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Critical points with lack of compactness and
singular dynamical systems

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Critical Points with Lack of Compactness and Singular Dynamical Systems (*) (**).

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*Sunto. - Si prova l'esistenza di punti critici di funzionali che non verificano la condizione (PS).
I teoremi astratti vengono applicati per trovare soluzioni periodiche di sistemi dinamici con
potenziali sia limitati sia con singolarità.*

0. - Introduction.

This paper has two main purposes: first, to prove the existence of T -periodic solutions of n -dimensional dynamical systems like

$$(0.1) \quad -\ddot{y} = \text{grad}_y V(t, y),$$

with potential V which (is T -periodic in t and) has singularities; second, to study the critical points of functionals whose Euler equations are (0.1). More precisely, the kind of functionals we are interested in are that ones of the form

$$(0.2) \quad f(u) = \frac{1}{2} (Au, u) + g(u),$$

where A is a linear selfadjoint operator acting on a Hilbert space E , (\cdot, \cdot) denotes the scalar product in E and g is a nonlinear C^1 map.

The main specific features of f are:

- (a) A has a kernel X with $n = \dim X < \infty$ and, roughly, both $g(x)$ and $\text{grad } g(x) \rightarrow 0$ as $x \in X$ and $\|x\|_E \rightarrow \infty$;
- (b) g (and hence f) are possibly defined on an open subset A of E .

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ANTONIO AMBROSETTI - VITTORIO COTI ZELATI: *Critical points with, etc.*

As a consequence of (a), f does not satisfy the compactness condition introduced by Palais and Smale (see § 1). Actually here we show that $c = 0$ is the only level where PS fails to hold.

The idea to overcome such a lack of compactness is the following: under some mild assumptions involving, roughly, the behaviour of $g|_X$ at infinity, we are able to prove that the level set $\{f < c\}$ is topologically equivalent to an n -dimensional sphere. This enables us to find critical points for f on A , via Morse type arguments. If A is a proper subset of E , a control on the behaviour of f on ∂A is needed; on the other hand, our approach permits to take advantage of the possibly rich topological structure of A : for example, if A has infinitely many non trivial homology groups, then we show that f has infinitely many critical points on A .

The abstract setting is contained in Part I, consisting of sections 1, 2 and 3: in § 1 we list the preliminaries, while § 2 and § 3 contain the theorems on the existence of critical points of f under two different kinds of assumptions at infinity for $g|_X$.

The results of Part I are applied in Part II to find periodic solutions of (0.1). More precisely, we assume $V: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ (Ω open subset of \mathbb{R}^n) is T -periodic in t and is such that:

$$(0.3) \quad V(t, y) \rightarrow 0 \quad \text{and} \quad \text{grad}_y V(t, y) \rightarrow 0 \quad \text{as } |y| \rightarrow \infty,$$

and look for T -periodic solutions of (0.1) as critical points of

$$(0.4) \quad f(u) = \frac{1}{2} \int_0^T |u'|^2 dt - \int_0^T V(t, u) dt$$

on $A = \{u \in H^{1,2}(S^1, \mathbb{R}^n); u(t) \in \Omega\}$. The functional f in (0.4) is of the form (0.2) and exhibits (a) and (b), so that the abstract results apply. After dealing in § 5 with bounded potentials (i.e. $\Omega = \mathbb{R}^n$ and $A = H^{1,2}$), we consider in § 6, 7, the situation we are mainly interested in: when Ω has a compact boundary $\partial\Omega$ and $V(t, y) \rightarrow -\infty$ as $y \rightarrow \partial\Omega$. In such a case A has a richer topological structure and the stronger critical points theorems of Part I can be employed provided a further condition on the behaviour of V near $\partial\Omega$ is assumed: namely that $\text{grad}_y V(t, y)$ is a «strong force» in the sense of GORDON [16]. The kind of results we can prove are illustrated by the following example: if $\Omega = \mathbb{R}^n \setminus \{0\}$ ($n \geq 2$), and $V(t, y)$ behaves like $-|x|^{-\alpha}$ with $\alpha > 2$ near $x = 0$ and like $\pm |x|^{-\beta}$ with $\beta > 0$ as $|x| \rightarrow \infty$, then (0.1) has infinitely many T -periodic solutions.

According to the Abstract Setting, we point out that in the applications to (0.1) only asymptotic conditions on V are required.

In the last section (§ 7) we shortly discuss extensions to cover autonomous systems, i.e. to the case when V does not depend on t .

Papers somewhat related to ours are [6; 9; 10; 12; 13; 16; 17; 18; ...]. We refer

to Remarks 3.6, 3.5 and 6.3 for comparison with those papers. Here we would like to spend few words to indicate the differences with [12] and [16].

The idea to overcome the lack of PS evaluating directly the topology of $\{f < c\}$ has been first used in [12], even if in a particular case. Actually, Part I here furnishes a general abstract tool which can be used to study the specific problem of [12].

In [16], the definition of «strong force» has been first introduced. The main difference with the present paper is that Gordon assumes $\partial\Omega$ is complicated enough, in such a way that the corresponding Δ splits in components where the functional f is coercive. In particular no lack of PS arises in [16]. For example, in the case listed before (i.e. $\Omega = \mathbb{R}^n \setminus \{0\}$), Gordon's result applies only if $n = 2$ (see also [9, 10, 18]).

Some results and the main ideas of this paper have been presented at the meeting «Recent developments in Hamiltonian systems», held at L'Aquila, Italy, June 1986 [2]. At that meeting we learned that GRECO [19] had meantime proved the existence of one periodic solution for (0.1) with singular potentials of the type we study in Theorems 6.3 and 7.1.

PART I: ABSTRACT SETTING

1. - Notations and preliminaries.

In this section we will state some basic tools and results in critical point and Morse theory. Such results are essentially known even if not in the specific way we shall need in the following.

Let E be a Hilbert space with scalar product (\cdot, \cdot) and norm $|\cdot|$. We will deal in the following with functionals f which are possibly defined on an open subset A of E . We denote by ∂A the (possibly empty) boundary of A . We also set $B_\infty = \{u \in E : |u| < R\}$.

If $f \in C^1(A; \mathbb{R})$, we set $f'(u) = \text{grad } f(u)$, $Z(f) = \{u \in A : f'(u) = 0\}$, and, for $c \in \mathbb{R}$, $Z_c(f) = \{u \in Z(f) : f(u) = c\}$, $\{f < c\} = \{u \in A : f(u) < c\}$, $\{f \leq c\} = \{u \in A : f(u) \leq c\}$, $\{a < f < b\} = \{u \in A : a < f(u) < b\}$, etc. We will also abbreviate $f^* = \{f < c\}$.

If S is a subset of A , we say that f satisfies the PS (Palais-Smale) condition on S if for every sequence $\{u_n\}$ in S such that $f(u_n)$ is bounded and $f'(u_n) \rightarrow 0$ there exists a converging subsequence $u_{n_k} \rightarrow u \in A$.

We will often refer to the steepest descent flow associated to a functional f . This is, essentially, the flow defined by the differential equation $x' = -f'(x)$. We will not enter here in the details of the construction of such a flow: we only recall its properties which we will need in the paper.

