in general if X is not reflexive. Example: Let X be the Banach space of all continuous functions $x : [0,1] \to \mathbb{R}$ with $x(0) = x(1) = 0$ and the norm $\|x\|_\infty = Max\{|x(t)| : t \in [0,1]\}$. Consider the continuous linear functional $f(x) = \int_0^1 x(t)dt$ and let C be unit closed ball in $X : \{x \in X : \|x\|_\infty \leq 1\}$. Clearly f does not support C. However see Theorem 7.2 below.

4) Let C be a closed bounded convex subset of a Banach space and let $x_0 \in \partial C$. It is not true in general that there exists a functional f supporting C at x_0. Example: let

$$C = \{\xi \in \ell^2 : \xi_j \geq 0,\ \ \|\xi\|^2 = \sum_{j=1}^{\infty} \xi_j^2 \leq 1\}.$$

We claim first that $C = \partial C$. Indeed, given $\xi \in C$ and $\epsilon > 0$, let n_0 be chosen such that $|\xi_{n_0}| < \epsilon/2$. The point

$$\tilde{\xi} = (\xi_1, \ldots, \xi_{n_0-1}, -\epsilon/2, \xi_{n_0+1}, \ldots) \notin C$$

and

$$\|\xi - \tilde{\xi}\| = |\xi_{n_0} + \frac{\epsilon}{2}| < \epsilon.$$

Next we show that the points $\hat{\xi} \in C$, with $\hat{\xi}_j > 0$ and $\|\hat{\xi}\| < 1$, are not support points. Indeed, fix one such $\hat{\xi}$ and suppose that there exists a functional f such that $Sup_C f = f(\hat{\xi})$. Since $0 \in C$ it follows that $0 \leq f(\hat{\xi})$. Also there exists a $t > 1$ such that $t\hat{\xi} \in C$. So $f(t\hat{\xi}) \leq f(\hat{\xi})$, which then implies that $f(\hat{\xi}) = 0$. By the Riesz representation theorem let $f = (\eta_1, \eta_2, \ldots) \in \ell^2$. So we have

$$\sum_{j+1}^{\infty} \eta_j \hat{\xi}_j = 0,$$

which implies that there exists a j_0 such that $\eta_{j_0} > 0$. Since the point $e_{j_0} = (0, \ldots, 0, 1, 0, \ldots) \in C$, [here 1 is in the j_0^{th} component and 0 in the remaining ones] and $f(e_{j_0}) = \eta_{j_0} > 0$ contradicting the fact proved above that $\mathrm{Sup}_C f = 0$.

5) However if the closed bounded convex set C has an interior, then all points on the boundary ∂C are support points. This is an immediate consequence of the Hahn Banach theorem: given any $x_0 \in \partial C$ there exists a functional $f \in X^*$ which separates x_0 and $\mathrm{Int}C$. As we saw in the example in 4 above, if the interior of C is empty then there are points in $C(= \partial C)$ which are not support points. However, see Theorem 7.1, which then provides a satisfactory answer to Problem 1.

6) For Problem 2, the example in 3) above provides a negative answer to the first question. The second question in answered in Theorem 7.2.

Theorem 7.1. (Bishop–Phelps [12]). *Let C be a closed convex subset of a Banach space. Then the set of support points of C are dense in ∂C.*

Theorem 7.2. (Bishop–Phelps [12]). *Let C be a closed bounded convex subset of a Banach space X. Then the set of continuous linear functionals which support C is dense in X^*.*

The proof of Theorem 7.1 relies on Theorem 7.4 which will follow from the result below, whose proof uses the Ekeland Variational Principle.

Theorem 7.3. (The Drop Theorem, Daneš [30]). *Let S be a closed subset of a Banach space X. Let $y \in X \backslash S$ and $R = \mathrm{dist}(y, S)$. Let r and ρ be positive real numbers such that $0 < r < R < \rho$. Then there exists $x_0 \in S$ such that*

$$(7.1) \qquad ||y - x_0|| \leq \rho \quad and \quad D(y, r; x_0) \cap S = \{x_0\}$$

where $D(y, r; x_0) = co(\overline{B_r}(y) \cup \{x_0\})$. [This set is called a "drop", in view of its evocative geometry].

Remark. By definition $\mathrm{dist}(y, S) = Inf\{||y - x|| : x \in S\}$. If X is reflexive this infimum is achieved, but in general this is not so. The notation "co" above means the convex hull. And $\overline{B_r}(y) = \{x \in X : ||x - y|| \leq r\}$.

Proof of Theorem 7.3. By a translation we may assume that $y = 0$. Let $F = \overline{B}_\rho(0) \cap S$ which is a closed subset of X, and consequently a complete metric space with a distance induced naturally by the norm of X. Define the following functional $\Phi : F \to I\!R$ by

$$\Phi(x) = \frac{\rho + r}{R - r} ||x||.$$

By Ekeland Variational Principle, given $\epsilon = 1$ there exists $x_0 \in F$ such that

$$(7.2) \qquad \Phi(x_0) < \Phi(x) + ||x - x_0||.$$

Such an x_0 satisfies the first requirement of (7.1) and now we claim that $\{x_0\} = D(0, r; x_0) \cap S$. Suppose by contradiction that there is another point $x \neq x_0$ in this intersection. So

$$(7.3) \qquad x \in S \quad \text{and} \quad x = (1 - t)x_0 + tv$$

for some $v \in \overline{B}_r(0)$ and $0 \leq t \leq 1$.

Clearly $0 < t < 1$. From (7.3): $||x|| \leq (1 - t)||x_0|| + t||v||$, which gives

$$(7.4) \qquad t(R - r) \leq t(||x_0|| - ||v||) \leq ||x_0|| - ||x||.$$

If follows from (7.2) and (7.3) that

$$\frac{\rho + r}{R - r}||x_0|| < \frac{\rho + r}{R - r}||x|| + ||x - x_0|| = \frac{\rho + r}{R - r}||x|| + t||x_0 - v||.$$

Using (7.4) to estimate t in the above inequality and estimating $||x_0 - v|| \leq \rho + r$, we obtain $||x_0|| < ||x|| + (||x_0|| - ||x||)$, which is impossible! □

Remark. The above theorem is due to Daneš, who gave in [30] a proof, different from the above one, using the following result of Krasnoselskii and Zabreiko [55]: Let X be a Banach space and let x and y be given points in X such that $0 < r < \rho < ||x - y||$. Then

$$\mathrm{diam}[D(x, r; y) \setminus B_\rho(x)] \leq \frac{2[||x - y|| + r]}{||x - y|| - r}(||x - y|| - \rho)^n.$$

The above proof is essentially the one in Brøndsted [17]. Relations between the Drop Theorem and the Ekeland Variational Principle have been pointed out by several people, Brézis and Browder [16], Daneš [31], Penot [61].

Theorem 7.4 (Browder [20]) *Let S be a closed subset of a Banach space X. Let $\epsilon > 0$ and $z \in \partial S$. Then there exist $\delta > 0$, a convex closed cone K with non-empty interior and $x_0 \in S$ such that*

$$(7.5) \qquad \|x_0 - z\| < \epsilon \quad \text{and} \quad S \cap (x_0 + K) \cap B_\delta(x_0) = \{x_0\}.$$

Remark. For the sake of geometric images, the above theorem means that: "a closed set S satisfies a local (exterior) cone condition on a dense set of ∂S".

Proof of Theorem 7.4. Choose $y \notin S$ such that $\|z - y\| \le \epsilon/3$. Then $R \equiv dist(y, S) \le \epsilon/3$. Take $\rho = \epsilon/2$ and choose $r < R$. By the Drop Theorem, there exists $x_0 \in S$ such that

$$(7.6) \qquad \|x_0 - y\| \le \epsilon/2 \quad \text{and} \quad D(y, r; x_0) \cap S = \{x_0\}$$

Since $\|x_0 - z\| \le \|x_0 - y\| + \|y - z\| \le \epsilon/2 + \epsilon/3$, the first assertion in (7.5) follows. For the second one take $\delta < \|x_0 - y\| - r$. It suffices to prove that the points

$$(7.7) \quad x = x_0 + t(v - x_0) \quad \text{with} \quad t \ge 0,\ v \in \overline{B}_r(y),\ \|x - x_0\| < \delta$$

are in the drop $D(y, r; x_o)$. Then we would take the cone K as the set of halflines with end point at 0 and passing through the points of the ball $\overline{B}_r(y - x_0)$, i.e.

$$K = \{u \in X : u = t(v - x_0)\ ,\ \ t \ge 0 \quad v \in \overline{B}_r(y)\}$$

So $K + x_0 = \{x_0 + t(v - x_0) : t \ge 0, v \in \overline{B}_r(y)\}$ as in (7.7). To prove the above claim, all we have to do is to show that the ts in (7.7) have to be ≤ 1, and so the x in (7.7) is indeed a point in the drop $D(y, r; x_0)$. We rewrite x

$$x = x_0 + t(y - x_0) + t(v - y) \Longrightarrow t(y - x_0) + t(v - y) = x - x_0$$

Estimating we obtain $t\|y - x_0\| - t\|v - y\| \le \|x - x_0\| < \delta$ or

$$t(\|y - x_0\| - r) < \delta < \|x_0 - y\| - r \Longrightarrow t < 1. \square$$

Proof of Theorem 7.1. Let $z \in \partial C$ and $\epsilon > 0$ be given. By Theorem 7.4 there exists $x_0 \in \partial C$, K and $\delta > 0$ such that

$$(7.8) \qquad C \cap (x_0 + K) \cap B_\delta(x_0) = \{x_0\}\ ,\quad \|x_0 - z\| < \epsilon.$$

Now we claim that in fact $C \cap (x_0 + K) = \{x_0\}$. Otherwise let $x \ne x_0$ with $x \in C \cap (x_0 + K)$; then $\overline{x} = x_0 + t(x - x_0)$ for small to $t > 0$ is $\ne 0$ and belongs to $C \cap (x_0 + x) \cap B_\delta(x_0)$, contradicting (7.8). Next, based in the assertion just proved we see that C and $U = Int(K + x_0)$ are disjoint. By the Hahn Banach theorem there exists a continuous linear functional $f \in X^*$ such that $Sup_C f \le Inf_{u \in U} f(u)$. So $Sup_C f \le f(x_0)$, and indeed there is equality because $x_0 \in C$. $\square$

To prove Theorem 7.2 we will use two lemmas due to Phelps [62].