PROPOSITION 1.1. - Let A be an open subset of a Hilbert space E , and let $f \in C^1(A; \mathbb{R})$ be bounded from below on A and such that $f(u) \rightarrow +\infty$ as $|u| \rightarrow \partial A$.

Then there exists $\eta \in C^1(\mathbb{R}^+; A)$ (the steepest descent flow) such that the function $f(\eta(t, u))$ is not increasing.

PROOF. - See [21].

We will say that a subset Y of a Hilbert space E is positively invariant under the steepest descent flow or, simply, positively invariant if $\eta(t, y) \in Y$, $\forall y \in Y$, $\forall t > 0$.

PROPOSITION 1.2. - Let A be an open subset of a Hilbert space E , and let $f \in C^1(A; \mathbb{R})$ be bounded from below on A and such that $f(u) \rightarrow +\infty$ as $|u| \rightarrow \partial A$. Let Y be a closed subset of A such that Y is positively invariant and PS holds for f in Y . Then it exists $u \in Y$ such that $f(u) = \min \{f(u) : u \in Y\}$.

PROOF. - The proof is the usual one (see [21], [20]). We only point out that the deformation arguments can be carried over in our setting because: (i) Y is positively invariant; (ii) since Y is closed and PS holds in Y , then $\forall a, b \in \mathbb{R}$, with $Z(f) \cap \{a < f < b\} \cap Y = \emptyset$, $\exists \delta > 0$ such that $\|f'(u)\| > \delta$, $\forall u \in \{a < f < b\} \cap Y$.

Essentially using the same arguments, one can prove:

PROPOSITION 1.3. - Let A be an open subset of a Hilbert space E , and let $f \in C^1(A; \mathbb{R})$ be such that $f(u) \rightarrow +\infty$ as $|u| \rightarrow \partial A$. If PS holds in $\{a < f < b\}$ (where $-\infty < a < b < +\infty$), and $Z(f) \cap \{a < f < b\} = \emptyset$, then f is a deformation retract of A . Moreover, if Y is a closed and positively invariant subset of A such that PS holds in $Y \cap \{a < f < b\}$ and $Z(f) \cap \{a < f < b\} \cap Y = \emptyset$, then $f|_Y$ is a deformation retract of $f|_Y$.

PROOF. - See [20, Lemma 3.3. - a), b)] and the proof of Proposition 1.2.

With $H_*(A)$, $(H_*(A, B), A \supset B)$ we will indicate the Singular Homology groups of the topological space A (of the couple of topological spaces (A, B)). From proposition 1.3, it follows that, if PS holds in $\{a < f < b\}$ (where $-\infty < a < b < +\infty$), and $Z(f) \cap \{a < f < b\} = \emptyset$, then

$$(1.1) \quad H_*(f) = H_*(f^*).$$

Moreover:

PROPOSITION 1.4. - Let $f \in C^1(A; \mathbb{R})$ be such that $f(u) \rightarrow +\infty$ as $|u| \rightarrow \partial A$ and let Y be a closed, positively invariant subset of A . Suppose: a) f satisfies PS in $\{a < f < b\} \cap Y$, with $-\infty < a < b < +\infty$; b) $Z(f) \cap \{a < f < b\} \cap Y$ is compact (if $b < +\infty$ this follows from a)); c) $Z(f) \cap \partial(\{a < f < b\} \cap Y) = \emptyset$; d) f is Fredholm of

index 0 in $Z(f) \cap \{a < f < b\} \cap Y$. Then $\exists q' \in N$ such that

$$(1.2) \quad H_+(f' \cap Y) \cong H_+(f'' \cap Y), \quad \forall q > q'.$$

PROOF. - Since PS holds in $\{a < f < b\} \cap Y$, and $Z(f) \cap \{a < f < b\} \cap Y$ is compact we can use [20, thm. 2.2] to deduce the existence of g for which: (i) a , b and c hold; (ii) g is close to f in the C^1 norm and differs from f only in a small neighborhood of the critical points; (iii) g has only a finite number of nondegenerate critical points in $\{a < g < b\} \cap Y$. From (ii) it follows that Y is positively invariant for g , too. From this fact, and well known results of Morse theory (see, for example [7, thm. B]) it follows (taking into account also the proof of Proposition 1.2) that (1.2) holds for g . Since from (ii) one deduces that $f' \cap Y = g' \cap Y$ and $f'' \cap Y = g'' \cap Y$, this Proposition follows. \square

From this Proposition we deduce

PROPOSITION 1.5. - Let A be an open set of a Hilbert space E , and let $f \in C^1(E; \mathbb{R})$ be Fredholm of index 0 and such that $f(u) \rightarrow +\infty$ as $u \rightarrow \partial A$. Moreover suppose f satisfies PS in the set $\{f > c\}$, $\forall c > 0$.

(i) if $\exists c^* : H_+(f') \neq H_+(f)$, then $Z(f) \cap \{f > c^*\} \neq \emptyset$;

(ii) if:

$$(A) \quad H_+(f) \neq 0 \quad \text{for infinitely many } q \in N,$$

while $\exists c^* > 0$, $\exists q_1 \in N$ such that $H_+(f') = 0$, $\forall q > q_1$, $\forall c \in]0, c^*]$, then f has infinitely many critical points.

PROOF. - (i) Suppose $Z(f) \cap \{f > c^*\} = \emptyset$. Then from (1.1) it follows $H_+(f) = H_+(f')$, $\forall q \in N$, a contradiction.

(ii) Suppose, by contradiction, that $Z(f)$ is finite and take $c \in]0, c^*]$ such that $Z_c(f) = \emptyset$. From the assumptions and using Proposition 1.4 with $a = c$ and $b = +\infty$ one finds $q_1 > 0$:

$$H_+(f) \cong H_+(f'), \quad \forall q > q_1,$$

a contradiction. \square

REMARK 1.6. - Actually a stronger result can be obtained: suppose the assumptions of Proposition 1.5 - (ii) hold. Further, suppose $Z_c(f) = \emptyset$ for some $c \in]0, c^*]$. Then f has infinitely many critical points u_n such that $f(u_n) \rightarrow +\infty$.

In fact if $\sup \{f(u) : u \in Z(f)\} = \sigma < +\infty$, then, applying Proposition 1.4 with $b = \sigma + 1$ and $a = c$, we reach a contradiction as before. \square

2. - Existence of critical points: a first case.

2.1. The PS condition.

We will deal here with functionals $f \in C^1(1; \mathbb{R})$ of the form

$$f(u) = \frac{1}{2} \langle Au, u \rangle + g(u),$$

where $A: E \rightarrow E$ and $g: 1 \rightarrow \mathbb{R}$ satisfy the assumptions listed below. First of all:

A1. A is a linear bounded selfadjoint operator in E with finite dimensional kernel;

$$X = \text{Ker } A, \quad \dim X = n < \infty,$$

$\forall x \in X$, the norm $\|x\|$ will be simply denoted (according to the notation of Part II), by $|x|$. Denoted by W the orthogonal complement to X , one has

$$E = X \oplus W.$$

If P indicates the orthogonal projection onto X , we will set

$$x_n = Pu, \quad w_n = u - x_n.$$

If no confusion arises, the subscript n will be omitted. We will also suppose;

$$A2. \quad \exists \alpha > 0 : \langle Au, u \rangle > \alpha |u|^2, \quad \forall u \in W.$$

On g we will assume:

$$g1. \quad g \in C^1(1; \mathbb{R}) \text{ and } \exists m > 0 : g(u) > -m, \quad \forall u \in 1;$$

$$g2. \quad u_n \in 1, u_n \text{ converges weakly to } u \in \partial 1 \text{ implies } g(u_n) \rightarrow +\infty;$$

$$g3. \quad \text{let } c^* = (2/\alpha)(m+1). \text{ Corresponding to } c^* \text{ there exists } r^* > 0 \text{ and } g_1, g_2 \in C^1(\mathbb{R}^n, \mathbb{R}) \text{ such that:}$$

$$(2.1) \quad g_i(x) > 0, \quad \forall x \in \mathbb{R}^n, i = 1, 2, |x| > r^*;$$

$$(2.2) \quad g_i(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, i = 1, 2;$$

and

$$(2.3) \quad g_1(Pu) < g(u) < g_2(Pu), \quad \forall u \in 1 \text{ with } |Pu| > r^* \text{ and } |w_n| < c^*.$$

REMARK 2.1. - We explicitly note that from (g1) and (g3) it follows that $\{u = Pu - w_n \in E : |Pu| > r^*, |w_n| < c^*\} \cap \partial 1 = \emptyset$.

First of all we derive from the preceding assumptions some consequences.

LEMMA 2.2. - (i) If $u \in f'$, $0 < \varepsilon < 1$, then $|w_\varepsilon| < c^\varepsilon$. (ii) There exists $\varepsilon^0 > 0$ such that if $u \in f'$, $0 < \varepsilon < \varepsilon^0$, then $|Pw_\varepsilon| > r^0$.

PROOF. - (i) Let $u \in f'$, with $0 < \varepsilon < 1$. Using (A2) and (g1) it follows:

$$\frac{1}{2} \alpha |w_\varepsilon|^2 - m < \frac{1}{2} (Aw_\varepsilon, w_\varepsilon) + g(u) = f(u) + \varepsilon.$$

Hence

$$|w_\varepsilon|^2 < (2/\alpha)(m + \varepsilon) < c^\varepsilon.$$

(ii) By contradiction, let $u_n \in A$ be such that $f(u_n) < 1/n$ and $|Pu_n| = r^0$. Letting $w_n = w_{\varepsilon_n}$, from the preceding (i) one has $|w_n| < c^{\varepsilon_n}$. Then (2.3) of (g3) implies:

$$g(u_n) > g_1(Pu_n).$$

Therefore it follows:

$$(2.4) \quad f(u_n) = \frac{1}{2} (Aw_n, w_n) + g(u_n) > g(u_n) > g_1(Pu_n).$$

Since $|Pu_n| = r^0$, then $g_1(Pu_n) > \delta = \min \{g_1(x) : |x| = r^0\}$. By (2.1) $\delta > 0$ and hence we deduce from (2.4):

$$f(u_n) > \delta > 0,$$

a contradiction which proves (ii). \therefore

COROLLARY 2.3. - For all $\varepsilon < \varepsilon^0$ one has

$$f' = F_1^0 \cup F_2^0$$

where

$$F_1^0 = \{u \in f' : |Pu_n| < r^0\},$$

$$F_2^0 = \{u \in f' : |Pu_n| > r^0\}.$$

In particular $F_1^0 \cap F_2^0 = \emptyset$. Moreover the sets F_1^0, F_2^0 are positively invariant under the steepest descent flow of f .

PROOF. - The first statement is a direct consequence of Lemma 2.2. The positive invariance of F_i^0 ($i = 1, 2$) follows from the fact that $\eta(t, u) \in f'$, $\forall t > 0, \forall u \in f'$ and by continuity, from (ii) of Lemma 2.2. \therefore

REMARKS 2.4. - (i) F_1^0 could possibly be empty, while $F_2^0 \neq \emptyset$ for $\varepsilon > 0$. In fact the set $\{x \in X : |x| > r\}$ is contained in F_2^0 provided $r > 0$ is large enough, because $g(x) \rightarrow 0$ as $|x| \rightarrow \infty$ (immediate consequence of (2.2) and (2.3)).

(ii) $f(u) > 0, \forall u \in F_2^0$. This follows from (g3) and Lemma 2.2 - (ii). \therefore

LEMMA 2.5. - Let $u_n \in A$ be such that $f(u_n) < 1/n$ and $|Pu_n| > r^0$. Then, setting $w_n = w_{\varepsilon_n}$, one has

$$(i) \quad |w_n| \rightarrow 0;$$

$$(ii) \quad |Pu_n| \rightarrow \infty.$$

PROOF. - From Lemma 2.2 - (i) one has $|w_n| < c^{\varepsilon_n}$. Since $|Pu_n| > r^0$ by assumption, we can use (g3) to infer that $g(u_n) > g_1(Pu_n) > 0$. Hence, using also (A2),

$$\frac{1}{2} \alpha |w_n|^2 < \frac{1}{2} (Aw_n, w_n) + g(u_n) = f(u_n) < 1/n$$

and (i) follows.

As for (ii), we remark that (2.4) still holds here. If $|Pu_n| < \text{const.}$, it would follow that $Pu_n \rightarrow x \in X$, for a subsequence, and $g_1(Pu_n) \rightarrow g_1(x)$. Since $g_1(x) > 0$ by (2.1), (2.4) implies $f(u_n) > \delta > 0$ for n large, a contradiction. \therefore

We are now in position to investigate the PS condition. An assumption on g' is in order:

g4. (i) $u_n \in A, w_n$ converges weakly to $u \in A$ implies $g'(u_n) \rightarrow g'(u)$ and

(ii) $g'(u_n) \rightarrow 0$ for all $u_n = w_n + x_n$ such that $|w_n| < \text{const.}$ and $|x_n| \rightarrow \infty$.

Let us point out that (g4) implies that

$$(2.5) \quad f' \text{ is Fredholm of index } 0.$$

In fact $f'(u) = Au + g'(u) = Au + Pu - Pu + g'(u)$ and $Au + Pu$ is a linear homeomorphism, while $g'(u) - Pu$ is compact.