Lemma 7.6. *Let S be a closed subset of a Banach space X. Let $f \in X^*$ be such that $\|f\|_{X^*} = 1$ and $\text{Sup}_S f < \infty$, and let $0 < k < 1$. Then the set K defined below is a closed convex cone*

$$(7.9) \qquad K = \{x \in X : k\|x\| \le f(x)\}.$$

Moreover, for all $z \in S$ there exists $x_0 \in S$ such that

$$(7.10) \qquad x_0 \in z + K \qquad S \cap (x_0 + K) = \{x_0\}.$$

Proof. The verification that K is a non-empty closed convex cone is straightforward. To prove the second assertion let $F = (z + K) \cap S$ and consider the functional $\Phi : F \to \mathbb{R}$ defined as $\Phi = -f\,|_F$. Take $\epsilon < k$ and use Ekeland Variational Principle: there exists $x_0 \in F$ such that

$$(7.11) \qquad -f(x_0) < -f(x) + \epsilon\|x_0 - x\|\ ,\ \ \forall x \in F,\ \ x \ne x_0.$$

Let $y \in S \cap (x_0 + K)$. First we claim that $y \in F$; indeed, since $y - x_0 \in K$ and $x_0 - z \in K$ it follows that $y - z \in K$. Next we show that $y = x_0$. Otherwise, from (7.11)

$$(7.12) \qquad -f(x_0) < -f(y) + \epsilon\|y - x_0\|$$

Since $y - x_0 \in K$ we have $k\|y - x_0\| \le f(y - x_0)$. This together with (7.12) gives a contradiction. $\square$

Lemma 7.7. *Let C be a closed convex subset of a Banach space X. Let $f \in X^*$, $\|f\|_{X^*} = 1$ and $0 < k < 1$. Let K be as in (7.9). Suppose that $x_0 \in C$ is such that*

$$(7.13) \qquad C \cap (x_0 + K) = \{x_0\}.$$

Then there exists $0 \neq g \in X^*$ *such that*

$$(7.14) \qquad \mathrm{Sup}_C g = g(x_0) \quad \|g - f\|_{X^*} \le k.$$

Proof. Consider the functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(x) = k\|x\| - f(x)$$

Clearly Φ is continuous and convex. Now apply the Hahn-Banach theorem to separate the sets:

$$C_1 = \{(x, r) \in X \times \mathbb{R} : \Phi(x) < r\}$$

$$C_2 = \{(x, r) \in X \times \mathbb{R} : x \in C - x_0 \,,\, r = 0\}.$$

C_1 is the interior of the epigraph, epiΦ, which is open and convex. C_2 is convex and closed. $C_1 \cap C_2 = \emptyset$; otherwise there exists $x \in C - x_0$ such that $\Phi(x) < 0$, i.e. $x \in K$. So $x + x_0 \in C$ and $x + x_0 \in x_0 + K$. Using (7.13) we obtain $x = 0$. But this is impossible since $(0,0) \notin C_1$. Then by the Hahn Banach theorem there exists $F \in (X \times \mathbb{R})^*$ such that

$$(7.15) \qquad \mathrm{Sup}_{C_2} F \le \mathrm{Inf}_{C_1} F$$

Observe that corresponding to F there are (unique) $g \in X^*$ and $t \in \mathbb{R}$ such that $F(x, r) = g(x) + tr$ for all $(x, r) \in X \times \mathbb{R}$. Since $(0,0) \in C_2 \cap \overline{C}_1$ it follows that the Sup and the Inf in (7.15) are both equal to 0. Since for all $x \in X$, $(x, \Phi(x)) \in \overline{C}_1$ we see that

$$(7.16) \qquad g(x) + t\Phi(x) \ge 0$$

This shows that t cannot be equal to 0, [otherwise $g = 0$ and so $F = 0$]. The point $(0,1) \in C_1$. So $0 \le F(0,1) = t$. Thus $t > 0$, and without loss of generality we may assume $t = 1$. If follows from (7.16) that $g(x) + k\|x\| - f(x) \ge 0$ for all $x \in \mathbb{R}$, and this gives the second assertion in (7.14). Now for $x \in C$, it follows that $(x - x_0, 0) \in C_2$ and so $g(x - x_0) \le 0$ which gives the first assertion in (7.14). □

Proof of Theorem 7.2. Given $0 \neq \hat{f} \in X^*$ and $\epsilon > 0$, let $f \equiv \hat{f}/\|\hat{f}\|_{X^*}$ and K as defined in (7.9) with $k = \epsilon/\|\hat{f}\|_{X^*}$. Choose a $z \in C$. Then by Lemma 7.6 there exists $x_0 \in C$ such that $S \cap (x_0 + K) = \{x_0\}$. Now by Lemma 7.7 there exists $0 \neq g \in X^*$ such that $\|g - f\|_{X^*} \le k$ and g supports C. Denote by $\hat{g} = \|\hat{f}\|_{X^*} g$. Then $\|\hat{g} - \hat{f}\| \le \epsilon$ and $\hat{g}$ supports C. Since the previous inequality is true for any $0 < \epsilon < \|\hat{f}\|_{X^*}$ the density follows. □

Chapter 8

Convex Lower Semicontinuous Functionals

Introduction. In Chapter 5 we introduced the concept of subdifferential of a convex lower semicontinuous function Φ. We observed that dom$\partial\Phi \subset$ domΦ. Here we prove a result of Brøndsted and Rockafellar [18] which show that dom$\partial\Phi$ is dense in domΦ. The main ingredients in the proof are the Ekeland Variational Principle again and a calculus of subdifferentials. The proof here follows Aubin–Ekeland [6] and differs from the original one and also from that in Ekeland–Temam [36].

Proposition 8.1. *Let* $\Phi, \Psi : X \to \mathbb{R} \cup \{+\infty\}$ *be two convex lower semicontinuous functionals defined in a Banach space* X *and such that* $\Phi \not\equiv +\infty$ *and* $\Psi \not\equiv +\infty$. *Then*

$$(8.1) \qquad \partial(t\Phi)(x) = t\partial\Phi(x), \quad \forall t > 0 \quad \forall x \in X$$

$$(8.2) \qquad \partial(\Phi + \Psi)(x) \subset \partial\Phi(x) + \partial\Psi(x) \quad \forall x \in X.$$

Moreover if there is $\overline{x} \in$ dom$\Phi \cap$ domΨ *where one of the functionals is continuous then there is equality in* (8.2).

Proof. (8.1) and (8.2) are straightforward. The last assertion is proved using the Hahn Banach theorem; see the details in Ekeland–Temann [36]. □

Subdifferential and Differentiability. The reader has surely observed that these two notions are akin. Indeed, subdifferentials were introduced to go through situations when one does not have a differential. This was precisely what ocurred in Chapter 5. We have the following result

Proposition 8.2. *Let $\Phi : X \to I\!R \cup \{+\infty\}$ be a convex lower semicontinuous function. Suppose that Φ is Gâteaux differentiable at a point $x_0 \in \mathrm{dom}\Phi$, i.e., there exists an element of X^*, denoted by $D\Phi(x_0)$, such that*

(8.3) $$\Phi(x_0 + tv) = \Phi(x_0) + t\langle D\Phi(x_0), v\rangle + o(t), \quad \forall v \in X.$$

[The above expression is to hold for small t; how small is t depends on v]. Then

(8.4) $$x_0 \in \mathrm{dom}\partial\Phi \quad \text{and} \quad D\Phi(x_0) \in \partial\Phi(x_0).$$

Moreover

(8.5) $$\partial\Phi(x_0) = D\Phi(x_0).$$

Proof. To prove (8.4) we have to show that

(8.6) $$\Phi(y) \geq \Phi(x_0) + \langle D\Phi(x_0), y - x_0\rangle, \quad \forall y \in X.$$

It $\Phi(y) = +\infty$ there is nothing to do. Assume that $y \in \mathrm{dom}\Phi$. So the whole segment connecting x_0 to y is in $\mathrm{dom}\Phi$, and we have

$$\Phi(x_0 + t(y - x_0)) \leq (1-t)\Phi(x_0) + t\Phi(y).$$

Using (8.3) we get, for small t:

$$\Phi(x_0) + t\langle D\Phi(x_0), y - x_0\rangle + o(t) \leq (1-t)\Phi(x_0) + t\Phi(y)$$

which implies (8.6) readily. Next, let us suppose that $\overline{\mu} \in \partial\Phi(x_0)$:

(8.7) $$\Phi(y) \geq \Phi(x_0) + \langle \overline{\mu}, y - x_0\rangle \quad y \in X.$$

Now given $v \in X$ we know by assumption (8.3) that $x_0 + tv$, for small t, is in $\mathrm{dom}\Phi$. So from (8.7)

$$\Phi(x_0 + tv) - \Phi(x_0) \geq t\langle\overline{\mu}, v\rangle.$$

Assuming $t > 0$, dividing through by t and passing to the limit we obtain $\langle D\Phi(x_0), v\rangle \geq \langle\overline{\mu}, v\rangle$ for all $v \in X$, which implies $\overline{\mu} = D\Phi(x_0)$. □

Theorem 8.3. (Brøndsted–Rockafellar [18]). *Let X be a Banach space and $\Phi : X \to I\!R \cup \{+\infty\}$ a convex lower semicontinuous function, such that $\Phi \not\equiv +\infty$. Then $\mathrm{dom}\partial\Phi$ is dense in $\mathrm{dom}\Phi$. More precisely, for any $\overline{x} \in \mathrm{dom}\Phi$, there exists a sequence (x_k) in X such that*

(8.8) $$\|x_k - \overline{x}\| \leq 1/k$$

(8.9) $$\Phi(x_k) \to \Phi(\overline{x})$$

(8.10) $$\partial\Phi(x_k) \neq \emptyset \quad \text{for all} \quad k.$$

Proof. The set

$$E = \{(x, a) \in X \times I\!R : \Phi(x) \leq a\}$$

[called the *epigraph* of Φ] is closed and convex. (Prove!) Since $E \neq X$ we can take $(x_0, a_0) \notin E$ and use Hahn Banach theorem: there exists $\mu \in X^*$ and $\alpha \in I\!R$ such that