Under the above assumptions, the PS condition fails to hold at the level $c = 0$. In fact, if $x_n \in X$ and $|x_n| \rightarrow \infty$ then $(x_n \in A, \text{ see Remark 2.1, and } f(x_n) = g(x_n) \rightarrow 0, \text{ while, as a consequence of (g4), one has } g'(x_n) \rightarrow 0. \text{ The following Lemma says that the preceding one is essentially the only situation in which PS fails to hold.}$

LEMMA 2.6. - (i) For all $\varepsilon > 0$ PS holds in the set $\{f > \varepsilon\}$; (ii) PS holds in every set where $|Pu| < \text{const.}$

PROOF. - (i) Let $\kappa_n \in I$ be such that

$$(2.6) \quad \varepsilon < f(u_n) = \frac{1}{2} (Aw_n, u_n) + g(u_n) < c_1 \quad (u_n = \kappa_n)$$

$$(2.7) \quad f'(u_n) = Aw_n + g'(u_n) \rightarrow 0.$$

From the right-hand side of (2.6) and (g1) we deduce that $\|u_n\| < c_2$ and hence u_n converges weakly, up to a subsequence, to u . We claim that $\|Pu_n\| < c_3$. In fact, otherwise, from (g4) we have that

$$(2.8) \quad g'(u_n) \rightarrow 0.$$

Multiplying (2.7) by u_n , it follows:

$$(Aw_n, u_n) = (f'(u_n), u_n) - (g'(u_n), u_n)$$

and hence

$$\varepsilon \|u_n\|^2 \leq \|f'(u_n)\| \|u_n\| + \|g'(u_n)\| \|u_n\|.$$

Since both $f'(u_n)$ and $g'(u_n)$ tend to 0 (see (2.7) and (2.8)), then $u_n \rightarrow 0$. We can now use the right-hand of (2.3) to find

$$(2.9) \quad f(u_n) < \frac{1}{2} (Aw_n, u_n) + g_2(Pu_n).$$

Since $u_n \rightarrow 0$ and $\|Pu_n\| \rightarrow \infty$, we have from (2.2)

$$\frac{1}{2} (Aw_n, u_n) + g_2(Pu_n) \rightarrow 0.$$

From (2.9) it finally follows that $f(u_n) \rightarrow 0$, in contradiction with the left hand side of (2.6), so that the claim is proved.

Now, if $\|Pu_n\| < c_3$, one has that $Pu_n \rightarrow \xi$ (up to a subsequence) and then $u_n = u_n + Pu_n$ converges weakly to $u + \xi = u$. Since $g(u_n) < c_1$, then by (g2) we infer that $u \in I$. Then from (g4) it follows that $g'(u_n) \rightarrow g'(u)$, and

$$Aw_n = f'(u_n) - g'(u_n) \rightarrow g'(u).$$

Then $u_n \rightarrow u$ and $u_n \rightarrow u$. This completes the proof of (i).

(ii) As in the proof of (i), $f(u_n) < c_1$ implies $\|u_n\| < c_2$. If, in addition, $\|Pu_n\| < \text{const}$, the PS follows as above.

2.b. The topology of f' .

We will always assume (11, 2) and (g1, 2, 3, 4). We recall that, by Corollary 2.3 one has

$$f' = \Gamma_1' \cup \Gamma_2', \quad \forall \varepsilon < \varepsilon^*$$

with $\Gamma_1' \cap \Gamma_2' = \emptyset$. The purpose of this subsection is to study the topology of Γ_1' . Let

$$\Pi(t, u) = t u_n + P u.$$

LEMMA 2.7. For all $0 < \varepsilon < \varepsilon^*$, there exists $\varepsilon' < \varepsilon$ such that

$$\Pi(t, u) \in \Gamma_1', \quad \forall t \in [0, 1], \forall u \in \Gamma_1'.$$

PROOF. - First we remark that $|P(\Pi(t, u))| = |Pu| > r^*$, $\forall t \in [0, 1], \forall u \in \Gamma_1'$. Then, arguing by contradiction, we let $t_n \in [0, 1], u_n \in \Gamma_1'$ be such that

$$(2.10) \quad 0 < \varepsilon < f(\Pi(t_n, u_n + x_n)) = \frac{1}{2} t_n^2 (Aw_n, u_n) + g(t_n u_n + x_n)$$

with $x_n = Pu_n$ and $u_n = u_n - x_n$. From the definition of Γ_1' we have

$$f(u_n) < 1/n \quad \text{and} \quad \|x_n\| > r^*.$$

Using Lemma 2.5 we get

$$(2.11) \quad u_n \rightarrow 0 \quad \text{and} \quad \|x_n\| \rightarrow +\infty.$$

Since $t_n \in [0, 1]$ and $P(t_n u_n + x_n) = x_n$, then (2.11) permits to use (g3) to estimate

$$g(t_n u_n + x_n) < g_2(x_n).$$

Since from (2.11) it follows that $(Aw_n, u_n) \rightarrow 0$ as well as $g_2(x_n) \rightarrow 0$, this gives a contradiction with (2.10).

Set $S^{-1} = \{x \in X : |x| = 1\}$. We can now state:

LEMMA 2.8. - If $Z(f) \cap \Gamma_2' = \emptyset$, $0 < \varepsilon < \varepsilon^*$, then S^{-1} is a deformation retract of Γ_1' .

PROOF. - Since $Z(f) \cap \Gamma_1' = \emptyset$ and by the positive invariance of Γ_2' under the steepest descent flow η (see Corollary 2.3), one has (see § 1, Proposition 1.3) that, $\forall \varepsilon' < \varepsilon$, Γ_1' is a deformation retract of Γ_2' . From the fact that $g(Pu) \rightarrow 0$ as $\|Pu\| \rightarrow +\infty$ we infer that $\exists r' > 0$ such that $X - B_{r'}$ is a subset of Γ_2' . Let $\varepsilon' < \varepsilon$ be such that Lemma 2.7 holds. By Lemma 2.5 - (ii) it is possible to take $\varepsilon' < \varepsilon$ in such a way that (Lemma 2.7 continues to hold and)

$$\|Pu\| > r', \quad \forall u \in \Gamma_1'.$$

Lastly, fix $g > r'$ in such a way that $\Gamma_1' \cap \partial B_g \cap X$ and let θ be the radial projection

$$\theta(t, x) = t g \frac{x}{|x|} + (1-t)x, \quad x \in X, x \neq 0, t \in [0, 1].$$

By the preceding remarks it is readily verified that $\theta(t, P_u) \in \Gamma_1^*$, $\forall t \in [0, 1]$, $\forall u \in \Gamma_1^*$. Hence $\partial B_\varepsilon \cap X$ turns out to be a deformation retract of Γ_1^* through the homotopy obtained combining η , H and θ , and the lemma follows. \square

From the preceding lemma we infer:

COROLLARY 2.9. - Under the hypothesis of Lemma 2.8 one has

$$(2.12) \quad H_*(\Gamma_1^*) = H_*(S^{n-1}).$$

2.6. Existence results.

Our first result is an immediate consequence of Lemma 2.8 and will be applied when, essentially, $A = E$.

THEOREM 2.10. - Suppose (A1, 2), (g1, 2, 3, 4) and

$$(2.13) \quad H_*(A) \neq H_*(S^{n-1}).$$

Then f has at least a critical point in A .

PROOF. - Since PS holds in $\{f > c\}$, $\forall c > 0$ (Lemma 2.6 - (i)), then, if $Z(f) = \emptyset$, we can use the steepest descent flow to obtain (see § 1, Proposition 1.3)

$$H_*(A) = H_*(f^c), \quad \forall c > 0.$$

Using Corollary 2.3 we have (see [15, Proposition 4.12])

$$(2.14) \quad H_*(A) = H_*(\Gamma_1^*) \oplus H_*(\Gamma_2^*), \quad \forall 0 < c < c^*.$$

If $\Gamma_1^* = \emptyset$, we can use Corollary 2.9 to find

$$H_*(A) = H_*(\Gamma_1^*) = H_*(S^{n-1}),$$

which contradicts (2.13).

If $\Gamma_1^* \neq \emptyset$, a critical point of f on A will be found as the minimum of f on Γ_1^* . Such a minimum exists because: (a) f is bounded from below on A (hence on Γ_1^*), since, as a consequence of (g1):

$$f(u) = \frac{1}{2} \langle Au, u \rangle + g(u) > -m, \quad \forall u \in A;$$

(b) PS holds on Γ_1^* (see Lemma 2.6 - (ii)); (c) Γ_1^* is positively invariant under the steepest descent flow η (see Corollary 2.3). Then Proposition 1.2 applies. \square

COROLLARY 2.11. - Suppose (A1, 2), (g1, 3, 4) and let $A = E$. Then $Z(f) \neq \emptyset$. If, in addition, $\Gamma_1^* \neq \emptyset$ for some $c < c^*$, then $\# Z(f) \geq 2$.

PROOF. - $Z(f) \neq \emptyset$ since for $A = E$ (2.13) holds and Theorem 2.10 applies. Let $\Gamma_1^* \neq \emptyset$ for some $c \in]0, c^*[$. According to the proof of Theorem 2.10, f has a minimum $u_1 \in \Gamma_1^*$. A second critical point can be found using a mountain pass type argument taking paths connecting the minimum and a point in Γ_1^* . Every such a path has to cross the surface $\{Pu\} = c^*$ where f takes values $> c^*$, see Lemma 2.2 - (ii). Then the mountain pass level $c > c^*$ is a critical value since PS holds in $\{f > c\}$, $\forall c > 0$. We leave the details to the reader. \square

REMARK 2.12. - In the case in which Corollary 2.11 applies one can be slightly more precise. Namely, if $f \in C^1(E, \mathbb{R})$ and is a Morse functional one can find at least 3 critical points provided $n \geq 2$. In fact the mountain pass critical point has Morse index = 1 [1] and to find a third critical point it suffices to argue by contradiction, using the Morse inequalities, as in Theorem 2.10. We do not carry over the details. \square

Our main existence theorem of this section is:

THEOREM 2.13. - Suppose (A1, 2), (g1, 2, 3, 4) and (A) hold. Suppose, also, that $g \in C^1(A; \mathbb{R})$. Then f has infinitely many critical points on A .

PROOF. - First of all, Corollary 2.3 yields

$$(2.15) \quad H_*(f^c) = H_*(\Gamma_1^*) \oplus H_*(\Gamma_2^*), \quad \forall c \in]0, c^*[, \forall q \in \mathbb{N}.$$

Next suppose, by contradiction, that $Z(f)$ is finite. Then $\exists c \in]0, c^*[$ such that

$$(2.16) \quad Z(f) \cap \Gamma_1^* = \emptyset$$

$$(2.17) \quad Z_c(f) \cap \Gamma_1^* = \emptyset$$

(2.16) allows us to use (2.11) to find

$$(2.18) \quad H_*(\Gamma_1^*) = \{0\}, \quad \forall q \neq 0, n-1,$$

Next let $Y = \Gamma_1^*$, $b = c$ and $a < -m$. From Lemma 2.6 - (ii) and (2.5) assumptions a), b) and d) of Proposition 1.4 hold. Moreover $\partial \Gamma_1^* = \{f = c\} \cap \Gamma_1^*$ and (2.17) yield c). Then from Proposition 1.4 we deduce the existence of $q_1 \in \mathbb{N}$ such that

$$(2.19) \quad H_*(\Gamma_1^*) = \{0\}, \quad \forall q > q_1.$$

Thus (2.15), (2.18) and (2.19) imply:

$$H_*(f^c) = \{0\}, \quad \forall c > \max(q_1, n).$$

We can now apply Proposition 1.5 (ii) and find a contradiction.

According to remark 1.6 we can prove here a stronger result.

THEOREM 2.14. - Suppose (A1, 2), (g1, 2, 3, 4) and (A) hold. Suppose, also, that $g \in C^1(A; R)$ and that

$$(2.20) \quad \exists R, \delta > 0 \text{ such that } (g'(u), Pu) < 0, \quad \forall u \text{ such that } \|u\| < \delta, |Pu| > R.$$

Then f has infinitely many critical points $u_n \in A$ such that $f(u_n) \rightarrow +\infty$.

PROOF. - We first remark that, under the assumption (2.20), $Z(f) \cap \Gamma_1^*$ is compact. In fact let $u_n \in Z(f) \cap \Gamma_1^*$. We know that $f(u_n) > 0, \forall n$. Actually we claim that $f(u_n) > \mu > 0, \forall n$. Otherwise, up to a subsequence, $f(u_n) \rightarrow 0$, and we deduce from Lemma 2.5 that $\exists N > 0$ such that $\forall n > N, |Pu_n| > R$ and $\|u_n\| < R$. For such n 's we therefore have

$$(g'(u_n), Pu_n) < 0, \quad \forall n > N.$$

But $u_n \in Z(f)$ implies

$$\begin{aligned} 0 &= (f'(u_n), Pu_n) \\ &= (Au_n, Pu_n) + (g'(u_n), Pu_n) \\ &= (g'(u_n), Pu_n) < 0, \end{aligned}$$

contradiction which proves the claim. Now the precompactness of $\{u_n\}$ follows from the PS. Set $\varepsilon = \inf \{f(u) : u \in Z(f) \cap \Gamma_1^*\} > 0$, and suppose, by contradiction, that there exists $\delta < +\infty$ such that $Z(f)$ is contained in f^δ . Then $Z(f)$ is compact and, as in proposition 1.4 we can find $g \in C^1(A; R)$ which has only finitely many nondegenerate critical points in $g^{q-1} = f^{\delta-1}$. Reasoning as in Proposition 1.4 we deduce the existence of $q_1, q_2 \in N$ such that

$$\begin{aligned} H_\varepsilon(1) &\cong H_\varepsilon(f^{q-1}), \quad \forall q \in N \\ &\cong H_\varepsilon(g^{q-1}), \quad \forall q \in N \\ &\cong H_\varepsilon(g^{q_1}), \quad \forall q > q_1. \end{aligned}$$

It is easy to see that Corollary 2.3 and Lemma 2.6 holds for g as well. Hence

$$\begin{aligned} H_\varepsilon(g^{q_1}) &\cong H_\varepsilon(\{u \in A : g(u) < \varepsilon/2, |Pu| < r^*\}) \oplus H_\varepsilon(\Gamma_1^{q_1}), \quad \forall q > q_1 \\ &\cong H_\varepsilon(\Gamma_1^{q_1}), \quad \forall q > q_1 \\ &\cong H_\varepsilon(S^{q-1}), \quad \forall q > q_1. \end{aligned}$$

This is a contradiction which proves the Theorem.

REMARK 2.15. - Using Lusternik-Schnirelman category it is possible to prove a result similar to Theorem 2.14, namely:

Suppose (A1, 2), (g1, 2, 3, 4) hold and let $\text{cat}_1 A = +\infty$. Then f has infinitely many critical points in A . Moreover there is a sequence of critical points $\{u_n\}$ such that $f(u_n) \rightarrow +\infty$.