(8.11) $$\Psi(x) \equiv \Phi(x) - \langle\mu, x\rangle - \alpha > 0 \quad \text{for all} \quad x \in X.$$

[there is a small step to get (8.11) from Hahn-Banach; see a similar situation in the proof of Proposition 5.4]. Now let us apply Theorem 4.2 to Ψ with $\epsilon/2 = \Psi(\overline{x}) - Inf_X \Psi$. So for each $\lambda = 1/k$, $k \in I\!N$, we obtain x_k such that (8.8) holds, and moreover

(8.12) $$\Psi(x_k) \leq \Psi(\overline{x})$$

(8.13) $$\Psi(x_k) < \Psi(x) + \epsilon k\|x_k - x\| \quad \forall x \neq x_k.$$

Next consider the functional $\Theta : X \to I\!R \cup \{+\infty\}$ defined by

$$\Theta(x) = \Psi(x) + \epsilon k\|x - x_k\|,$$

which is convex and lower semicontinuous. From (8.13) it follows that x_k is the (unique) global minimum of Θ. So $0 \in \partial\Theta(x_k)$. Let $\Gamma : X \to I\!R$ be the convex lower semicontinuous functional defined by $\Gamma(x) = \|x - x_k\|$. By Proposition 8.1 it follows that there are $z^* \in \partial\Psi(x_k)$ and $w^* \in \partial\Gamma(x_k)$ such that $0 = z^* + \epsilon k w^*$. Using Proposition 8.1 once more and Proposition 8.2 we see that there exists $x^* \in \partial\Phi(x_k)$ such that $z^* = x^* - \mu$, proving (8.10). To prove (8.9) we rewrite (8.12) in terms of Φ:

(8.14) $$\Phi(x_k) \leq \Phi(\overline{x}) + \langle\mu, x_k - \overline{x}\rangle.$$

Using (8.9) and (8.14) we have $\limsup \Phi(x_k) \leq \Phi(\overline{x})$. On the other hand, since Φ is weakly lower semicontinuous we obtain $\Phi(\overline{x}) \leq \liminf \Phi(x_k)$. There two last inequalities prove (8.12). □

The Duality Mapping. Let X be a Banach space. In the theory of monotone operators $T : X \to 2^{X^*}$ a very important role is played by the so-called duality mapping. It essentially does in Banach space the job done by the identity in Hilbert spaces. The *duality* mapping $J : X \to 2^{X^*}$ is defined for each $x \in X$ by $J0 = 0$ and

$$(8.15) \qquad Jx = \{\mu \in X^* : \langle \mu, x\rangle = ||x||^2,\ ||\mu|| = ||x||\}, \quad \text{for} \quad x \neq 0,$$

where we use the same notation for norms in both X and X^*. Expression (8.15) is equivalent to.

$$Jx = \{\mu \in X^* : \langle \mu, x\rangle \geq ||x||^2,\ \ ||\mu|| \leq ||x||\}.$$

We can show directly from the definition, using Hahn-Banach, that Jx is a non-empty w^*-closed convex subset of X^*, for each $x \in X$. However, we will prove that and much more using the results on subdifferentials proved in Chapter 5, after we prove the following proposition.

Proposition 8.4. *Let X be a Banach space, and $\Phi : X \to \mathbb{R}$ the functional defined by $\Phi(x) = \frac{1}{2}||x||^2$. Then*

$$(8.16) \qquad \partial\Phi = J.$$

Remark. A study of generalized duality mappings can be found in Browder [23].

Proof. Let $\mu \in \partial\Phi(x)$. It suffices to consider $x \neq 0$, since $\partial\Phi(0) = 0$ and $J0 = 0$. Then

$$(8.17) \qquad \frac{1}{2}||y||^2 \geq \frac{1}{2}||x||^2 + \langle \mu, y - x\rangle \quad \forall y \in X.$$

Let $t > 0$ and $v \in X$ be arbitrary, and replace in (8.17) y by $x + tv$:

$$t\langle \mu, v\rangle \leq \frac{1}{2}||x + tv||^2 - \frac{1}{2}||x||^2 \leq t||x||\,||v|| + \frac{1}{2}t^2||v||^2.$$

Dividing through by t and passing to the limit

$$\langle \mu, v\rangle \leq ||x||\,||v|| \quad \forall v \in X.$$

This implies $||\mu|| \leq ||x||$. On the other hand, let $y = tx$ in (8.17):

$$(8.18) \qquad \frac{1}{2}(t^2 - 1)||x||^2 \geq (t - 1)\langle \mu, x\rangle.$$

For $0 < t < 1$ we obtain from (8.18) $\frac{1}{2}(t + 1)||x||^2 \leq \langle \mu, x\rangle$. Letting $t \to 1$ we have $\langle \mu, x\rangle \geq ||x||^2$, completing the proof that $\mu \in Jx$. The other way around, assume now that $\mu \in Jx$, and we claim that (8.17) holds. Let us estimate the right side of (8.17) using the properties of μ:

$$\langle \mu, y\rangle - \langle \mu, x\rangle + \frac{1}{2}||x||^2 \leq ||x||\,||y|| - ||x||^2 + \frac{1}{2}||x||^2,$$

and clearly the right side of the above inequality is $\leq \frac{1}{2}\,||y||^2$. □

Remark 1. If $\mu \in J(x_0)$ it follows from the above that μ *supports the ball* of radius $||x_0||$ around 0 at the point x_0. Indeed if $||y|| \leq ||x||$, then $\langle \mu, y\rangle \leq \langle \mu, x\rangle$. I.e. $Sup\{\langle \mu, y\rangle : ||y|| \leq ||x_0||\} = \langle \mu, x_0\rangle$.

Conversely, if a functional μ supports the unit ball at a point x_0, then $\mu \in J(||\mu||x_0)$. Indeed, from

$$\langle \mu, x_0\rangle = Sup_{||y|| \leq 1}\langle \mu, y\rangle$$

we obtain $\langle \mu, x_0\rangle = ||\mu||$, which proves the claim.

Remark 2. Let us call $\mathcal{R}(J) = \cup\{Jx : x \in X\}$. If X is a reflexive Banach space, the above remark says that $\mathcal{R}(J) = X^*$. If X is not reflexive there is a result of R. C. James [49] which says that there are functionals which do not support the unit ball. So for non reflexive Banach spaces $\mathcal{R}(J) \neq X^*$. However the Bishop Phelps theorem proved in Chapter 7 says that $\mathcal{R}(J)$ is *dense* in X^*, for all Banach spaces.

Gâteaux Differentiability of the Norm. As in the previous section let $\Phi(x) = \frac{1}{2}||x||^2$. Since $\Phi : X \to \mathbb{R}$ is continuous and convex we have by Proposition 5.4 that

$$(8.19) \qquad \lim_{t \downarrow 0} \frac{\Phi(x + ty) - \Phi(x)}{t} = \mathrm{Max}_{\mu \in Jx}\langle \mu, y\rangle, \quad t > 0.$$

It follows from (8.19) that

$$(8.20) \qquad \lim_{t \uparrow 0} \frac{\Phi(x + ty) - \Phi(x)}{t} = \mathrm{Min}_{\mu \in Jx}\langle \mu, y\rangle, \quad t < 0.$$

The functional Φ is Gâteaux differentiable if and only if the two limits in (8.19) and (8.20) are equal and in fact there is a continuous linear functional, noted by $D\Phi(x)$ and called the Gâteaux derivative at x, such there limits are equal to $\langle D\Phi(x), y\rangle$. So the existence of the Gâteaux derivative of Φ at a certain point x implies that

$$\mathrm{Max}_{\mu\in Jx}\langle \mu, y\rangle = \mathrm{Min}_{\mu\in Jx}\langle \mu, y\rangle \quad \text{for all} \quad y \in Y.$$

Clearly this implies that Jx is a singleton, and $Jx = D\Phi(x)$. The converse is clearly true: if Jx is singleton then (8.19) and (8.20) imply that Φ is Gâteaux differentiable at x and $D\Phi(x) = Jx$. So we have proved.

Proposition 8.5 *Let X be a Banach space. $\Phi(x) = \frac{1}{2}\,||x||^2$ is Gâteaux differentiable at a point x if and only if Jx is a singleton. Moreover $Jx = D\Phi(x)$. In particular, the duality mapping is singlevalued, $J : X \to X^*$, if and only if $\Phi(x) = \frac{1}{2}\,||x||^2$ is Gâteaux differentiable at all points $x \in X$.*

Remark 1. Which geometric properties of the Banach space X give a singlevalued duality mapping? For the definitions below let $B_1 = \{x \in X : ||x|| \le 1\}$ and $\partial B_1 = \{x \in X : ||x|| = 1\}$. In the terminology introduced in Chapter 7, we see from the Hahn Banach theorem that all points of ∂B_1 are support points, i.e. given $x \in \partial B_1$ there exists a functional $\mu \in X^*$ such that

$$\mathrm{Sup}_{B_1}\mu = \langle \mu, x\rangle.$$

In geometric terms we say that the ball B_1 has a hyperplane of support at each of its boundary points. A Banach space X is said to be *strictly convex* if given $x_1, x_2 \in \partial B_1$, with $x_1 \ne x_2$, then $||tx_1 + (1-t)x_2|| < 1$ for all $0 < t < 1$. Another way of saying that it is: each hyperplane of support touches ∂B_1 at a unique point. (Or still ∂B_1 contains no line segments). A Banach space X is said to be *smooth* if each point $x \in \partial B_1$ possesses only one hyperplane of support. Examples in $\mathbb{R}^2$ with different norms: (i) the Euclidean norm $||x||^2 = x_1^2 + x_2^2$ is both strictly convex and smooth; (ii) the sup norm $||x|| = \mathrm{Sup}\{|x_1|, |x_2|\}$ is neither strictly convex non smooth; (iii) the norm whose unit ball is $\{(x_1, x_2) : -1 \le x_1 \le 1,\ x_1^2 - 1 \le x_2 \le 1 - x_1^2\}$ is strictly convex but not smooth: (iv) the norm whose unit ball is the union of the three sets next is smooth but not strictly convex:

$$\{(x_1, x_2) : -1 \le x_1, x_2 \le 1\},\ \{(x_1, x_2) : x_1 \ge 1, (x_1 - 1)^2 + x_2^2 \le 1\}$$

and

$$\{(x_1, x_2) : x_1 \le -1, (x_1 + 1)^2 + x_2^2 \le 1\}.$$

Remark 2. With the terminology of the previous remark, Proposition 8.5 states: " *$\Phi(x) = \frac{1}{2}\,||x||^2$ is Gâteaux differentiable if and if X is smooth*". No condition on reflexivity is asked from X.

Remark 3. If X^* is strictly convex then J is singlevalued. Indeed, for each $x_0 \in X$, Jx_0 is a convex subset of the set $\{\mu \in X^* : ||\mu|| = ||x_0||\}$, which is a singleton in the case when X^* is strictly convex. So X^* strictly convex implies that X is smooth, in view of Proposition 8.5 and Remark 2 above.

Remark 4. There is a duality between strict convexity and smoothness in finite dimensional Banach spaces: X is strictly convex [resp. smooth] if and only if X^* is smooth [resp. strictly convex]. Such a result does not extend to all Banach spaces, see Beauzamy [8: p. 186] for an example. However this is true for reflexive Banach spaces. This was first proved by Šmulian [75], and it follows readily from Remark 3 above and Remark 5 below.

Remark 5. X^* smooth implies X strictly convex. Indeed suppose that X is not strictly. Then there are $x_1, x_2 \in \partial B_1$ such that $\bar{x} = \frac{1}{2}(x_1 + x_2) \in \partial B_1$. $J\bar{x} \in \partial B_1^*$ where $\partial B_1^* = \{\mu \in X^* : ||\mu|| = 1\}$. We now claim that x_1 and x_2 viewed as elements of X^{**} are two support functionals of B_1^* at $J\bar{x}$, contradicting the smoothness of X^*. In fact

$$1 = \langle J\bar{x}, \bar{x}\rangle = \frac{1}{2}\langle J\bar{x}, x_1\rangle + \frac{1}{2}\langle J\bar{x}, x_2\rangle \tag{8.21}$$

implies $\langle J\bar{x}, x_1\rangle = 1$ and $\langle J\bar{x}, x_2\rangle = 1$, proving the claim.

Fréchet Differentiability. We know that a functional $\Phi : X \to \mathbb{R}$ which is Gâteaux differeintiable is not in general Fréchet differentiable. However

Proposition 8.6. *Let X be a Banach space, A functional $\Phi : X \to \mathbb{R}$ is continuously Fréchet differentiable (i.e. C^1) if and only if it is continuously Gâteaux differentiable.*

Proof. One of the implications is obvious. Let us assume that Φ is Gâteaux differentiable in a neighborhood V of point $u_0 \in X$ and that the mapping $x \in V \mapsto D\Phi(x) \in X^*$ is continuous. We claim that $D\Phi(u_0)$ is the Fréchet derivative at u_0, and, indeed:

$$\Phi(u_0 + v) - \Phi(u_0) - \langle D\Phi(u_0), v\rangle = o(v). \tag{8.22}$$

The real-valued function $t \in [0,1] \mapsto \Phi(u_0 + tv)$ is differentiable for small v. So by the mean value theorem $\Phi(u_0 + v) - \Phi(u_0) = \langle D\Phi(u_0 + \tau v), v\rangle$, which

holds for small v and some $\tau \in [0,1]$. So we could estimate the left side of (8.22) by $|D\Phi(u_0 + \tau v) - D\Phi(u_0)|$, and using the continuity of the Gâteaux derivative we finish. □

It is clear that a functional $\Phi : X \to \mathbb{R}$ could be Fréchet differentiable without being C^1. However this is not the case if $\Phi(x) = \frac{1}{2}||x||^2$.

Proposition 8.7. *Let X be a Banach space. If $\Phi(x) = \frac{1}{2}||x||^2$ is Fréchet differentiable [which implies Gâteaux differentiable and $\partial\Phi = J$, where J is singlevalued] then $J : X \to X^*$ is a continuous mapping.*

Remark. Without additional assumptions on the Banach space, a singlevalued duality mapping $J : X \to X^*$ is continuous from the strong topology of X to the w^*-topology of X^*. For simplicity let us sketch the proof using sequences. Since the w^*-topology needs not to be metrizable, filters should be used, see Beauzamy [24 ; p. 177]. Let $x_n \to x$ in X. Since $||Jx_n||_{X^*} \le$ const, there is $\mu \in X^*$ such that

$$Jx_n \overset{w^*}{\rightharpoonup} \mu.$$

We claim that $\mu = Jx$. First, from $\langle Jx_n, x_n\rangle = ||x_n||^2$ we conclude $\langle \mu, x\rangle = ||x||^2$. Next given $\epsilon > 0$ there exists $u \in X$ with $||u|| = 1$ such that $||\mu|| \le \langle \mu, u\rangle + \epsilon \le \langle Jx_n, u\rangle + \epsilon \le ||x_n|| + \epsilon$ for large n. Passing to the limit, and since $\epsilon > 0$ is arbitrary $||\mu|| \le ||x||$. □

Proof of Proposition 8.7. Suppose by contradiction that there is a sequence $x_n \to x$ and $r > 0$ such that $||Jx_n - Jx|| > 2r$, for all n. So for each n there exists $y_n \in X$, $||y_n|| = 1$ such that

$$\langle Jx_n - Jx, y_n\rangle > 2r. \tag{8.23}$$

Using the Fréchet differentiability of Φ we can find $\delta > 0$ such that

$$|\Phi(x + y) - \Phi(x) - \langle Jx, y\rangle| \le r||y|| \quad \text{for} \quad ||y|| \le \delta. \tag{8.24}$$

On the other hand we have

$$\Phi(x + \delta y_n) - \Phi(x_n) \ge \langle Jx_n, x + \delta y_n - x_n\rangle. \tag{8.25}$$

Now using (8.24) we estimate

$$\langle Jx_n - Jx, \delta y_n\rangle \le \Phi(x + \delta y_n) - \Phi(x_n) + \langle Jx_n, x_n - x\rangle - \langle Jx, \delta y_n\rangle$$

which is equal to

$$(8.26)\quad \Phi(x+\delta y_n) - \Phi(x) - \langle Jx, \delta y_n\rangle + \Phi(x) - \Phi(x_n) + \langle Jx_n, x_n - x\rangle.$$

The first three terms in (8.26) we estimate using (8.24). So starting with (8.23) we get

$$2r\delta < r\delta + \Phi(x) - \Phi(x_n) + \|x_n\|\,\|x_n - x\|$$

which implies that $\Phi(x_n)$ does not converge to $\Phi(x)$. This contradicts the continuity of Φ. □

As a consequence of the previous propositions we have the following characterization of Fréchet differentiability of the norm:

Proposition 8.8. *Let X be a Banach space. $\Phi(x) = \frac{1}{2}\|x\|^2$ is Fréchet differentiable if and only if the duality mappings is singlevalued and continuous.*

Remark. Which geometric properties of the Banach space X give a continuous singlevalued duality mapping? We start with a condition introduced by Šmulian [75]. X^* satisfies *condition* (S) if for each $x \in \partial B_1$ we have

$$(8.27)\qquad \lim_{\delta\to 0} \operatorname{diam} A_x(\delta) = 0,$$

where $A_x(\delta) = \{\mu \in X^* : \langle \mu, x\rangle \geq 1-\delta\} \cap B_1^*$.

Proposition 8.9. *Let X be a Banach space. The duality mapping is singlevalued and continuous if and only if X^* satisfies (S).*

Proof. (i) First assume (S). Suppose that there exists $x \in \partial B_1$ such that Jx contains $\mu_1 \neq \mu_2$. Clearly $\mu_1,\ \mu_2 \in A_x(\delta)$, for all $\delta > 0$, and $\|\mu_1 - \mu_2\| > 0$ negates (8.27). To show the continuity of J, it suffices to prove that if $x_n \to x$, with $\|x_n\| = 1$, then $Jx_n \to Jx$. We know that

$$Jx_n \overset{w^*}{\rightharpoonup} Jx,$$

so it is enough to show that (Jx_n) is a Cauchy sequence. Given $\epsilon > 0$, choose $\delta > 0$ such that $\operatorname{diam} A_x(\delta) \leq \epsilon$. We know that $\langle Jx_n, x\rangle \to \langle Jx, x\rangle = 1$. So there exists n_0 such that $Jx_n \in A_x(\delta)$ for all $n \geq n_0$. Using condition (S) we conclude that $\|Jx_n - Jx_m\| \leq \epsilon$ for all $n,\ m \geq n_0$.

(ii) Conversely, assume by contradiction that (S) does not hold, for some

$x \in \partial B_1$. So there exists $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$ we can find $\mu_n,\ \mu_n'$ in B_1^* with the properties

$$\langle \mu_n, x\rangle \geq 1 - \frac{1}{n},\quad \langle \mu_n', x\rangle \geq 1 - \frac{1}{n},\quad \|\mu_n - \mu_n'\| \geq \epsilon_0.$$

We have seen, Remark 2 after Proposition 8.4, that $\mathcal{R}(J)$ is dense in X^*. So we can find x_n, y_n in X such that

$$(8.28)\quad \langle Jx_n, x\rangle \geq 1 - \frac{2}{n},\quad \langle Jy_n, x\rangle \geq 1 - \frac{2}{n},\quad \|Jx_n - Jy_n\| \geq \frac{\epsilon_0}{2},$$

$$(8.29)\qquad \|Jx_n\| \leq 1 + \frac{1}{n},\qquad \|Jy_n\| \leq 1 + \frac{1}{n}.$$

Observe that the sequences (Jx_n), (Jy_n) are bounded in X^*. By the Banach Alaoglu Theorem, there exists μ and μ' in X^* such that

$$Jx_n \overset{w^*}{\rightharpoonup} \mu,\qquad Jy_n \overset{w^*}{\rightharpoonup} \mu'.$$