The proof is based on the fact that under such assumptions $\text{cat}_1 f^r$ is finite since: it is finite in Γ_1^* since Γ_1^* is positively invariant and f is bounded from below and satisfies PS there, while Γ_1^* is a subset of $\Sigma^* = \{u \in A : \|u_n\| < c^*, |Pu| > r^*\}$ and $\text{cat}_\infty \Sigma^* = 2$ imply $\text{cat}_1 \Gamma_1^* < \text{cat}_1 \Sigma^* < \text{cat}_\infty \Sigma^* = 2$. We remark that $\text{cat}_1 A = +\infty$ implies (A1).

3. - Existence of critical points: a second case.

We deal here with a functional $f \in C^1(A, R)$ of the form

$$f(u) = \frac{1}{2}(Au, u) + g(u)$$

with A satisfying (A1, 2) and g satisfying (g1, 2), (g4) and

g5. $\forall \varepsilon > 0, \exists r > 0$ such that for all $u \in A$ with $|Pu| > r$ and $\|u_n\| < c$ one has

$$(3.1) \quad g(u) < 0$$

and

$$(3.2) \quad (g'(u), u) > 0.$$

As in § 2 we start investigating the PS condition.

LEMMA 3.1. - The PS condition holds in $\{f > \varepsilon\}, \forall \varepsilon > 0$, and in $f^\delta, \forall \delta > 0$.

PROOF. - As in Lemma 2.6, if $u_n \in A$ satisfies (2.6) and (2.7) one finds $\|u_n\| < c$, (where $\kappa_n = \kappa_n$). If $Pu_n \rightarrow \infty$, one finds again that $\kappa_n \rightarrow 0$. We can now use (3.1) of (g5) to get

$$f(u_n) = \frac{1}{2}(Au_n, u_n) + g(u_n) < \frac{1}{2}(Au_n, u_n).$$

Hence $f(u_n) \rightarrow 0$, in contradiction with (2.6). Thus $|Pu_n| < \text{const.}$ and the conclusion follows as in Lemma 2.6. This proves that the PS holds in $\{f > \varepsilon\}, \forall \varepsilon > 0$. The same argument works for the second statement as well.

As a consequence of (3.2) one has:

LEMMA 3.2. - For all $\varepsilon > 0$, $\exists R > 0$ such that

$$(f'(u), u) > 0, \quad \forall u \in f', \|u\| > R.$$

In particular $f' \cap B_R$ is positively invariant under the steepest descent flow of f .

PROOF. - Arguing by contradiction, suppose there exist $\varepsilon' > 0$ and a sequence $u_n \in f'$ such that $\|u_n\| \rightarrow +\infty$ and $(f'(u_n), u_n) < 0$. From $u_n \in f'$ we infer readily that $\|u_n\| < \text{const.}$ Then $\|Pu_n\| > \|u_n\| - \|u_n\| > \|u_n\| - \varepsilon$, hence $\|Pu_n\| \rightarrow +\infty$. Using (3.2) of (g5) one has

$$(f'(u_n), u_n) = \frac{1}{2}(\Delta u_n, u_n) + (g'(u_n), u_n) > (g'(u_n), u_n) > 0,$$

a contradiction. \therefore

We are in position to state:

THEOREM 3.3. - Suppose (A1, 2) and (g1, 2, 4, 5) hold and that $g \in C^1(A; R)$. Then:

- (i) $Z(f) \neq \emptyset$;
- (ii) if (A) holds, then $\exists u_n \in A$ such that $f'(u_n) = 0$ and $f(u_n) \rightarrow +\infty$.

PROOF. - (i) f is bounded from below on A and

$$\inf_A f < \inf_I f < \inf_I g < 0$$

because of (3.1). Since PS holds on $f^{-\delta}$, $\forall \delta > 0$ (Lemma 3.1), then f attains its minimum on some $\bar{u} \in I$:

$$f(\bar{u}) = \min \{f(u) : u \in A\}.$$

- (ii) Fixed $\varepsilon > 0$, let us take $R > 0$ according to Lemma 3.2. Set

$$S(t, u) = \begin{cases} u & \text{if } \|u\| < R \\ (1-t)u + tRu/\|u\| & \text{if } \|u\| > R. \end{cases}$$

We claim

LEMMA 3.4. - $f' \cap B_R$ is a deformation retract of f' through S .

PROOF. - By a direct calculation one has

$$(3.3) \quad \frac{d}{dt} f(S(t, u)) = (f'(S(t, u)), S(t, u)) \frac{(R/\|u\|) - 1}{1 + ((R/\|u\|) - 1)}.$$

Since $\|S(t, u)\| > R$ whenever $\|u\| > R$, then, using Lemma 3.2, it follows that

$$\frac{d}{dt} f(S(t, u)) < 0 \quad \text{whenever } \|u\| > R \text{ and } S(t, u) \in f'.$$

In particular $u \in f'$ implies that $(d/dt)f(S(t, u))|_{t=0} < 0$. Thus $S(t, u) \in f'$ for $t \in [0, \mu]$ for some $\mu > 0$. If the Lemma is not true, it would exist $\tau > 0$ such that $S(\tau, u) \in f'$ and $(d/dt)f(S(t, u))|_{t=\tau} = 0$. This is clearly a contradiction which proves the Lemma. \therefore

PROOF OF THEOREM COMPLETED. - The proof is similar to that of Theorem 2.14. In fact also here we have that, $\forall b < +\infty$, $Z(f) \cap f^b$ is compact. This follows, essentially, from Lemma 3.2. More precisely, let $u_n \in Z(f) \cap f^b$. Take a subsequence u_{n_k} such that $f(u_{n_k}) \rightarrow c$. If $c \neq 0$, the precompactness follows from PS. If $f(u_{n_k}) \rightarrow 0$, from Lemma 3.2 we deduce that $\|u_{n_k}\| < R$ and it is easy to find a converging subsequence. The proof now follows by contradiction as in theorem 2.14, the only difference being that here $H_n(g^{q_n}) \cong H_n(g^{q_n} \cap B_R) \cong \{0\}$, $\forall q > q_n$ (g being, as before, a C^1 function, having only nondegenerate critical points, which coincide with f outside a neighborhood of the critical points). \therefore

REMARK 3.5. - Also here, as in Remark 2.15, one can show that $\# Z(f) = +\infty$ provided $\text{cat}_1 A = +\infty$. \therefore

REMARK 3.6. - Among papers dealing with lack of PS, [6] have studied functionals with strong resonance at infinity. They have used different methods based on linking arguments and obtained results which are different from ours in generality and form.

Different questions concerning the lack of PS are investigated in [4, 23, ...]. \therefore

PART II: APPLICATIONS

4. - Applications: general framework.