(As usual take subsequences if necessary). Passing to the limit in (8.28) we obtain $\langle \mu, x\rangle \geq 1,\ \ \langle \mu', x\rangle \geq 1$. From (8.29) it follows that in fact we have $\langle \mu, x\rangle = \langle \mu', x\rangle = 1$, which implies $\mu = \mu' = Jx$. On the other hand, from the last assertion in (8.28) we can find $z \in B_1$ such that

$$\langle Jx_n - Jy_n, z\rangle \geq \frac{\epsilon_0}{4}.$$

Passing to the limit in this inequality we come to a contradiction. □

Remark. Now we give a geometric condition which is sufficient to having the continuity and singlevaluedness of J. Some definitions. A Banach space X is said to be *uniformly convex* (Clarkson [26]) if given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $x,\ y \in \partial B_1$ and $\|\frac{1}{2}(x+y)\| \ngeq 1-\delta$ then $\|x-y\| \nleq \epsilon$. A Banach space X is said to be *locally uniformly convex* (Lovaglia [58]) if given $\epsilon > 0$ and $x_0 \in \partial B_1$ there exists $\delta = \delta(\epsilon, x_0)$ such that if $x \in B_1$ and $\|\frac{1}{2}(x + x_0)\| \ngeq 1 - \delta$ then $\|x - x_0\| \nleq \epsilon$. The previous two definitions can be given in terms of sequences as follows. X is *uniformly convex* if given any two sequences (x_n) and (y_n) in B_1 such that $\|\frac{1}{2}(x_n + y_n)\| \to 1$ then $\|x_n - y_n\| \to 0$. X is locally uniformly convex if given any point $x_0 \in \partial B_1$ and any sequence (x_n) in B_1 such that $\|\frac{1}{2}(x_0 + x_n)\| \to 1$ then $x_n \to x_0$. A Banach space X is said to satisfy *Property H* (Fan–Glicksberg [40]) if X is strictly convex and

$$(8.30)\qquad x_n \rightharpoonup x_0,\quad \|x_n\| \to \|x_0\| \ \Rightarrow\ x_n \to x_0.$$

Hilbert spaces are uniformly convex. A uniformly convex Banach space is locally uniformly convex. A locally uniformly convex Banach space satisfies Property (H). The first assertion is easily verified using the fact that $x+y$ is orthogonal to $x-y$, for $x, y \in \partial B_1$. The second is trivial. And the third is proved as follows. It is clear that a locally uniformly convex Banach space is strictly convex. To prove (8.20) we may assume that $||x_n|| \leq 1$ and $||x_0|| = 1$. We claim that $x_n \rightharpoonup x_0$ and $||x_n|| \to ||x_0||$ implies that $||\frac{1}{2}(x_n + x_0)|| \to 1$. Once this is done, the fact that X us supposed locally uniformly convex implies that $x_n \to x$. Suppose by contradiction that a subsequence of (x_n), denoted by (x_n) again, is such that $||\frac{1}{2}(x_n + x_0)|| \leq t < 1$, for all n. Let $\mu \in Jx_0$. Then

$$\langle \mu, \frac{1}{2}(x_0 + x_n)\rangle \leq \langle \mu, tx_0\rangle$$

which implies $\frac{1}{2} + \frac{1}{2}\langle \mu, x_n\rangle \leq t$. Passing to the limit we come to a contradiction.

Proposition 8.10. *Let X be a Banach space and suppose that X^* is locally uniformly convex. Then the duality mapping is singlevalued and continuous.*
Proof. Singlevaluedness is clear. Now let $x_n \to x_0$ in X. As in Proposition 8.9 we may suppose $||x_n|| = 1$. We know that

$$Jx_n \overset{w^*}{\rightharpoonup} Jx_0.$$

Suppose by contradiction (passing to a subsequence) that $\frac{1}{2}||Jx_n + Jx_0|| \leq t < 1$. Then

$$(Jx_n + Jx_0, x_n + x_0) \leq 4t.$$

On the other hand the left side of the above inequality is equal to

$$(Jx_n, x_n) + (Jx_0, x_n) + (Jx_n, x_0) + (Jx_0, x_0) = 2 + (Jx_0, x_n) + (Jx_n, x_0)$$

which converges to 4. Impossible! □

Proposition 8.11. *Let X be a reflexive Banach space. Then J is singlevalued and continuous if and only if X^* satisfies Property H.*
Proof. It can easily be seen that, in the case of reflexive spaces, Property H on X^* implies the said properties on J. To prove the converse, we use Propostion 8.9 and show that Property S implies Property H on X^*; of course reflexivity is used again. Suppose by contradiction that there exists a sequence

$$\mu_n \rightharpoonup \mu_0, \quad ||\mu_n|| \to ||\mu_0||, \quad \mu \not\to \mu_0.$$

We may suppose without loss of generality that $||\mu_n|| \leq 1$ and $||\mu_0|| = 1$. So there exist $\epsilon_0 > 0$ and sequences (j_n) and (k_n) going to ∞ such that

$$||\mu_{j_n} - \mu_{k_n}|| \geq \epsilon_0. \tag{8.31}$$

Since J is singlevalued continuous and onto, we can find x_0 in X with $||x_0|| = 1$ such that $\mu_0 = Jx_0$. Consider the set $A_{x_0}(\delta)$ defined in (8.27) and choose $\delta > 0$ such that $\operatorname{diam} A_{x_0}(\delta) < \epsilon_0/2$. Clearly $\mu_n \in A_{x_0}(\delta)$ for large n. But this contradicts (8.30). □

Uniform Fréchet Differentiability of the Norm. In this section we limit ourselves to Proposition 8.12 below. We refer to the books of Beauzamy [8] and Diestel [32] for more on this subject. A concept of uniform smothness can be introduced and be shown to enjoy a duality with uniform convexity, just like smoothness and strict convexity do.

Proposition 8.12. *The duality mapping J is singlevalued and uniformly continuous on bounded subsets of X if and only if X^* is uniformly convex.*
Proof. (i) We first prove that X^* uniformly convex implies the said properties on J. From Propostion 8.10 it follows that J is singlevalued and continuous. To prove uniform continuity we preceed by contradiction. Assume that the exists $\epsilon_0 > 0$ and sequences (x_n), (y_n) in some fixed bounded subset of X such that

$$||x_n - y_n|| < \frac{1}{n} \quad \text{and} \quad ||Jx_n - Jy_n|| \geq \epsilon_0.$$

We may assume that $||x_n|| = ||y_n|| = 1$. Now we claim that $\frac{1}{2}||Jx_n + Jy_n|| \to 1$ as $n \to \infty$, arriving then at a contradiction through the use of the uniform convexity of X^*. To prove the claim just look at the identity

$$\langle Jx_n + Jy_n, x_n\rangle = \langle Jx_n, x_n\rangle + \langle Jy_n, y_n\rangle + \langle Jy_n, x_n - y_n\rangle$$

and estimate to obtain

$$2 \geq \langle Jx_n + Jy_n, x_n\rangle \geq 2 - 2||x_n - y_n||.$$

Now we assume that J is singlevalued and uniformly continuous on bounded sets. By the fact that $\mathcal{R}(J)$ is dense in X^* it suffices to show that given $\epsilon > 0$

there exists a $\delta > 0$ such that

$$\|Jx - Jy\| \geq \epsilon \Rightarrow \frac{1}{2}\|Jx + Jy\| \leq 1 - \delta \tag{8.32}$$

for $x, y \in \partial B_1$. First we write an identity for $u, v \in \partial B_1$

$$\langle Jx + Jy, u\rangle = -\langle Jx - Jy, v\rangle + \langle Jx, u + v\rangle + \langle Jy, u - v\rangle. \tag{8.33}$$

Observe that the sup of the left side with respect to $u \in \partial B_1$ is the norm of $Jx + Jy$, and the sup of the first term in the right side with respect to $v \in \partial B_1$ gives the norm of $Jx - Jy$. Next let $0 < \epsilon' < \epsilon$ and choose $\overline{v} \in \partial B_1$ such that

$$\langle Jx - Jy, \overline{v}\rangle \geq \epsilon'. \tag{8.34}$$

Now choose $0 < \xi < \epsilon'$. By the uniform continuity of J there is $\eta > 0$ such that

$$\|Jz_1 - Jz_2\| \leq \xi \quad \text{if} \quad \|z_1 - z_2\| \leq 4\eta. \tag{8.35}$$

Now take $v = \eta\overline{v}$ in (8.33) and let us estimate separately the three terms in the right side of (8.33). The first is trivially estimated by $-\epsilon'\eta$. The other two are estimated as follows:

$$\langle Jx, u + v\rangle + \langle Jy, u - v\rangle \leq \|u + v\| + \|u - v\|.$$

Let $s = (u + v)/\|u + v\|$ and $d = (u - v)/\|u - v\|$, and write the identity

$$\|u + v\| + \|u - v\| = \langle Js, u\rangle + \langle Jd, u\rangle + \langle Js - Jd, v\rangle. \tag{8.36}$$

Then estimate (8.36) using (8.35) and get

$$\|u + v\| + \|u - v\| \leq 2 + \xi\|v\| \leq 2 + \xi\eta.$$

Finally (8.33) is estimated by $-\epsilon'\eta + 2 + \xi\eta$, for all $u \in \partial B_1$. So

$$\frac{1}{2}\|Jx + Jy\| \leq 1 - \frac{\epsilon' - \xi}{2}\eta.$$

and (8.32) is proved. □

Remark. The functional Φ is said to be ***uniformly Fréchet differentiable*** if it is Fréchet differentiable and if given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\Phi(x + u) - \Phi(x) - \langle\Phi'(x), u\rangle| \leq \epsilon\|u\|$$

for all $x \in B_1$ and all $\|u\| \leq \delta$. It is easy to see that the uniform differentiability of Φ is equivalent to J being singlevalued and uniformly continuous on bounded sets. So the norm of a Banach space is uniformly Fréchet differentiable iff X^* is uniformly convex.

Chapter 9

Normal Solvability

Introduction. Let X and Y be Banach spaces. Let $f : X \to Y$ be a given function. This section is devoted to questions relative to the solvability of the equation $f(x) = y$, when $y \in Y$ is given. The function f is supposed to be Gâteaux differentiable and we would like to have sufficient conditions for the solvability of the above equation stated in terms of the properties of the Gâteaux derivative Df_x. Parallelling the Fredholm theory for compact operators these conditions will naturally involve Df_x^*. For the sake of later referencing we start by recalling some results from the theory of linear operators. Then we go to the so-called normal solvability results of F. E. Browder and S. I. Pohožaev. Some results of W. O. Ray and I. Ekeland are also discussed. We close the section with a comparative study of the results here with the classical inverse mapping theorem.