We will apply the abstract results of Part I to find T -periodic solutions of n -dimensional second order systems. Precisely, let Ω be an open subset of R^n , $n \geq 2$ (even if some of the results below will be true even when $n = 1$) and let $V: R \times \Omega \rightarrow R$

be such that

$$(V1) \quad V \in C^1(\mathbb{R} \times \Omega), \quad V(t+T, y) = V(t, y), \quad \forall (t, y) \in \mathbb{R} \times \Omega.$$

Set $V'(t, y) = \nabla_y V(t, y)$. We look for T -periodic solutions of

$$(4.1) \quad -\ddot{y} = V'(t, y).$$

Set $E = H^{1,2}(S^1, \mathbb{R}^n)$ where $S^1 = \mathbb{R}/[0, T]$, with scalar product

$$(u, v) = \int_0^T \langle \dot{u}', \dot{v}' \rangle dt + \int_0^T \langle u, v \rangle dt$$

and norm

$$\|u\| = \left(\int_0^T |\dot{u}'|^2 + \int_0^T |u|^2 \right)^{1/2} \quad (*)$$

Here and below $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean scalar product and norm in \mathbb{R}^n . Let $A: E \rightarrow E$ be defined by

$$(Au, v) = \int_0^T \langle \dot{u}', \dot{v}' \rangle dt.$$

(41) trivially holds with $X = \text{Ker } A = \mathbb{R}^n$. Remark that here

$$Pu = (1/T) \int_0^T u$$

and $W = \{u \in E: \int_0^T u = 0\}$. The Poincaré inequality implies that also (42) holds. Let

$$A = \{u \in E: u(t) \in \Omega, \forall t \in \mathbb{R}\},$$

and define

$$g(u) = - \int_0^T V(t, u(t)) dt.$$

The T -periodic solutions of (4.1) are the critical points of the functional

$$f(u) = \frac{1}{2} (Au, u) + g(u).$$

Remark that f is C^1 (C^2) provided V is C^2 in y (resp. C^3 in y).

(*) From now on, we will write \int for $\int_0^T dt$.

5. - Bounded potentials.

In this section we will take $\Omega = \mathbb{R}^n$ and V satisfying (V1) and:

$$(V2) \quad V(t, y) \rightarrow 0 \text{ as } |y| \rightarrow +\infty \text{ uniformly in } t \text{ and } \exists r_1 > 0, V(t, y) < 0, \forall |y| > r_1;$$

$$(V3) \quad V'(t, y) \rightarrow 0 \text{ as } |y| \rightarrow +\infty \text{ uniformly in } t.$$

We check the assumptions of theorem 2.10 and Corollary 2.11 by showing

LEMMA 5.1. - (i) If (V2) holds, then (g1) and (g3) are true. (ii) If (V3) holds, then (g4) is true.

PROOF. - (i) (g1) is trivial. As for (g3), recall that $\forall w \in W$ one has

$$\|w\|_{L^\infty} \leq c_2 \|w\|_E.$$

Let $r^* = c_2 c^* > r_1$ and set

$$g_1(y) = (1/T) \min \left\{ - \int_0^T V(t, \xi) dt : t \in \mathbb{R}, |y| - c_2 c^* < |\xi| < |y| + c_2 c^* \right\},$$

$$g_3(y) = (1/T) \max \left\{ - \int_0^T V(t, \xi) dt : t \in \mathbb{R}, |y| - c_2 c^* < |\xi| < |y| + c_2 c^* \right\}.$$

Properties (2.1) and (2.2) of g are immediate consequence of (V3). Moreover, from

$$-|Pu| + \|w_0\|_{L^\infty} \leq |u(t)| \leq \|w_0\|_{L^\infty} + |Pu|$$

it follows that $\forall u \in A$ with $\|w_0\|_{L^\infty} < c^*$

$$|Pu| - c_2 c^* < |u(t)| < |Pu| + c_2 c^*$$

from which (g3) immediately follows.

(ii) It is well known that g' is compact. Let $u_n = w_n + x_n$ be such that

$$\|w_n\|_{L^\infty} < \text{const}, \quad |x_n| \rightarrow +\infty.$$

From this and

$$u_n(t) > x_n, \quad |w_n(t)| > |x_n| - c_2 \|w_n\|_{L^\infty}$$

it follows that $u_n(t) \rightarrow +\infty$ uniformly. Then from (V3) one deduces:

$$(g'(u_n), v) = - \int_0^T \langle V'(t, u_n), v \rangle dt \rightarrow 0, \quad \forall v \in E,$$

and the lemma follows. \square

THEOREM 5.2. - Suppose $\Omega = \mathbb{R}^n$ and V satisfies (V1, 2, 3). Then (4.1) has at least one T -periodic solution.

PROOF. - We remark that in the present case ($\Omega = \mathbb{R}^n$), (g2) can be neglected. The result follows, taking into account the discussion in § 4 and Lemma 5.1, from Theorem 2.10. \therefore

The following is an example in which Corollary 2.11 applies:

THEOREM 5.3. - Suppose $\Omega = \mathbb{R}^n$ and V satisfies (V1, 2, 3, 4). In addition suppose

$$(5.1) \quad \exists \xi \in \mathbb{R}^n: \int V(t, \xi) > 0.$$

Then (4.1) has at least 2 T -periodic solutions.

PROOF. - It suffices to note that $f(\xi) = -\int V(t, \xi) < 0$. Hence, by Remark 2.4, (ii) $\xi \in \Gamma^+$, $\forall \varepsilon > 0$. \therefore

REMARK 5.4. - According to Remark 2.12 one could improve the preceding result by showing that (4.1) has a third solution, provided V is C^1 in y , $n \geq 2$ and (4.1) has only non-degenerate solutions. \therefore

REMARK 5.5. - Periodic solutions for dynamical systems with bounded potentials have been studied in the following papers: [8, 11, 12, 22, 23, ...].

Papers [8, 11, 23] deal with even or periodic potential; they both use linking argument to prove existence of one (or more) solutions.

Papers [12, 22] deal with potential which are of the kind we have studied here, but they obtain results different from ours. In particular, [22] proves existence of only the solution corresponding to the minimum under the hypothesis of Theorem 5.3, while [12], using a method similar to the one used here, proves analogous results but under different assumptions. \therefore

6. - Strong forces.

Now we deal with with potentials V with singularities. Let

$$\Omega = \mathbb{R}^n \setminus K$$

with K compact. On the behaviour of V near K we will suppose:

(SF) there exist $\varepsilon > 0$ and $U \in C^1(\Omega; \mathbb{R})$ such that, setting $U' = \text{grad } U$, one has

$$(6.1) \quad U(y) \rightarrow -\infty \quad \text{as } y \rightarrow \bar{y} \in K, \quad y \in \Omega;$$

$$(6.2) \quad V(t, y) \leq -U'(y)^2, \quad \forall t \in \mathbb{R}, \quad \forall y \in K, \quad y = \{y \in \Omega: \text{dist}(y, K) < \varepsilon\}.$$

Condition (SF) (= Strong Force) has been first introduced by GORDON [16]. If $K = \{0\}$, (SF) implies that $V(t, y) \approx -|y|^{-\alpha}$ with $\alpha \geq 2$ as $y \rightarrow 0$.

LEMMA 6.1. - If (SF) holds, then (g2) is true.

PROOF. - [16], [18]. \therefore

In order to use Theorems 2.13, 2.14 and 3.3, we prove

LEMMA 6.2. - If $\Omega = \mathbb{R}^n \setminus K$, with K compact, then (A) holds.

PROOF. - The result is possibly well known, but we do not know a precise reference and thus we report here a sketch of the proof for completeness.