What is Normal Solvability Let $L : X \to Y$ be a bounded linear operator from a Banach space X to a Banach space Y. The equation $Lx = y$ is said to be ***normally solvable*** (in the sense of Hausdorff) if

$$y \in N(L^*)^\perp \Rightarrow y \in \mathcal{R}(L). \tag{9.1}$$

Here $L^* : Y^* \to X^*$ is the adjoint operator defined as follows for each $\mu \in Y^*$, $L^*\mu \in X^*$ is given by $\langle L^*\mu, x\rangle = \langle \mu, Lx\rangle$ for all $x \in X$. The other notations in (9.1) are: (i) $N(T)$ to denote the kernel of an operator $T : X \to Y$, i.e. $N(T) = \{x \in X : Tx = 0\}$; (ii) the range of T, $\mathcal{R}(T) =$

$\{y \in Y : \exists x \in X \text{ s.t. } Tx = y\}$, and the (right) polar of a subspace $B \subset X^*$, $B^{\perp} = \{x \in X : \langle \mu, x \rangle = 0, \ \forall \mu \in B\}$. Later on we also use the (left) polar of a subspace $A \subset X$ defined as

$$A^{\perp} = \{\mu \in X^* : \langle \mu, x \rangle = 0 \quad \forall x \in A\}.$$

The well-known Fredholm alternative for compact linear operators $T : X \to X$ in a Banach space X gives that the operator $L = I - T$ is normally solvable. For a general bounded linear operator $L : X \to Y$, one can prove that

$$N(L^*)^{\perp} = \overline{\mathcal{R}(L)}. \tag{9.2}$$

So all operators L with a closed range are normally solvable.

Some Results From The Linear Theory. We now recall some theorems from the theory of bounded linear operators in Banach spaces. The reader can find the proofs in many standard texts in Functional Analysis, see for instance Yosida [79], Brézis [15].

Theorem 9.1. (Closed range Theorem). *Let $L : X \to Y$ be a bounded linear operator, X and Y Banach spaces. The following properties are equivalent:* (i) *$\mathcal{R}(L)$ is closed,* (ii) *$\mathcal{R}(L^*)$ is closed,* (iii) *$\mathcal{R}(L) = N(L^*)^{\perp}$,* (iv) *$\mathcal{R}(L^*) = N(L)^{\perp}$.*

Theorem 9.2. (Surjectivity Theorem). *Let $L : X \to Y$ be a bounded linear operator. The following conditions are equivalent* (i) *$\mathcal{R}(L) = Y$,* (ii) *$N(L^*) = \{0\}$ and $\mathcal{R}(L^*)$ is closed,* (iii) *there exists a constant $C > 0$ such that $\|y^*\| \le C\|L^*y^*\|$.*

Theorem 9.3. (Surjectivity Theorem for the adjoint). *Let $L : X \to Y$ be a bounded linear operator. The following conditions are equivalent:* (i) *$\mathcal{R}(L^*) = X^*$,* (ii) *$N(L) = \{0\}$ and $\mathcal{R}(L)$ is closed,* (iii) *there exists a constant $C > 0$ such that $\|x\| \le C\|Lx\|$.*

Normal Solvability of Nonlinear Operators. Now we describe a non-

linear analogue of Theorem 9.2. This result was proved by Pohožaev [63] (see also [64], [65]) for reflexive Banach spaces and $f(X)$ weakly closed, and by Browder [20], [21], for general Banach spaces. The Browder papers mentionned above contain much more material on normal solvability besides the simple results presented here.

Theorem 9.4 *Let $f : X \to Y$ be a Gâteaux differentiable function between Banach spaces X and Y. Assume that $f(X)$ is closed. Let us use the notation Df_x for the Gâteaux derivative at a point $x \in X$. Assume that $N(Df_x^*) = \{0\}$ for all $x \in X$. Then f is surjective.*

The above result follows from a more general one, (namely Theorem 9.5) due also to Browder. The proof below follows the same spirit of Browders original proof. However it uses a more direct approach [directness is a function of the arrangement one sets in one's presentation!] through the Drop Theorem (Theorem 7.3), proved in Chapter 7 via the Ekeland Variational Principle.

Theorem 9.5. *Let X and Y be Banach spaces, and $f : X \to Y$ a Gâteaux differentiable function. Assume that $f(X)$ is closed. Let $y \in Y$ be given and suppose that there are real numbers $\rho > 0$ and $0 \le p < 1$ such that*

$$f^{-1}(B_\rho(y)) \neq \emptyset \tag{9.3}$$

$$\operatorname{Inf}\{\|y - f(x) - z\| : z \in \overline{\mathcal{R}(Df_x)}\} \le p\|y - f(x)\|, \tag{9.4}$$

for all $x \in f^{-1}(B_\rho(y))$. Then $y \in f(X)$.

Remark. If (9.3) and (9.4) holds simultaneously for each $y \in Y$, then f is surjective. Observe that a large ρ gives (9.3), but then (9.4) is harder to be attained.

Remark 2. Proof of Theorem 9.4. $N(Df_x^*) = \{0\}$ implies, by (9.2), that (9.4) is attained with $p = 0$ and arbitrary ρ. So, for each given y, take ρ such that $\operatorname{dist}(y, f(X)) < \rho$, and take $p = 0$. Therefore Theorem 9.5 implies Theorem 9.4.

Remark 3. The thesis of Theorem 9.4 still holds if in the hypotheses we replace $N(Df_x^*) = \{0\}$ by $\mathcal{R}(Df_x)$ dense in Y. Theorem 9.4 contains a result of Kačurovskii [35], who considered continuously Fréchet differentiable mappings f and assumed that $\mathcal{R}(Df_x) = Y$ for all $x \in X$. The proof uses a Newton–Kantorovich method of successive approximations. See Remark 1 after Theorem 9.8.

Remark 4. A Gâteaux differentiable mapping $f : X \to Y$ is said to be a *Fredholm mapping* if $Df_x : X \to Y$ is a Fredholm (linear) operator for each $x \in X$. We recall that a bounded linear operator $L : X \to Y$ is Fredholm if $N(L)$ is finite dimensional and $\mathcal{R}(L)$ is closed and has finite codimension. The index $i(L)$ is defined as $i(L) = \dim N(L) - \operatorname{codim}\mathcal{R}(L)$. We observe that $i(Df_x)$ for a Fredholm mapping f is locally constant. Since X is connected we can then define

$$i(f) = i(Df_x) \quad \text{for some} \quad x \in X$$

since the right side is independent of x. Now if in Theorem 9.4 we assume that f is a Fredholm mapping of index 0, then condition $N(Df_x^*) = \{0\}$ can be replaced by $N(Df_x) = \{0\}$.

Proof of Theorem 9.5. Let $S = f(X)$. Suppose by contradiction that $y \notin S$. Let $R = \operatorname{dist}(y, S)$ and choose $r, \rho > 0$ such that $r < R < \rho$ and $p\rho < r$. Observe that if (9.3) and (9.4) hold for some ρ_0 then it also holds for any other ρ, with $R < \rho \le \rho_0$. Then use the Drop Theorem: there exists $u_0 \in S$

(9.5) $\qquad \|u_0 - y\| < \rho \quad \text{and} \quad S \cap D(y; r; u_0) = \{u_0\}.$

Now let $x_0 \in X$ be such that $f(x_0) = u_0$, ~~whose existence follows from a (9.3) and the first assertion in (9.5)~~. Then (9.4) implies

$$\operatorname{Inf}\{\|y - f(x_0) - z\| : z \in \overline{\mathcal{R}(Df_{x_0})}\} \le p\|y - f(x_0)\| < r.$$

So there exists $x \in X$ such that

(9.6) $\qquad \|y - f(x_0) - Df_{x_0}(x)\| < r,$

and approximating the Gâteaux derivative by the Newton quocient one has for small $t > 0$:

$$\|w_t\| \equiv \|y - f(x_0) - \frac{f(x_0 + tx) - f(x_0)}{t}\| < r.$$

Thus the vector $y - w_t \in D(y, r; u_0)$, and the same is true for $(1 - t)u_0 + t(y - w_t)$ with $0 < t < 1$ and t small. But this last statement simply says that

(9.7) $\qquad f(x_0 + tx) \in D(y, r; u_0), \quad \forall t > 0 \ \text{small}.$

The second assertion in (9.5) and (9.7) imply that

$$f(x_0 + tx) = u_0 \quad \forall t > 0 \quad \text{small}.$$

which gives $Df_{x_0}(x) = 0$. Going back to (9.6) we get $\|y - f(x_0)\| < r$, which is impossible. □

Some Surjectivity Results. Both theorems 9.4 and 9.5 have as hypothesis the statement that $f(X)$ is a closed set. This is a global assumption whose verification may cause difficulties when applying those theorems. It would be preferable to have local assumptions instead. That it is the contents of the next result which is due to Ekeland, see Bates-Ekeland [7]; see also Ray-Rosenholtz [69] for a slightly more general result. Observe that the function f is assumed to be continuous in the next theorem. This implies that the graph of f closed, but asserts nothing like that about $f(X)$.

Theorem 9.6. *Let X and Y be Banach spaces and $f : X \to Y$ a continuous mapping, which is Gâteaux differentiable. Assume:*

(9.8) $\qquad \mathcal{R}(Df_x) = Y, \quad \forall x \in X$

(9.9) $\qquad \exists k > 0 \ s.t. \ \forall x \in X, \ \forall y \in Y, \ \exists z \in (Df_x)^{-1}(y)$

with the property:

$$\|z\| \le k\|y\|.$$

Then f is surjective.

Proof. If suffices to prove that $0 \in f(X)$. Define the functional $\Phi : X \to \mathbb{R}$ by $\Phi(x) = \|f(x)\|$. Clearly Φ satisfies the conditions for the applicability of the Ekeland Variational Principle. So given $\epsilon > 0$ there exists $x_\epsilon \in X$ such that

(9.10) $\qquad \|f(x_\epsilon)\| \le \operatorname{Inf}_X \|f(x)\| + \epsilon$

(9.11) $\qquad \|f(x_\epsilon)\| < \|f(x)\| + \epsilon\|x - x_\epsilon\| \quad \forall x \ne x_\epsilon.$

Take in (9.11) $x = x_\epsilon + tv$, where $t > 0$ and $v \in X$ are arbitrary. Let $\mu_t \in Y^*$ such that

(9.12) $\qquad \|\mu_t\| = 1, \quad \|f(x_\epsilon + tv)\| = \langle \mu_t, f(x_\epsilon + tv)\rangle;$

see Remark 1 after the proof of Proposition 8.4: $\mu_t \in J(f(x_\epsilon + tv)/\|f(x_\epsilon + tv\|)$. We observe that $\|f(x_\epsilon)\| \ge \langle \mu_t, f(x_\epsilon)\rangle$. Altogether, we can write (9.11) as

(9.13) $\qquad \dfrac{\langle \mu_t, f(x_\epsilon + tv)\rangle - \langle \mu_t, f(x_\epsilon)\rangle}{t} \ge -\epsilon\|v\|.$

By the Banach–Alaoglu theorem (i.e., the w^*-compactness of the unit ball in Y^*) and the fact that

$$\frac{1}{t}[f(x_\epsilon + tv) - f(x_\epsilon)] \to Df_{x_\epsilon}(v) \quad \text{(strongly) in} \quad Y$$

we can pass to the limit as $t \to 0$ in (9.12) and (9.13) and obtain

$$(9.14) \qquad \|\mu_0\| = 1, \quad \|f(x_\epsilon)\| = \langle \mu_0, f(x_\epsilon)\rangle$$

$$(9.15) \qquad \langle \mu_0, Df_{x_\epsilon}(v)\rangle \geq -\epsilon\|v\| \quad \text{for all} \quad v \in X.$$

Now using hypothesis (9.8) and (9.9) we can select a $v \in X$ such that $Df_{x_\epsilon}(v) = -f(x_\epsilon)$ and $\|v\| \leq k\|f(x_\epsilon)\|$. All this gives

$$\langle \mu_0, f(x_\epsilon)\rangle \leq \epsilon k\|f(x_\epsilon)\|.$$

So if we start with an ϵ such that $\epsilon k < 1$, the last inequality contradicts (9.14), unless $f(x_\epsilon) = 0$. □

Remark 1. The passage to the limit in the above proof requires a word of caution. If X is separable then the w^*-topology of the unit ball in X^* is metrizable. So in this case we can use sequences in the limiting questions. Otherwise we should use filters. See Dunford–Schwartz [35, p. 426].

Remark 2. Let $L : X \to Y$ be a bounded linear operator with closed range. Then there exists a constant $k > 0$ such that for each $y \in \mathcal{R}(T)$ there is an $x \in X$ with properties that $y = Lx$ and $\|x\| \leq k\|y\|$. This is a classical result of Banach and it can be proved from the Open Mapping Theorem in a straightforward way: consider the operator $\hat{T} : X/N(T) \to \mathcal{R}(T)$. In this set-up it is contained in Theorem 9.3 above. Now let us see which implications this has to Theorem 9.6 above. Condition (9.8) implies that the inequality in (9.9) holds with a k depending on x. Viewing a generalization of Theorem 9.6 let us define a functional $k : X \to \mathbb{R}$ as follows. Assume that $f : X \to Y$ has a Gâteaux derivative with the property that $\mathcal{R}(Df_x)$ is the whole of Y. For each $x \in X$, $k(x)$ is defined as a constant that has the property

$$(9.16) \qquad \|x\| \leq k(x)\|y\| \quad \forall y \in Y \quad \text{and some} \quad x \in (Df_x)^{-1}y.$$

We remark that for each $x \in X$, the smallest value possible for $k(x)$ is the norm of the T^{-1} where $T : X/N(Df_x) \to Y$.

Theorem 9.7. *Let X and Y be Banach spaces and $f : X \to Y$ a continuous mapping which is Gâteaux differentiable. Assume*

$$(9.17) \qquad \mathcal{R}(Df_x) = Y, \quad \forall x \in X$$

$$(9.18) \qquad \forall R > 0 \quad \exists c = c(R) \text{ s.t. } k(x) \leq c, \quad \forall \|x\| \leq R.$$

$$(9.19) \qquad \|f(x)\| \to \infty \quad \text{as} \quad \|x\| \to \infty.$$

Then f is surjective.

Proof. It suffices to prove that $0 \in f(X)$. Define $\Phi : X \to \mathbb{R}$ by $\Phi(x) = \|f(x)\|$. Let $\rho = \|f(0)\|$. It follows from (9.19) that there exists $R > 0$ such that

$$(9.20) \qquad \|f(x)\| \geq \frac{3}{2}\rho \quad \text{if} \quad \|x\| \geq R.$$

Choose an $\epsilon > 0$ such that $\epsilon c(R) < 1$ and $\epsilon \leq \rho/2$. By the Ekeland Variational Principle there exists $x_\epsilon \in X$ such

$$(9.21) \qquad \|f(x_\epsilon)\| \leq \operatorname{Inf}_X \Phi + \epsilon \leq \rho + \epsilon \leq 3\rho/2.$$

$$(9.22) \qquad \|f(x_\epsilon)\| < \|f(x)\| + \epsilon\|x - x_\epsilon\|, \quad \forall x \neq x_\epsilon$$

It follows from (9.20) and (9.21) that $\|x_\epsilon\| \leq R$. Now we preceed as in the proof of Theorem 9.6 and conclude that $f(x_\epsilon) = 0$. □

Remark 1. If $X = Y$ and f = identity + compact is a continuously Fréchet differentiable operator, the surjectivity of f has been established by Kačurovskii [50] under hypothesis (9.19) and $N(Df_x) = \{0\}$ for all $x \in X$. Since Df_x is also of the form identity + compact, such a condition is equivalent to (9.17); this is a special case of the situation described in Remark 4 after the statement of Theorem 9.5. So Kačurovskii result would be contained in Theorem 9.7 provided one could prove that in his case condition (9.18) holds. Is it possible to do that? In the hypotheses of Kačurovskii theorem, Krasnoselskii [54] observed that f is also injective.

Remark 2. Local versions of Theorem 9.7 have been studied by Cramer and Ray [28], Ray and Walker [69].

Comparison with the Inverse Mapping Theorem. The classical inverse mapping theorem states: "Let X and Y be Banach spaces, U an open neighborhood of x_0 in X, and $f : U \to Y$ a C^1 function. Assume that $Df_{x_0} : X \to Y$ is an isomorphism (i.e., a linear bounded injective operator from X onto Y, and then necessarily with a bounded inverse). Then there exists an open neighborhood V of $x_0 / V \subset U$, such that $f|_V : V \to f(V)$ is a diffeormorphism". The injectivity hypothesis can be withdrawn from the theorem just stated provided the thesis is replaced by f being an open mapping in a neighborhood of x_0. More precisely we have the following result due to Graves [48]. If you have the book by Lang [56], the result is proved there.

Theorem 9.8. *Let X and Y be Banach spaces, U an open neighborhood of x_0 in X, and $f : U \to Y$ a C^1 function. Assume that $Df_{x_0} : X \to Y$ is surjective. Then there exists a neighborhood V of x_0, $V \subset U$, with the property that for every open ball $B(x) \subset V$, centered at x, $f(V)$ contains an open neighborhood of $f(x)$.*

Remark 1. If the mapping $f : X \to Y$ is defined in the whole of X, and it is C^1 with $\mathcal{R}(Df_x) = Y$ for all $x \in X$, Graves theorem says that $f(X)$ is open in Y. If we have as an additional hypothesis that $f(X)$ is closed, it follows then that $f(X) = Y$, in view of the connectedness of Y. Now go back and read the statement of Theorem 9.4. What we have just proved also follows from Theorem 9.4, using relation (9.2). Observe that $\mathcal{R}(Df_x) = Y$ is much stronger a condition that $N(Df_x^*) = \{0\}$. The latter will be satisfied if $\mathcal{R}(Df_x)$ is just dense in Y. We remark that the proof of Graves theorem via an iteration scheme uses the fact that $\mathcal{R}(Df_x)$ is the whole of Y. We do not know if a similar proof can go through just with hypothesis that $\mathcal{R}(Df_x)$ is dense in Y.

Remark 2. Graves theorem, Theorem 9.8 above, can be proved using Ekeland Variational Principle. Since few seconds are left to close the set, we leave it to the interested reader.

The following global version of the inverse mapping theorem is due to Hadamard in the finite dimensional case. See a proof in M. S. Berger [10] or in J. T. Schwartz *NYU* Lecture Notes [73]. More general results in Browder [19].

Theorem 9.9. *Let X and Y be Banach spaces and $f : X \to Y$ a C^1 function. Suppose that $Df_x : X \to \mathbb{R}$ is an isomorphism. For each $R > 0$, let*

$$\varsigma(R) = \operatorname{Sup}\{\|(Df_x)^{-1}\| : \|x\| \leq R\}.$$

Assume that

$$\int^{\infty} \frac{dr}{\varsigma(r)} = \infty.$$

[In particular this is case if there exists, constant $k > 0$ such that $\|(Df_x)^{-1}\| \leq k$ for all $x \in X$]. Then f is a diffeomorphism of X onto Y.

Remark. Go back and read the statement of Theorem 9.6. The ontoness of the above theorem, at least in the particular case, is contained there.

Bibliography

[1] H. Amann and P. Hess — A multiplicity result for a class of elliptic boundary value problems - Proc. Royal Soc. Edinburgh 84A (1979), 145-151.

[2] A. Ambrosetti and G. Prodi — Analisi non lineare, I Quaderno - Pisa (1973).

[3] A. Ambrosetti and G. Prodi — On the inversion of some differentiable mappings with singularities between Banach spaces - Ann. Mat. Pura ed Appl. 93 (1972), 231-246.

[4] A. Ambrosetti and P. H. Rabinowitz — Dual variational methods in critical point theory and applications - J. Functional Anal 14 (1973), 349-381.

[5] J. Appell — The superposition operator in function spaces - a survey, Report No. 141 - Institut für mathematik Augsburg (1987).

[6] J.-P. Aubin and I. Ekeland — Applied Nonlinear Analysis - John Wiley and Sons (1984).

[7] P. W. Bates and I. Ekeland — A saddle point theorem - Differential Equations. Academic Press (1980), 123-126.