Let $p \in K$ and $R > 0$ be such that K is contained in B_R . Set $\Omega_1 = \mathbb{R}^n \setminus B_R$, $\Omega_2 = \mathbb{R}^n \setminus \{p\}$ and $A_1 = \{u \in E: u(t) \in \Omega_1, \forall t \in \mathbb{R}\}$, $A_2 = \{u \in E: u(t) \in \Omega_2, \forall t \in \mathbb{R}\}$. Clearly $A_1 \supset A \supset A_2$, and since A_1 is a deformation retract of A , A_2 is a retract of A . Then ([15, pag. 37])

$$(6.3) \quad H_*(A) \cong H_*(A_1) \oplus H_*(A_2).$$

Since it is well known [7, (3.10)] that (A) holds for A_1 , then (6.3) implies that (A) holds for A as well. \therefore

We can now prove

THEOREM 6.3. - Suppose $\Omega = \mathbb{R}^n \setminus K$ and let $V \in C^1(\mathbb{R} \times \Omega; \mathbb{R})$ satisfy (V1, 2, 3) and (SF). Then (4.1) has infinitely many T -periodic solutions.

PROOF. - Since V is bounded from above, (g1) holds. Moreover (g3, 4) continue to hold as in Lemma 5.1. Then the result follows from Lemmas 6.1, 6.2 and Theorem 2.13. \therefore

We can also prove

THEOREM 6.4. - Let the assumptions of Theorem 6.3 be satisfied. If, moreover, $\exists R, \delta > 0$ such that $|\xi| > R, |\eta| < \delta$ imply $\langle V'(t, \xi + \eta), \xi \rangle > 0$, then there exists a sequence u_n of T -periodic solutions of (4.1) such that $f(u_n) \rightarrow +\infty$.

PROOF. - It is easy to show, using arguments already used several times, that (2.20) holds and the result then follows from Theorem 2.14. \therefore

As a last application, we consider V satisfying (V1), (V3) and

$$(V4) \quad V(t, y) \rightarrow 0 \quad \text{as } |y| \rightarrow +\infty, \quad \text{uniformly in } t, \quad \text{and } \exists r_1 > 0: \langle V'(t, y), y \rangle < 0, \quad \forall |y| > r_1, \quad \forall t.$$

We show:

LEMMA 6.5. - If V satisfies (V4), then (g5) holds.

PROOF. - From (V4) it follows readily that $V(t, y) > 0, \forall y > r_1, \forall t$. This together with arguments already used in Lemma 5.1, yields (g5).

THEOREM 6.6. - If $\Omega = \mathbb{R}^n \setminus K$ and $V \in C^1(\mathbb{R} \times \Omega; \mathbb{R})$ satisfies (V1, 3, 4) and (SF), then (4.1) has infinitely many T -periodic solutions; moreover there exists a sequence u_n of such solutions such that $f(u_n) \rightarrow +\infty$.

PROOF. - Apply Theorem 3.3, taking into account Lemmas 6.1, 6.2 and 6.5.

REMARK 6.7. - If

$$D = \{y \in \Omega: V'(t, y) = 0, \forall t \in \mathbb{R}\}$$

is not empty, each $y \in D$ is a (constant) solution of (4.1). If D is compact the arguments of Theorems 6.3, 6.4 and 6.6 can be carried over to show that (4.1) has actually infinitely many non-constant solutions. We remark that D is compact if (V4) holds.

REMARK 6.8. - Besides [16], already discussed in the introduction, papers [3, 5, 9, 10, 13, 14, 18] deal with singular potentials.

[3, 5] consider potentials defined in a bounded well Ω , with $V \rightarrow +\infty$ as $y \rightarrow \partial\Omega$.

[18] studies cases when V can have singularities both with $V \rightarrow +\infty$ and $V \rightarrow -\infty$. The existence of one T -periodic solution is proved, assuming further condition at $y = 0$.

[9, 10] are close to [16] and either $n = 2$ or geometrical conditions are assumed which permit to avoid the lack of PS.

[13] studies potentials roughly of the type $|y|^{-\alpha} - |y|^{-\beta}$, case which is different from ours because the corresponding functional is not bounded from below.

In all these papers, (SF) is assumed. The only work where (SF) is violated is [14], but Ω is a bounded well and $V \rightarrow -\infty$ as $x \rightarrow \partial\Omega$. For a discussion of the problems arising when (SF) does not hold, see [17].

We explicitly remark that, in analogy with what seen in § 5, an existence result can be stated for bounded potentials verifying (V4) instead of (V2), namely:

THEOREM 6.9. - If V satisfies (V1, 3, 4) and $\Omega = \mathbb{R}^n$, then (4.1) has at least one T -periodic solution.

PROOF. - Apply Theorem 3.3 - (i). Notice that in this case, however, f attains negative values and has a global (negative) minimum.

7. - Autonomous systems.

If V does not depend on t , our problem becomes to find non-constant periodic solutions of a given period T of the equation

$$(7.1) \quad -\ddot{y} = V'(y).$$

As before, the T -periodic solution of (7.1) are the critical points of the functional

$$f_T(u) = \frac{1}{2} \int_0^T |\dot{u}|^2 dt - \int_0^T V(u) dt.$$

All the setting and the results of the preceding sections apply to the present case, too. But now:

(i) the critical points of V on Ω (i.e. the $y \in \Omega$ such that $V'(y) = 0$) are also critical points of f_T on A corresponding to constant solution of (7.1) (hence periodic of any given period T);

(ii) if $u \in Z(f_T)$, then also $S^1 u = \{u(t + \theta), \theta \in S^1\}$ is contained in $Z(f_T)$. In particular $Z(f_T)$ is infinite whenever it contains a non-constant solution.

Thus additional arguments are required to deduce from the abstract results of § 2, 3 informations on the solutions of (7.1).

To overcome (i), one usually makes assumptions on the set of the critical points of V . In particular, the question arises if $\Omega = \mathbb{R}^n$ - case in which the existence of one or two solutions for (7.1) has been proved (Theorems 5.2, 5.3, 6.9). This kind of arguments have been discussed in [13, Step 4], where we refer to for statements and details.

If V has singularities, we can take advantage of the fact that now, for all given T , f_T has infinitely many critical points. Suppose that

$$(7.2) \quad Z(V) \text{ is compact.}$$

Then, using the same arguments of theorems 2.14 and 3.3, we can show that, for any fixed $T > 0$, f_T has at least one (actually infinitely many) critical points $u \in Z(V)$. As solution of (7.1) u has minimal period $\tau = T/k$ for some integer $k \geq 1$. Take $T' = \tau/2$. As above, $f_{T'}$ has a critical point v which is a solution of (7.1) with period $T' = \tau/2$ hence also a T -periodic solution of (7.1). Moreover $\{v(t)\}_{t \in \mathbb{R}} \neq \{u(t)\}_{t \in \mathbb{R}}$, since τ was the minimal period of u . Repeating this argument, one obtains:

THEOREM 7.1. - Suppose $V \in C^1(\Omega; \mathbb{R})$ satisfies (SF), (7.2) and either (V1, 2, 3) or (V1, 3, 4). Then $\forall T > 0$ (7.1) has infinitely many, non-constant, distinct T -periodic solutions.

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