[8] B. Beauzamy — Introduction to Banach spaces and their geometry - Notas de Matemática, 86, North Holland (1982).

[9] H. Berestycki — Le nombre de solutions de certains problèmes semilineaires - J. Fctl. Anal.

[10] M. S. Berger — Nonlinearity and Functional Analysis - Academic Press (1977).

[11] M. Berger and E. Podolak — On the solutions of a nonlinear Dirichlet problem - Indiana Univ. Math. J. 24 (1975), 837-846.

[12] E. Bishop and R. R. Phelps — The support functionals of a convex set - Proc. Symp. Pure Math. Amer. Math. Soc., Vol 7 (1962), 27-35.

[13] H. Brezis — The Ambrosetti-Rabinowitz *MPL* via Ekeland's minimization principle. (Manuscript).

[14] H. Brezis — Opérateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert - Notas de Matematica No. 50. (North-Holland (1973).

[15] H. Brezis — Analyse fonctionelle - Masson (1983).

[16] H. Brezis and F. E. Browder — A general principle on ordered sets in nonlinear functional analysis - Adv. Math. 21 (1976), 355-364.

[17] A. Brøndsted — On a lemma of Bishop and Phelps - Pac. J. Math. 55 (1974), 335-341.

[18] A. Brøndsted and R. T. Rockafellar — On the subdifferentiability of convex functions - Proc. AMS 16 (1965), 605-611.

[19] F. E. Browder — Nonlinear operators and nonlinear equations of evolution in Banach spaces - Proc. Symp. Pure Math. Amer. Math. Soc. Vol XVIII, Part 2 (1976).

[20] F. E. Browder — Normal solvability for nonlinear mappings into Banach spaces - Bull Amer. Math. Soc. 77 (1971), 73-77.

[21] F. E. Browder — Normal solvability and the Fredholm alternative for mappings in infinite dimensional manifolds - J. Fctl. Anal. 8 (1971), 250-274.

[22] J. Caristi — Fixed point theorems for mappings satisfying unwardness conditions - Trans. Amer. math. Soc. 215 (1976), 241-251.

[23] J. Caristi and W. A. Kirk — Mapping Theorems in Metric and Banach spaces - Bull. Acad. Pol. Sci. XXIII (1975), 891-894.

[24] K. C. Chang — Solutions of asymptotically linear operator equations via Morse theory - Comm. Pure App. Math. XXXIV (1981), 693-712.

[25] K. C. Chang — Variational methods and sub - and supersolutions - Scientia Sinica A XXVI (1983), 1256-1265.

[26] J. A. Clarkson — Uniformly convex spaces - Transactions AMS 40 (1936), 396-414.

[27] D. G. Costa, D. G. de Figueiredo and J. V. A. Gonçalves - On the uniqueness of solution for a class of semilinear elliptic problems - J. Math. Anal. Appl. 123 (1987), 170-180.

[28] W. J. Cramer Jr. and W. O. Ray — Solvability of nonlinear operador equations - Pac. J. Math. 95 (1981), 37-50.

[29] E. N. Dancer — On the ranges of certain weakly nonlinear elliptic partial differential equations - J. Math. Pures et Appl. 57 (1978), 351-366.

[30] J. Daneš — A geometric theorem useful in nonlinear functional analysis - Boll. Un. Mat. Ital. 6 (1972), 369-375.

[31] J. Daneš — Equivalence of some geometric and related results of nonlinear functional analysis - Comm. Math. Univ. Carolinae 26 (1985), 443-454.

[32] J. Diestel — Geometry of Banach spaces - Lecture Notes in Mathematics 485, Springer Verlag (1975).

[33] C. L. Dolph — Nonlinear integral equations of Hammerstein type - Trans. AMS 66 (1949), 289-307.

[34] J. Dungundji — Topology - Allyn & Bacon (1966).

[35] N. Dunford and J. T. Schwartz — Linear Operators, Part I: General Theory, Interscience Publishers, Inc. New York (1957).

[36] I. Ekeland and R. Temam — Convex Analysis and Variational Problems - North-Holland Publishing Company (1976).

[37] I. Ekeland — Sur les problèmes variationnels, CR Acad. Sci. Paris 275 (1972), 1057-1059.

[38] I. Ekeland — On the variational principle, J. Math. Anal. Appl. 47 (1974), 324-353.

[39] I. Ekeland — Nonconvex minimization problems, Bull. Amer. Math. Soc. 1 (1979), 443-474.

[40] K. Fan and I. Glicksberg — Some geometrical properties of the spheres in a normed linear space - Duke Math. J. 25 (1958), 553-568.

[41] D. G. de Figueiredo — On the superlinear Ambrosetti-Prodi problem - Nonl. Anal. TMA 8 (1984), 655-665.

[42] D. G. de Figueiredo and J.-P. Gossez — Conditions de non-resonance pour certains probl{emes elliptiques semilinéaires - C.R.A.S. Paris 302 (1986), 543-545.

[43] D. G. de Figueiredo and S. Solimini - A variational approach to superlinear elliptic problems - Comm. in PDE 9 (1984), 699-717.

[44] S. Fučik — Remarks on a result by A. Ambrosetti and G. Prodi - Boll. Un. Mat. Ital. 11 (1975), 259-267.

[45] T. Gallouet and O. Kavian — Resultats d'existence et de non-existence pour certains problèmes demi-linéaires a l'infini - Ann. Fac. Sci. Toulouse, III (1981), 201-246.

[46] D. Gilbarg and N. S. Trudinger — Elliptic Partial Differential Equations - Springer Verlag (1977).

[47] J.-P. Gossez — Existence of optimal controls for some nonlinear processes - J. Optim. Th. and Appl. 3(1969), 89-97.

[48] L. M. Graves — Some mapping theorems - Duke Math. J. 17 (1950), 111-114.

[49] R. C. James — Weak compactness and reflexivity - Israel J. Math. 2 (1964), 101-119.

[50] R. J. Kačurovskii — Generalization of the Fredholm theorems and of theorems on linear operators with closed range to some classes of nonlinear operators - Sov. Math. Dokl. Vol. 12 (1971), 487-491.

[51] R. Kannan and R. Ortega — Superlinear elliptic boundary value problems - Csech Math. J.

[52] J. Kasdan and F. Warner — Remarks on some quasilinear elliptic equations - Comm. Pure Appl. Math. XXVIII (1975), 367-397.

[53] ~~M. A. Krasnoselskii~~ — On several new fixed point principles - Sov. Math. Dokl. Vol. 14 (1973), 259-261.

[54] M. A. Krasnoselsii — Topological methods in the theory of nonlinear integral equations - MacMillan, New York (1964).

[55] M. A. Krasnoselskii and P. P. Zabreiro — On the solvability of nonlinear operator equations - Funkcional. Anal. i Priložen. 5 (1971), 42-44.

[56] S. Lang — Real Analysis - Addison-Wesley Publ. C. (1969).

[57] A. Laser and P. J. McKenna — On multiple solutions of a nonlinear Dirichlet problem - Differential Equations, Acadmic Press (1980), 199-214.

[58] A. R. Lovaglia — Locally uniformly convex Banach spaces - Transactions AMS 78 (1955), 225-238.

[59] C. A. Magalhaes — Doctoral dissertation (University of Brasilia).

[60] J. Mawhin, J. R. Ward Jr. and M. Willem — Variational methods and semilinear elliptic equations, Sem. Math. Louvain, Rap. 32 (1983).

[61] J.-P. Penot — The drop theorem, the petal theorem and Ekeland's Variational Principle - Nonlinear Analysis TMA 10 (1986), 813-822.

[62] R. R. Phelps — Support cones in Banach spaces and their applications - Adv. Math. 13 (1974), 1-19.

[63] S. I. Pohožaev — Normal solvability of nonlinear equations - Soviet Mat. Dokl. 10 (1969), 35-38.

[64] S. I. Pohožaev — On nonlinear operators having weakly closed range and quasilinear elliptic equations - Math. USSR Sbornik 7 (1969), 227-250.

[65] S. I. Pohožaev — Normal solvability Banach spaces - Funkcional Anal. i Priložen 3 (1969), 80-84.

[66] P. H. Rabinowitz — Some minimax theorems and applications to nonlinear partial differential equations - Nonlinear Analysis: a collection of papers in honor of E. H. Rothe, (L. Cesari, R. Kannan and H. F. Weinberger, editors) Academic Press (1978), 161-177.

[67] P. H. Rabinowitz — Some critical point theorems and applications to semilinear elliptic partial differential equations - Ann. Scuola Norm Sup. Pisa, Ser. IV, 5 (1978), 215-223.

[68] P. H. Rabinowitz — Variational methods for nonlinear eigenvalue problems, (The Varenna Lectures) - Eigenvalues of nonlinear problems (G. Prodi, editor) C.I.M.E., Edizioni Cremonese Roma (1974), 140-195.

[69] W. O. Ray and A. M. Walker — Mapping theorems for Gateaux differentiable and accretive operators - Nonlinear Anal. TMA 6 (1982), 423-433.

[70] I. Rosenheltz and W. O. Ray — Mapping theorems for differentiable operators - Bull. Acad. Polon. Sci.

[71] B. Ruf - On nonlinear problems with jumping nonlinearities - Ann. Math. Pura App. 28 (1981), 133-151.

[72] L. Schwartz — Théorie des distributions, Vol. I - Hermann, Paris (1957).

[73] J. T. Schwartz — Nonlinear functional analysis — NYU Lecture Notes (1963-1964).

[74] G. Scorsa - Dragoni — Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto at un'altra variabile - Rend. Sem. Mat. Univ. Padova 17 (1948), 102-106.

[75] V. L. Šmulian — Sur la derivatilité de la norme dans l'espace de Banach - Dokl. Akad. Nauk 27 (1940), 643-648.

[76] S. Solimini - Existence of a third solution for a class of b.v.p. with jumping nonlinearities - J. Nonlinear Anal. 7 (1983), 917-927.

[77] G. Stampacchia — Le problème de Dirichlet pour les équations elliptiques du second ordre à coeficients descontinus - Ann. Inst. Fourier 15 (1965), 189-258.

[78] M. M. Vainberg — Variational methods in the study of nonlinear operators - Holden Day (1964).

[79] K. Yosida — Functional Analysis - Springer Verlag (1974).